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AN EFFICIENT COMPUTATION OF THE SOLUTION OF THE BLOCK DECOUPLING PROBLEM WITH COEFFICIENT ASSIGNMENT OVER A RING

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The paper presents procedures to check solvability and to compute solutions to the Block Decoupling Problem over a Noetherian ring and procedures to compute a feedback law that assigns the coefficients of the compensated system while mantaining the decoupled structure over a Principal Ideal Domain. The algorithms have been implemented using MapleV® and CoCoA [7].

1. INTRODUCTION

Systems over rings have recently received a renewed attention, since they appear to be useful in describing various classes of systems such as, for instance, delay differential systems and systems depending on parameters. A number of control problems such as the Disturbance Decoupling Problem, the Model Matching Problem and the Block Decoupling Problem for systems over rings are known to be solvable in theory using geometric methods [8, 9, 10, 13, 14, 16].

An obstacle to the practical implementation of such methods is represented by the fact that the algorithms usually employed for linear systems over a field do not work when the coefficients of the systems belong to a ring. However, new geometric algorithms have recently been found (see [2, 3]), and tools for symbolic computer algebraic computations, such as MapleV®, Matematica® and CoCoA [7], allow us to implement them.

In this paper we describe a number of procedures, implemented using MapleV and CoCoA, that check the solvability conditions of the Block Decoupling Problem, and, in case of positive answer, compute the state feedback which achieves the decoupling for systems over a Noetherian ring.

In the case of a Principal Ideal Domain, if the coefficients or the poles of the closed loop system have to be assigned in order to assure, for instance, stability, a further procedure computes a feedback which achieves the coefficient assignment while maintaining the decoupled structure.

The paper is based on the results of [10] for the solution of the Block Decoupling Problem using the geometric approach and on the results of [11] and [5] on the

Coefficient Assignment Problem. An application to the Block Decoupling Problem with stability for delay differential systems with a finite number of incommensurable delays is given in the examples.

2. PRELIMINARIES AND STATEMENT OF THE PROBLEM

Let Σ be the system defined over a ring R (commutative, with identity, without zero divisors) by

$$\begin{cases} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{cases} \tag{1}$$

where $x(\cdot)$ belongs to the free state module $\mathcal{X} = R^n$, $u(\cdot)$ belongs to the free input module $\mathcal{U} = R^m$, $y(\cdot)$ belongs to the free output module $\mathcal{Y} = R^p$, and A, B, C are matrices of suitable dimensions with entries in R.
A good reference for the reader who is not familiar with algebraic notions such zero divisors, Principal Ideal Domain, Noetherian ring etc. is [15].

Let us assume that the output of (1) splits into k blocks, $k \geq 2$. Writing $y_i \in \mathcal{Y}_i := \mathcal{R}^{p_i}, i = 1, \ldots, k$ with $\sum_{i=1}^k p_i = p$ and $\mathcal{Y} = \mathcal{Y}_1 \oplus \ldots \oplus \mathcal{Y}_k$, the output equations of (1) read as

$$y_i(t) = C_i x(t), \qquad i = 1, \ldots, k, \tag{2}$$

where $C_i : \mathcal{X} \to \mathcal{Y}_i, i = 1, \ldots, k$ are matrices of suitable dimensions with entries in R. Then our problem can be stated as follows.

Problem Statement. Given a system Σ of the form (1), (2), the *Block Decoupling Problem* for Σ, shortly *BDP*, consists in finding, if possible, suitable integers n_a and $m_i, i = 1, \ldots, k$, and a dynamic state feedback law of the form

$$\begin{cases} x_a(t+1) &= A_1 x(t) + A_2 x_a(t) + \sum_{i=1}^k G_{ai} v_i(t) \\ u(t) &= Fx(t) + Hx_a(t) + \sum_{i=1}^k G_i v_i(t) \end{cases} \tag{3}$$

where $x_a \in \mathcal{X}_a := R^{n_a}$, $v_i \in R^{m_i}, i = 1, \ldots, k$, A_1, A_2, F, H, G_i and G_{ai} are matrices of suitable dimensions with entries in the ring R, such that in the compensated system $\Sigma_{F,G}$

$$\Sigma_{F,G} = \begin{cases} x(t+1) &= (A+BF)x(t) + BHx_a(t) + \sum_{i=1}^k BG_i v_i(t) \\ x_a(t+1) &= A_1 x(t) + A_2 x_a(t) + \sum_{i=1}^k G_{ai} v_i(t) \\ y_i(t) &= C_i x(t), \quad i = 1, \ldots, k \end{cases} \tag{4}$$

each block input v_i completely controls y_i, but has no influence on the output y_j for $j \neq i$, $i = 1, \ldots, k$.

In case it is also required that the coefficients of the characteristic polynomial of $\begin{pmatrix} A+BF & BH \\ A_1 & A_2 \end{pmatrix}$ are assigned, we will speak of *BDP* with *Coefficient Assignment* and will modify accordingly the previous definition.

For systems with coefficients in a field, the solvability of the BDP can be characterized in terms of controllability subspaces (see [17]). For systems with coefficients in a ring, it is more convenient, in order to avoid problems connected with the definition of controllability submodules (see [9]), to use pre-controllability submodules.

Definition 1. (see [9]) Given a system Σ described over a ring R by equations of the form (1), a submodule $\mathcal{R}$ is a pre-controllability submodule if

i) $\mathcal{R}$ is (A, B)-invariant, i.e. $A\mathcal{R} \subseteq \mathcal{R} + \operatorname{Im} B$;

ii) $\mathcal{R}$ is the minimum element of the family

$$\mathcal{S}_{\mathcal{R}} = \{S \subseteq X \text{ such that } S = \mathcal{R} \cap (AS + \operatorname{Im} B)\}.$$

Pre-controllability submodules which satisfy the strong condition of being (A, B)-invariant submodules of feedback type, i.e. such that $(A + BF)\mathcal{R} \subseteq \mathcal{R}$ for some static state feedback $F : \mathcal{X} \to \mathcal{U}$, are controllability submodules in classical sense. In general (A, B)-invariance does not imply (A, B)-invariance of feeback type and, to this regards, it is of crucial importance the following result.

Proposition 1. [9] Let Σ be a system defined by (1) over a Noetherian ring R. Then, an (A, B)-invariant submodule of the state module $\mathcal{X} = R^n$ which is a direct summand of $\mathcal{X}$ is an (A, B)-invariant submodule of feedback type.

An important consequence of the above Proposition is that, by using a suitable extension of the system Σ, one can always expand an (A, B)-invariant submodule into an (A, B)-invariant submodule of feedback type in the extended system (see [9]). In other words, this means that an (A, B)-invariant submodule can be made invariant by using a dynamic feedback.

We can state the following result about the solvability of the BDP, whose proof can be found in [10].

Theorem 1. [10] Assume that the system Σ, defined over a Noetherian ring R by (1) and (2), is reachable. Let $\mathcal{R}_i^*$ denote the maximum pre-controllability submodule of Σ contained in $\mathcal{K}_i := \cap_{j=1, j\neq i}^{k} \operatorname{Ker} C_j$. Then, the Block Decoupling Problem with Coefficient Assignment is solvable for the system Σ if and only if

$$\mathcal{R}_i^* + \operatorname{Ker} C_i = \mathcal{X}, \quad i = 1, \dots, k. \tag{5}$$

In practical terms, to check the above solvability conditions for the BDP one has to compute a number of pre-controllability submodules. Since classical geometric algorithms (see [17]) do not converge in a finite number of steps over a ring, new "ad hoc" algorithms for the computation of maximal pre-controllability submodules have been introduced.

Proposition 2. [2] Let Σ be a system defined by (1) over a PID R and let $\mathcal{V} \subseteq \mathcal{X}$ be a submodule. Then, the sequence $\{\mathcal{R}_i^1\}$ defined by

$$\begin{cases} \mathcal{R}_0^1 := \mathcal{S}^*(\operatorname{Im} B) \cap \mathcal{V} \\ \mathcal{R}_k^1 := \mathcal{S}^*(\operatorname{Im} B) \cap \mathcal{V} \cap A^{-1}(\mathcal{R}_{k-1}^1 + \operatorname{Im} B), \end{cases} \tag{6}$$

where $\mathcal{S}^*(\operatorname{Im} B)$ is the maximum submodule of $\mathcal{X}$ containing $\operatorname{Im} B$ and verifying $A(S \cap \operatorname{Ker} C) \subseteq S$, converges in a finite number of steps towards $\mathcal{R}^*(\mathcal{V})$, the maximum pre-controllability submodule contained in $\mathcal{V}$.

Recently, the following more general algorithm was obtained.

Proposition 3. [3] Let Σ be a system defined by (1) over a Noetherian ring R and let $\mathcal{V} \subseteq \mathcal{X}$ be a submodule. Then the sequence $\{\mathcal{R}_i^2\}$ defined by

$$\begin{cases} \mathcal{R}_0^2 := \mathcal{S}^*(\operatorname{Im} B) \cap \mathcal{V} \cap A^{-1}(\operatorname{Im} B) \\ \mathcal{R}_k^2 := \mathcal{S}^*(\operatorname{Im} B) \cap \mathcal{V} \cap A^{-1}(\mathcal{R}_{k-1}^2 + \operatorname{Im} B), \end{cases} \tag{7}$$

where $\mathcal{S}^*(\operatorname{Im} B)$ is the maximum submodule of $\mathcal{X}$ containing $\operatorname{Im} B$ and verifying $A(S \cap \operatorname{Ker} C) \subseteq S$, converges in a finite number of steps towards $\mathcal{R}^*(\mathcal{V})$, the maximum pre-controllability submodule contained in $\mathcal{V}$.

The above algorithms and the results mentioned in Theorem 1 allow us to compute practically a solution to the BDP with Coefficient Assignment when one exists.

3. THE ALGORITHMS

The procedures we describe in this section allow one to check the solvability conditions of the BDP for a system Σ described by (1) and (2) over a Noetherian ring. In case of positive answer a feedback which decouples the system is computed. If the ring is a PID, it is also possible to assign the coefficients, or the poles, of the closed loop system and a feedback is computed which achieves coefficients or pole assignment, while mantaining the decoupled structure.

3.1. Checking the solvability conditions

PID case. In order to check conditions (5) one has to compute the elements of the sequence $\{\mathcal{R}_i^1\}$ using Algorithm (6). This requires the computation of the sum of two submodules, of the intersection of two submodules and of the inverse image of a submodule by a linear map. Over a PID a submodule $\mathcal{V}$ of a free module R^n is free and can be described by a generating-matrix V, namely a matrix whose columns are a minimal set of generators for $\mathcal{V}$. It can be shown that

i) Let $S = P\left(V_1 \mid V_2 \right) Q = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}$, be the Smith Form of the matrix $\left(V_1 \mid V_2 \right)$ with P and Q unimodular, V_1 and V_2 generating matrices of $\mathcal{V}_1$ and $\mathcal{V}_2$ respectively. Writing $Q = \left(Q_1 \quad Q_2 \right)$, we have that the matrix $W = P\left(V_1 \mid V_2 \right) Q_1$ is a generating matrix for the submodule $\mathcal{V}_1 + \mathcal{V}_2$;

ii) denoting by K_{12} a generating matrix for the Kernel of $(\ V_1 \mid -V_2 \ )$, a generating matrix for $\mathcal{V}_1 \cap \mathcal{V}_2$ is given by the matrix $(\ \mathbf{I}_n \mid 0 \ ) K_{12}$;

iii) denoting by K_{A1} a generating matrix for the Kernel of $(\ A \mid -V_1 \ )$, a generating matrix for $A^{-1}(\mathcal{V}_1)$, the inverse image of $\mathcal{V}_1$ by A, is given by the matrix $(\ \mathbf{I}_n \mid 0 \ ) K_{A1}$.

Using MapleV, each of the elementary procedures mentioned above, including the construction of the Smith Form, can be implemented. Then, the elements of the sequence $\{\mathcal{R}_i^1\}$ can be computed using Algorithm (6) and conditions (5) can be practically checked.

Remark that the computations are carried on in a symbolic way and therefore no numerical approximation is involved.

Noetherian case. In the more general case of systems over a Noetherian ring R, more sophisticated tools are needed, since submodules of a free module are not necessarily free and the Smith Form is no longer available. In general, a submodule $\mathcal{V}$ of R^n can be characterized by a generating matrix V whose columns form a Gröbner Basis for $\mathcal{V}$ (see [6]). In order to deal with the general case, we'll need to consider the module of all the solutions of an homogeneous system of linear equations $Vw = 0$ with coefficients in the ring, called the Syzygy module of V. More precisely, let us state the following Definition.

Definition 2. [1] Given the vectors $v_1,\dots,v_s \in R^n$, a syzygy of the $n \times s$ matrix $V = [v_1 \cdots v_s]$ is a vector $w \in R^s$ such that $Vw = 0$. The set of all such syzygies is called the Syzygy module of V.

To construct the elements of the sequence $\{\mathcal{R}_i^2\}$ using Algorithm (7) one has to compute the sum of two submodules and the intersection of two ideals or of two submodules and to determine if an element, or a vector, belongs, respectively, to an ideal or to a submodule (see [1] for details).

If A is an $n \times n$ matrix with entries in a Noetherian ring R, $\mathcal{V}_1$, $\mathcal{V}_2$ are two submodules of R^n and the columns of V_i are a Gröbner basis for $\mathcal{V}_i$, $i = 1, 2$, it can be shown that

i) a Gröbner basis for the submodule $\mathcal{V}_1 + \mathcal{V}_2$ and for $\mathcal{V}_1 \cap \mathcal{V}_2$ can be computed directly from V_i, $i = 1, 2$;

ii) a Gröbner basis for $A^{-1}(\mathcal{V}_1)$, the inverse image of $\mathcal{V}_1$ by A, can be obtained by computing the Syzygy module of the matrix $(\ A \mid -V_1 \ )$ and by taking the first n coordinates of each syzygy.

The package CoCoA [7], devoted to computations in commutative algebra, allows one to compute the Gröbner basis of a submodule and the Syzygy module of a set of homogeneous linear equations. Then, Algorithm (7) can be directly implemented using CoCoA and Gröbner bases R_i for the submodules $\{\mathcal{R}_i^*\}$ are computed. The integers n_i are the number of vectors in R_i. The solvability conditions (5) of the

3.2. Constructing the decoupling feedback

Assume that the solvability conditions (5) are satisfied and that R_i is a generating matrix for $\mathcal{R}_i^*$, $i = 1, \dots, k$. Following the lines in the proof of Theorem 1 (see [10]), we extend the system Σ to a system Σ_e described by the matrices:

$$A_e = \begin{pmatrix} A & 0_{n \times n_a} \\ 0_{n_a \times n} & 0_{n_a \times n_a} \end{pmatrix}, B_e = \begin{pmatrix} B & 0_{n \times n_a} \\ 0_{n_a \times m} & \mathbf{I}_{n_a \times n_a} \end{pmatrix}$$

and

$$C_e = \begin{pmatrix} C & 0_{p \times n_a} \end{pmatrix} \tag{8}$$

where $n_a = n_1 + n_2 + \cdots + n_k$ and the integers n_i, $i = 1, \dots, k$, are the number of elements of a Gröbner Basis R_i for the submodules $\mathcal{R}_i^*$, $i = 1, \dots, k$. The columns of the matrices $\begin{pmatrix} R_i \\ 0 \\ \vdots \\ \mathbf{I}_{n_i \times n_i} \\ 0 \end{pmatrix}$ span the submodules $\mathcal{R}_{ei}^*$, which are (A_e,B_e)-invariant of feedback type, since they are direct summands of R^{n+n_a} (compare with Proposition 1.

The matrix $R_e = \begin{pmatrix} R_1 & \dots & R_k & \mathbf{I}_{n \times n} \\ \mathbf{I}_{n_1 \times n_1} & 0 & \dots & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \mathbf{I}_{n_k \times n_k} & 0 \end{pmatrix}$ is unimodular and we have $R_e^{-1} = \begin{pmatrix} 0 & \mathbf{I}_{n_1 \times n_1} & \dots & 0 \\ 0 & \ddots & 0 & \mathbf{I}_{n_k \times n_k} \\ \mathbf{I}_{n \times n} & -R_1 & \dots & -R_k \end{pmatrix}$.

The first step towards the construction of a Decoupling Feedback is now the computation of matrices $M_i \in R^{m \times n_i}$ and $L_i \in R^{n_i \times n_i}$ such that

$$AR_i = R_i L_i + BM_i,$$

whose existence is guaranteed by the (A, B)-invariance of $\mathcal{R}_i^*$.

This is achieved computing a generating matrix $\begin{pmatrix} X_{i,1} \\ X_{i,2} \\ X_{i,3} \end{pmatrix}$ for the Syzygy module of the matrix

$$\begin{pmatrix} A & -R_i & -B \end{pmatrix}, \tag{9}$$

where the number of rows in $X_{i,1}$, $X_{i,2}$, $X_{i,3}$ is respectively n, n_i and m. Since the columns of $\begin{pmatrix} R_i \\ L_i \\ M_i \end{pmatrix}$ are also elements of the Syzygy module of the matrix (9), a matrix K_i, with n_i columns, can be found by solving a linear system such that

$$R_i = X_{i,1} K_i, \quad L_i = X_{i,2} K_i \quad \text{and} \quad M_i = X_{i,3} K_i.$$

A generating matrix $\begin{pmatrix} G_{ie}^{i,1} \\ G_{ie}^{0,1} \end{pmatrix}$ for the Syzygy module of the matrix $[\ R_i \mid -B\ ]$

gives the matrices $G_{ie}^{i,1}$ and $G_{ie}^{0,1}$ such that

$$R_i G_{ie}^{i,1} = B G_{ie}^{0,1}, \quad i = 1, \dots, k,$$

Then, a static state feedback which solves the Block Decoupling Problem for Σ_e is given by (F_e, G_e)

$$F_e = \begin{pmatrix} F_1 & -M_1 - F_1R_1 & -M_2 - F_1R_2 & \dots & -M_k - F_1R_k \\ 0 & L_1 & 0 & \dots & 0 \\ 0 & 0 & L_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & L_k \end{pmatrix}, \tag{10}$$

(the matrix F_1 can be chosen arbitrarily to satisfy further conditions if needed) and

$$G_e = \begin{pmatrix} G_{1e}^{(01)} & G_{2e}^{(0,1)} & \dots & G_{ke}^{(0,1)} \\ G_{1e}^{(1,1)} & 0 & \dots & 0 \\ 0 & G_{2e}^{(2,1)} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & G_{ke}^{(k,1)} \end{pmatrix}. \tag{11}$$

In fact, (see [10]), we have that

$$R_e^{-1}(A_e + B_eF_e)\,R_e = \begin{pmatrix} L_1 & 0 & \dots & 0 & 0 \\ 0 & L_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & L_k & 0 \\ 0 & 0 & \dots & 0 & A + BF_1 \end{pmatrix} \tag{12}$$

and

$$R_e^{-1}B_eG_e = \begin{pmatrix} G_{1e}^{(1,1)} & 0 & \dots & 0 & 0 \\ 0 & G_{2e}^{(3,1)} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & G_{ke}^{(k,1)} & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{13}$$

The static state feedback (F_e, G_e) which decouples Σ_e gives rise to a dynamic feedback of the form (3) for Σ.

The procedure to compute the matrices $L_i, M_i, G_{ie}^{0,i}, G_{ie}^{i,1}$ implemented using CoCoA is much simpler than the one written with MapleV, which anyway can be used only for systems defined over a PID. In fact, even if also MapleV allows the use of Gröbner Bases, CoCoA is much more efficient and the time required to perform the computations is ten times smaller. On the other hand, in building up the Decoupling Feedback F_e, G_e, MapleV is faster, being more suitable to matrix manipulation.

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3.3. Assigning the coefficients

In the PID case, in addition to decouple one can also stabilize or assign the coefficients of the resulting closed loop system. For this, the integers $\bar{n}, \bar{n}_i$ should be chosen as follows: $\bar{n} = 0$ if $m = 1$ and $\bar{n} = 1$ otherwise, and $\bar{n}_i = 0$ if $\dim(G_{ie}^{(2i-11)}) = 1$, $\bar{n}_i = 1$ otherwise (see [10]). After having achieved the decoupled form, one can assign arbitrarily the coefficients of (A, B) and of each subsystems $(L_i, G_{ie}^{(2i-11)})$, maintaining the decoupled structure, by following the procedure described in [12]. Implementation of such procedure requires essentially the same elementary operations we have already used and can therefore be performed in MapleV or CoCoA.

4. EXAMPLES

To illustrate the procedures described above we will apply them to a couple of delay differential systems.

Example 1. Let us consider the delay-differential system Σ_1' given by:

$$\begin{cases} \dot{x}_1(t) &= x_1(t) + x_2(t) + u_1(t) \\ \dot{x}_2(t) &= x_1(t-\partial) + x_3(t) + u_2(t) \\ \dot{x}_3(t) &= x_2(t) + u_1(t-\partial) \\ y_1(t) &= x_1(t) \\ y_2(t) &= x_2(t-\partial) \end{cases} \tag{14}$$

where h represents a delay. By introducing the delay operators Δ, defined for any function $f(t)$ by $\Delta f(t) = f(t-h)$, we can formally associate to Σ_1' the system Σ_1 over the PID ring $\mathcal{R}[\Delta]$ of polynomials in one indeterminate. Σ_1 is then defined by equation of the form (1), with

$$A := \begin{pmatrix} 1 & 1 & 0 \\ \Delta & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \Delta & 0 \end{pmatrix} \quad \text{and} \quad C := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Delta & 0 \end{pmatrix}.$$

We search for a dynamic feedback law of the form (3) which decouples the two outputs of the system. The algorithm (6) implemented using MapleV, following the lines described above, gives generating matrices

$$R_1 := \begin{pmatrix} 0 & -1 \\ 0 & 0 \\ -\Delta & -\Delta \end{pmatrix} \quad \text{and} \quad R_2 := \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1-\Delta \end{pmatrix}$$

needed to decouple the system. Following the procedure described above we obtain the matrices F_e, G_e (with $F_1 = 0$)

$$F_e := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \Delta & 2\Delta & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1+\Delta \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \quad G_e := \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Since

$$R_e^{-1} = \begin{bmatrix} 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ -\Delta & -\Delta & 0 & 1-\Delta & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

one has

$$R_e^{-1}(A_e + B_e F_e)\, R_e = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1+\Delta & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \Delta & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$R_e^{-1} B_e G_e = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C_e R_e = \begin{bmatrix} 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\Delta & 0 & 0 & \Delta & 0 \end{bmatrix}.$$

The resulting closed loop system is

$$\left\{ \begin{array}{rcl} x_1(t) & = & -x_2(t) \\ \dot{x}_2(t) & = & x_2(t) + v_1(t) \\ \dot{x}_3(t) & = & -x_4(t) + x_4(t-\partial) + v_2(t) \\ \dot{x}_4(t) & = & -x_3(t) \\ \dot{x}_5(t) & = & x_5(t) + x_6(t) \\ \dot{x}_6(t) & = & x_5(t-h) + x_7(t) \\ \dot{x}_7(t) & = & x_6(t) \\ y_1(t) & = & -x_2(t) + x_5(t) \\ y_2(t) & = & -x_3(t-\partial) + x_6(t-\partial). \end{array} \right. \tag{15}$$

The closed loop transfer matrix of the system is then

$$T(s) = \begin{pmatrix} -\dfrac{1}{s-1} & 0 \\ 0 & -\dfrac{e^{-\partial s}s}{-1+e^{-\partial s}+s^2} \end{pmatrix}$$

and the system Σ' is decoupled.
Suppose now that we want to assign all the coefficient at 2. As $\mathcal{R}_i^* \cap \operatorname{Im} B$ is of dimension 1 for $i = 1, 2$, it follows that $\bar{n}_i = 0$ for $i = 1, 2$ as the subsystems $(L_i, G_{ie}^{(2i-11)})$ are cyclic. Thus, $\bar{n}$ must be equal to 1 to make the subsystem (A, B) cyclic. Now, one should take a dynamic extension of $n_a = 5$. The program then computes the new matrices F_e and G_e, giving rise to the closed loop transfer function

$$T(s) = \begin{pmatrix} \dfrac{-s}{4+4s+s^2} & 0 \\ 0 & \dfrac{-e^{\partial s}s}{4+4s+s^2} \end{pmatrix}.$$

The coefficients of the matrix F_e have very involved expressions, so they will not be displayed here. The computation of this example were performed with MapleV on a PC 166 MHz in 26 seconds.

Example 2. We consider now a system with two incommensurable delays, which can be modeled by a system over the ring $\mathbb{R}[\Delta_1, \Delta_2]$, which is not a PID but is Noetherian. Let the delay-differential system Σ_2' be given by the equations

$$\begin{cases} \dot{x}_1(t) &= x_3(t) + u_1(t-h_2) \\ \dot{x}_2(t) &= x_1(t-4h_1) + x_2(t-h_2) \\ \dot{x}_3(t) &= x_1(t-3h_2) + +x_2(t-h_2) + x_3(t) + u_2(t) \\ y_1(t) &= x_2(t) - x_3(t-h_1) \\ y_2(t) &= x_1(t), \end{cases} \tag{16}$$

where h_1, h_2 are two incommensurable delays. Introducing the delay operators Δ_1, Δ_2, defined by $\Delta_i f(x) := f(x-h_i)$, $i = 1, 2$, we can associate to Σ_2' the system Σ_2 defined by the following matrices:

$$A := \begin{pmatrix} 0 & 0 & 1 \\ \Delta_1^4 & \Delta_2 & 0 \\ \Delta_2^3 & \Delta_2 & 1 \end{pmatrix}, \quad B := \begin{pmatrix} \Delta_2 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad C := \begin{pmatrix} 0 & 1 & -\Delta_1 \\ 1 & 0 & 0 \end{pmatrix}$$

and consider the problem of decoupling the two outputs of Σ_2. By CoCoA we compute Gröbner Bases of the submodules $\mathcal{R}_i^*$, $i = 1, 2$,

$$R_1 = \begin{pmatrix} 0 \\ 0 \\ \Delta_2 \end{pmatrix}, \quad R_2 = \begin{pmatrix} \Delta_2 & 0 \\ 0 & \Delta_1^4\Delta_2 \\ 0 & \Delta_1^3\Delta_2 \end{pmatrix}.$$

The Block Decoupling Problem is solvable here only in a weak sense, (see [10]) since Σ_2 is only weakly reachable. In fact, a Gröbner Basis of the reachability submodule of Σ_2 is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \Delta_1^4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

The dimension n_a of the extension required to achieve decoupling is: $n_a = 1+2 = 3$ and the decoupling feedback is given by the matrices F_e, G_e:

$$F_e := \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & f_1 & f_2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta_1^3 \\ 0 & 0 & 0 & 0 & 1 & \Delta_2 \end{bmatrix}, \quad G_e := \begin{bmatrix} 0 & 1 \\ \Delta_2 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

with $f_1 = \Delta_1^3\Delta_2 - \Delta_2^4$, $f_2 = \Delta_1^3\Delta_2^2 - \Delta_1^3\Delta_2 - \Delta_1^4\,\Delta_2^2$.
The decoupled system is then given by

$$W_e^{-1}(A_e + B_e F_e)\,W_e = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta_1^3 & 0 & 0 & 0 \\ 0 & 1 & \Delta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \Delta_1^4 & \Delta_2 & 0 \\ 0 & 0 & 0 & \Delta_2^3 & \Delta_2 & 1 \end{bmatrix},$$

$$W_e^{-1}B_e G_e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C_e W_e = \begin{bmatrix} -\Delta_1\Delta_2 & 0 & 0 & 0 & 1 & -\Delta_1 \\ 0 & \Delta_2 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The transfer function matrix of the compensated system is

$$T(s) = \begin{bmatrix} \frac{\Delta_1\Delta_2}{-1+s} & 0 \\ 0 & \frac{\Delta_2(\Delta_2 - s)}{\Delta_1^3 + s\Delta_2 - s^2} \end{bmatrix}.$$

5. CONCLUSIONS

The paper shows how to practically check the solvability conditions and compute solutions for the Block Decoupling Problem and the Coefficient Assignment Problem The proposed procedures are based on the results obtained using the geometric approach in [10] and perform symbolic computer algebraic computations using MapleV® and CocoA. Examples are provided concerning delay differential systems.

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