KYBERNETIKA — VOLUME 35 (1999), NUMBER 6, PAGES 707-720

SCOPE AND GENERALIZATIONS OF THE THEORY OF LINEARLY CONSTRAINED LINEAR REGULATOR

PAOLO D'ALESSANDRO AND ELENA DE SANTIS

A previous paper by the same authors presented a general theory solving (finite horizon) feasibility and optimization problems for linear dynamic discrete-time systems with polyhedral constraints. We derived necessary and sufficient conditions for the existence of solutions without assuming any restrictive hypothesis. For the solvable cases we also provided the inequative feedback dynamic system, that generates by forward recursion all and nothing but the feasible (or optimal, according to the cases) solutions. This is what we call a dynamic (or automatic) solution. The crucial tool for the development of the theory was the conical approach to linear programming, illustrated in detail in a recent book by the first author. Here we extend this theory in two different directions. The first consists in generalizations for more complex constraint structures. We carry out two cases of mixed input state constraints, yielding the dynamic solution for both of them. The second case is particularly interesting because it appears at first sight hopeless, but, again, resort to the conical approach provides the key to overcome the difficulty. The second direction consists in evaluating the possibility of obtaining at least one solution to problems in the present class, by means of linear, instead of inequative, feedback. We illustrate three mechanisms that exclude any linear solution. In the first the linear feedback cannot handle cases where the origin is in the constraining set for the state. In the second the linear feedback lacks the initial condition independence of the inequative solution. In the third the linear feedback cannot control the geometric multiplicity of eigenvalues of the system, and this prevents stabilization, when the constraint structure is such that we cannot allow the state to converge to the origin. These results clearly strengthen the significance and relevance of the theory of linear (optimal) regulator.

1. INTRODUCTION

The literature on constrained systems is rather broad. We cite e.g. the papers [1] and the references therein and the paper [5], where the problem of stabilizing a linear discrete time system under combined input and state constraints was addressed. In [2] we presented a general theory solving (finite horizon) feasibility and optimization problems for constrained linear dynamic discrete-time systems. The assumed constraints and functionals were linear. The general time-varying system and constraint case was dealt with. We derived necessary and sufficient conditions for the existence of solutions without assuming any restrictive hypotheses. For the solvable case we also derived the structure of inequative feedback, which, coupled with the given system, generates all and nothing but the feasible (or optimal according to the cases) solutions. This is what we call a dynamic (and automatic) solution.

This theory parallels that of the quadratic regulator and fills a gap in dynamic optimization, which in all textbooks starts from the quadratic instead of the linear case. Crucial to our results was the machinery of the conical approach to linear programming presented in [3].

In the present paper we make more definite the contours of the theory in terms of generalizations and significance. More specifically we address two important questions. Are those considered in [2] the only cases that can be solved providing at the same time a dynamic solution? Notice in this respect that the condition of providing a dynamic solution is essential, because we know that any formulable finite horizon linear case can be solved statically within the spaces of time-functions signals. This fact is trivial and we do not need to dwell on an illustration.

The second question is: can we solve the problems addressed in [2], using a linear instead of inequative feedback? Notice that this approach necessarily implies to give up the set of all solutions because the solution of the feedback system becomes unique. Thus this drawback could be accepted only in trade of a numerically more simple solution.

We solve here these two problems in a significant, albeit not systematic, manner. For the first question we show that we are able to derive dynamic and complete solutions for two cases of mixed constraints. For the second question we produce three examples that show that the linear feedback solution does not exist in general.

The significance of the results we presents goes way beyond answering the above two questions. In the first case we show that the machinery in [3] is powerful enough to generate dynamic solutions for mixed constraints, where they look at first sight impossible. Moreover the structure of these dynamic solutions contains some novelties with respect to the case of separate input and state constraints. In the second case we show that the power of the inequative feedback goes way beyond that of linear feedback to solve cases that are precluded this latter. Thus it is not a simple matter of one solution versus all the solutions. In fact, to cite just a relevant example, we propose important control problems, that cannot be solved via eigenvalue assignment by linear feedback, because linear feedback has no control on multiplicity of eigenvalues. We show that these same problems are easily solved by inequative feedback.

The paper is organized as follows: in the first part of Section 2 we recall and generalize to the case of input constraints the results in [2]. The main new results which answer our first question are in Subsection 2.1. In Section 3, examples are offered to answer our second question. We will expand a little on the conceptual content of our results in the conclusion.

2. GENERALIZATIONS PRESERVING DYNAMIC SOLUTIONS

In this session we will explore more general regulator problems, that still allow to derive a dynamical solution like those introduced in [2].

For simplicity we will consider the only feasibility. The extension to optimization follows easily along the lines illustrated in [2]. We initiate recalling briefly the derivation and structure of the regulator, but, unlike [2], we make reference to time invariant systems and constraints. This hypothesis simplifies to some extent notations and, at the same time evidences where, how and what time-variances are generated by the solution mechanism. Our exposition goes as directly as possible to the point, since formal statements in theorem-proof paradigm can be found in [2].

Thus consider the time invariant dynamic system

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t+1) &= Cx(t+1), \qquad t \geq 0 \end{aligned} \tag{1}$$

with time-invariant state constraints

$$Wx(t) \leq M, \quad 0 < t \leq T \quad \text{or} \quad \mathrm{x(t) \in D}. \tag{2}$$

We assume for the present purposes a given initial state $x(0) = \overline{x}$. In [2] we defined S_t, $0 \leq t < T$ (the set of admissible states at time t) to be the set of all states x, for which there exists at least one input steering the system from x at time t to states at $t+1, t+2, \ldots, T$, that satisfy the constraint at the corresponding instant of time. The system is feasible if and only if S_0 is non-void and $x(0) = \overline{x} \in S_0$.

At each time t, S_t is the set of states x that satisfy the constraint:

$$Ax + Bu \in S_{t+1} \cap D = E_{t+1}$$

which defines the backward in time recursion for the sets of admissible states, with initial (or final, if we look to time orientation) condition $E_T = D = \{x : W\,x \leq M\}$. Notice that, although we considered an entirely stationary case (both system's equation and constraints are stationary), the actual nature of the problem is time-varying since the E_t's pose a time-varying constraint. For reasons that will be evident shortly, it is convenient to write $W = W_T$ and $M = M_T$. Substituting for $x(T)$ in the expression of E_T:

$$W_T(Ax(T-1) + Bu(T-1)) \leq M_T$$

whence:

$$W_T Bu(T-1) \leq M_T - W_T Ax(T-1).$$

This is a first instance of the inequative feedback equation. At this point we apply to the feedback equation the dual conical feasibility condition in its matrix form, stated in [3]. Thus, if Q_T is a matrix whose rows are the generators of the pointed polyhedral cone $R(W_T B)^{\perp} \cap P$, where the symbol P denotes the nonnegative orthant of the suitable Euclidean space, the equation is feasible if and only if

$$Q_T(M_T - W_T Ax(T-1)) \geq 0$$

or

$$Q_T W_T Ax(T-1) \leq Q_T M_T$$

which defines S_{T-1} and shows that it is a polyhedron. Therefore E_{T-1} is a polyhedron too and we can write $E_{T-1} = \{x : W_{T-1}\, x \leq M_{T-1}\}$ with:

$$W_{T-1} = \begin{pmatrix} Q_T W_T A \\ W \end{pmatrix} \quad \text{and} \quad M_{T-1} = \begin{pmatrix} Q_T M_T \\ M \end{pmatrix}.$$

Now, in view of the backward recursion, it is clear that all the E_t are polyhedra and we can write $E_t = \{x : W_t\, x \leq M_t\}$ with the above final conditions $W_T = W$ and $M_T = M$. Thus it is obvious that to solve completely the problem it only remains to rewrite the above formulas for the generic t.

The inequative feedback equation is given by:

$$W_t B u(t-1) \leq M_t - W_t A x(t-1), \quad t = 1, \dots, T.$$

Thus, provided the constrained dynamical system is feasible, the set of all and nothing but the admissible solutions is given by the closed loop system:

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ W_{t+1} B u(t) &\leq M_{t+1} - W_{t+1} A x(t) \end{aligned}$$

with $x(0) = \overline{x}$ and $t = 0, \dots, T-1$. Notice that this system is time-varying although the original system was stationary.

The polyhedra S_t are defined by:

$$Q_{t+1} W_{t+1} A x(t) \leq Q_{t+1} M_{t+1}$$

where Q_{t+1} is a matrix whose rows are the generators of the pointed polyhedral cone $R(W_{t+1}B)^{\perp} \cap P$. Thus

$$W_t = \begin{pmatrix} Q_{t+1} W_{t+1} A \\ W \end{pmatrix} \quad \text{and} \quad M_t = \begin{pmatrix} Q_{t+1} M_{t+1} \\ M \end{pmatrix}.$$

This equation defines a backward recursion for the coefficient matrices W_t and vectors M_t with initial conditions $W_T = W$ and $M_T = M$. The recursion allows us to compute all such matrices and vectors and hence also the sequence of polyhedra $\{S_t\}$ and $\{E_t\}$. Finally the system is feasible if and only if S_0 is non-void and $x(0) = \overline{x} \in S_0$.

We claimed in [2], without expanding on this point for the sake of brevity, that it is easy to generalize this theory to the case in which there is also a (pointwise in time) constraint for the input. We carry on this exercise for the present stationary case. Thus consider the problem obtained from the previous one just adding the further constraint:

$$Zu(t) \leq N. \tag{3}$$

To get to the solution, it suffices to couple this equation with the feedback equation. Maintaining the same symbols as before, in particular for the new coefficient and vectors for the sets E_t, the new feedback equation becomes:

$$\begin{pmatrix} W_{t+1} B \\ Z \end{pmatrix} u(t) \leq \begin{pmatrix} M_{t+1} \\ N \end{pmatrix} - \begin{pmatrix} W_{t+1} A \\ 0 \end{pmatrix} x(t).$$

If now Q_{t+1} is a matrix, whose rows are the generators of $R\begin{pmatrix} W_{t+1}B \\ Z \end{pmatrix}^{\perp} \cap P$, the new sets S_t will be defined by:

$$Q_{t+1}\begin{pmatrix} W_{t+1}A \\ 0 \end{pmatrix} x(t) \leq Q_{t+1}\begin{pmatrix} M_{t+1} \\ N \end{pmatrix}$$

and the new matrices and vectors defining the polyhedra E_t are given by

$$W_t = \begin{pmatrix} Q_{t+1}\begin{pmatrix} W_{t+1}A \\ 0 \end{pmatrix} \\ W \end{pmatrix}$$

$$M_t = \begin{pmatrix} Q_{t+1}\begin{pmatrix} M_{t+1} \\ N \end{pmatrix} \\ M \end{pmatrix}.$$

A complete statement for the solution of the problem follows by a verbatim repetition of the one for the previous cases substituting these new formulas wherever it applies. This is omitted for the sake of brevity.

2.1. The case of mixed input and state constraints

From now on we drop the additional input constraint, to avoid more cumbersome notation, but it will be clear from the present analysis how it is possible to extend our results to incorporate such constraints.

Next we turn to addressing the following question. Can we still obtain a dynamic solution in presence of mixed input-state constraints?

Actually there are various possible structures for mixed constraints, according to the role played by the time variable. The easiest case to handle arises when the constraints reflect the way time appears in the constraint given by the dynamical equation. This corresponds to coupling to the dynamic system (1) the following time-invariant constraints:

$$Wx(t+1) + Zu(t) \leq M, \quad t = 0, \dots, T-1. \tag{4}$$

This case is new with respect to the theory in [2]. As first move, inspired by the previous cases we pose:

$$W_T = W, \quad Z_T = Z \quad \text{and} \quad M_T = M.$$

Next we substitute for $x(T)$ the expression given by the dynamic system (1) in the constraint at time $(T-1)$ and obtain:

$$(W_T B + Z_T)\, u(T-1) \leq M_T - W_T A x(T-1)$$

which has the form of a particular instance of the feedback equation. Applying the cited dual conical feasibility condition, if Q_T is a matrix, whose rows are the generators of $R(W_T B + Z_T)^{\perp} \cap P$, this equation is feasible if and only if:

$$Q_T W_T A x(T-1) \leq Q_T M_T$$

which defines for the present case the polyhedron of admissible states S_{T-1}. Next we couple this inequality with the constraint at time $T-2$. In this way we obtain a new constraint:

$$W_{T-1}x(T-1) + Z_{T-1}u(T-2) \leq M_{T-1}$$

where:

$$\begin{aligned} W_{T-1} &= \begin{pmatrix} Q_T W_T A \\ W \end{pmatrix} \\ Z_{T-1} &= \begin{pmatrix} 0 \\ Z \end{pmatrix} \\ M_{T-1} &= \begin{pmatrix} Q_T M_T \\ M \end{pmatrix}. \end{aligned}$$

At this point we can carry on the backward recursion, which obviously leads to identical formulas except that the generic time t substitutes the specific time index.

The polyhedra S_t are defined, for $t = 0, \ldots, T-1$, by the inequalities:

$$Q_{t+1}W_{t+1}Ax(t) \leq Q_{t+1}M_{t+1}$$

where Q_{t+1} is a matrix whose rows are the generators of the pointed polyhedral cone $R(W_{t+1}B + Z_{t+1})^{\perp} \cap P$. The backward recursion is regulated by the equations (valid for $t = 0, \ldots, T-1$):

$$\begin{aligned} W_t &= \begin{pmatrix} Q_{t+1} W_{t+1} A \\ W \end{pmatrix} \\ Z_t &= \begin{pmatrix} 0 \\ Z \end{pmatrix} \\ M_t &= \begin{pmatrix} Q_{t+1} M_{t+1} \\ M \end{pmatrix} \end{aligned}$$

with final (or initial) conditions $W_T = W$, $Z_T = Z$ and $M_T = M$. The constrained system is feasible if and only if S_0 is non-void and $x(0) = \overline{x} \in S_0$. If that is the case, then all and nothing but the solutions of the constrained system are given by the solutions of the inequative feedback system:

$$\begin{gathered} x(t+1) = Ax(t) + Bu(t) \\ (W_{t+1}B + Z_{t+1})\, u(t) \leq M_{t+1} - W_{t+1}Ax(t) \end{gathered}$$

with initial condition $x(0) = \overline{x}$ and defined in the time interval $[0, T-1]$. Notice that in this case there is no definition of the E_t's. Also it is needless to say that the feedback system becomes time-varying although we started from a time invariant constrained system.

The analysis of the present case seems to suggest that, if we had specified a different kind of mixed constraints, we would not have succeeded in deriving a dynamic solution. To some surprise a more careful examination shows that is not so. In fact we will now provide a dynamical solution for the case where the input and state variable, appearing in the constraints, are evaluated at the same instant of time. The key argument relates to the constraining mechanism at the end of the temporal span of the problem, which we assumed finite.

To be more precise we assume that the dynamic system (1) is coupled with the following constraints:

$$Wx(t) + Zu(t) \leq M \quad t = 1, \ldots, T. \tag{5}$$

Let us look at the way the constraint acts at time T. Because $u(T)$ influences the state at time $T+1$, its value is immaterial, provided we can select a state satisfying the inequality. In other words we need to find the set of all $x(T)$ for which there exists an $u(T)$, that satisfies $Wx(T) + Zu(T) \leq M$. This is but one of the problems to which the dual conical theory developed in [3] gives a complete solution. First we rewrite the inequality as:

$$Zu(T) \leq M - Wx(T).$$

From [3] we know that, if Q is a matrix whose rows are the generators of the pointed polyhedral cone $R(Z)^{\perp} \cap P$ the set of all and nothing but the bound vectors, that make this inequality feasible, are those satisfying:

$$Q(M - Wx(T)) \geq 0$$

or

$$QWx(T) \leq QM. \tag{6}$$

The polyhedron defined by this latter inequality is the sought set of states. In this way we substitute to the final constraint another constraint, that bounds the state alone. That substitution is what makes the dynamic solution possible. What we have done is nothing but to project the polyhedron, defined by the constraint in the input-state product space, on the only state space. As already explained, this is justified by the fact that for each state in the projection, there is an input that, paired with the state, satisfies the original constraint. Who is such an input is irrelevant because our horizon terminates at time T. Notice that, consequently, we have to deal with a time-variant constraint system from scratch. The constraint coincides with the given one for $t = 1, \ldots, T-1$, and is given by (6) for $t = T$.

Let us now investigate whether there exists a dynamic solution for this constrained system.

To the purpose of developing the backward recursion that coupled with the inequative feedback dynamic system solves the problem. It is convenient to state the following positions:

$$\begin{aligned} W_T &= W \\ Z_T &= Z \\ M_T &= M \\ Q_{T+1} &= Q. \end{aligned}$$

Notice that now Q_{T+1} is a matrix whose rows are the generators of $R(Z_T)^{\perp} \cap P$. Next we substitute for $x(T)$ in the final constraint

$$Q_{T+1} W_T x(T) \leq Q_{T+1} M_T$$

to obtain:

$$Q_{T+1} W_T B u(T-1) \leq Q_{T+1} M_T - Q_{T+1} W_T x(T-1). \tag{7}$$

Notice that this equation (which is the counterpart of the last step feedback equation of the first problem we examined) has the form of a mixed constraint. Because at $T-1$ we are given a mixed constraint, we pair the two constraint to obtain the present form of the last step feedback equation. Thus the latter inequality, paired with:

$$Zu(T-1) \leq M - Wx(T-1)$$

yields the inequative feedback equation:

$$\begin{pmatrix} Q_{T+1} W_T B \\ Z \end{pmatrix} u(T-1) \leq \begin{pmatrix} Q_{T+1} M_T \\ M \end{pmatrix} - \begin{pmatrix} Q_{T+1} W_T A \\ W \end{pmatrix} x(T-1).$$

Using self-evident positions, we rewrite this as:

$$Z_{T-1} u(T-1) \leq M_{T-1} - W_{T-1} x(T-1).$$

Next let Q_T be a matrix, whose rows are the generators of the (pointed polyhedral) cone $R(Z_{T-1})^{\perp} \cap P$, then, applying the cited dual conical feasibility condition [3], we obtain the inequality defining E_{T-1}:

$$Q_T W_{T-1} x(T-1) \leq Q_T W_{T-1}.$$

Notice that this inequality corresponds precisely to the final constraint. Thus, substituting the value of $x(T-1)$, given by the system's dynamic equation, the recursion may start all over again. Consequently the dynamic solution exists and has the structure appended below.

The inequative feedback system has the form:

$$\begin{aligned} Z_t u(t) &\leq M_t - W_t x(t) \\ x(t+1) &= Ax(t) + Bu(t). \end{aligned}$$

The polyhedra E_t are defined by:

$$Q_{t+1} W_t x(t) \leq Q_{t+1} M_t$$

where Q_{t+1} is a matrix whose rows are the generators of the pointed polyhedral cone $R(Z_t)^{\perp} \cap P$. Moreover:

$$\begin{aligned} Z_t &= \begin{pmatrix} Q_{t+2} W_{t+1} B \\ Z \end{pmatrix} \\ M_t &= \begin{pmatrix} Q_{t+2} M_{t+1} \\ M \end{pmatrix} \\ W_t &= \begin{pmatrix} Q_{t+2} W_{t+1} A \\ W \end{pmatrix} \end{aligned}$$

with initial conditions:

$$\begin{aligned} Z_T &= Z \\ M_T &= M \\ W_T &= W. \end{aligned}$$

Finally notice that E_1 is defined by

$$Q_2 W_1 x(1) \leq Q_2 M_1.$$

Substituting for $x(1)$

$$Q_2 W_1 B u(0) \leq Q_2 M_1 - Q_2 W_1 A \overline{x}.$$

If Q_1 is the matrix whose rows are the generators of the pointed polyhedral cone $R(Q_2 W_1 B)^{\perp} \cap P$, then S_0 is defined by

$$Q_1 Q_2 W_1 A \overline{x} \leq Q_1 Q_2 M_1.$$

The constrained system is feasible if and only if S_0 is non-void and $x(0) = \overline{x} \in S_0$. If this is the case the above inequative feedback system yields all and nothing but the feasible solutions of the constrained dynamic system.

We could carry on the study of generalizations along these lines. However we stop here for space reasons: it is enough here to have made the point that the approach introduced in [2] can applied to a much larger span of problems than those studied therein.

In the final paragraph we turn to examine a question in a sense dual to the previous one. Instead of studying generalization we would like to see whether other approaches can overlap to that of [2], producing in certain cases special solutions (i. e. not capable of yielding all the admissible trajectories) enjoying a simpler feedback structure.

3. EXAMPLES

In [2] we have showed that in the case of time invariant systems with time variant or time invariant state constraints the admissible inputs at some time t are described in general by linear inequalities, with coefficients dependent on time but independent of the initial state. In the following examples we show that in general a linear state feedback control law solving the feasibility problem does not exist. In the first example we consider the case of time variant constraints, with the origin belonging to some of the constraining sets. In the second we study the case of time variant constraints, where the origin belongs to no one of the constraining sets. In the last example the constraints and the inequalities describing the admissible input are time invariant: this because the state constraining set is invariant controllable. We recall that given a set Σ and a system $x(t+1) = Ax(t) + Bu(t)$, we say that Σ is invariant controllable if $\forall\, x \in \Sigma,\ \exists\, u : Ax + Bu \in \Sigma$.

3.1. Example 1

Let us consider the time invariant discrete time system

$$x(t+1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t)$$

with time variant state constraints:

$$x(t) \geq t \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad t \geq 0 \quad \text{or} \quad x(t) \in D_t, \quad t \geq 0.$$

It is trivial to see that in this special case

$$E_t = D_t, \quad t \geq 0.$$

In fact for a given T,

$$D_T = \left\{ x(T) : -x(T) \leq -T \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

Substituting for $x(T)$ we have

$$-Ax(T-1) - Bu(T-1) \leq -T \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

or also

$$-Bu(t) \leq -T \begin{pmatrix} 1 \\ 1 \end{pmatrix} + Ax(t).$$

The matrix Q whose rows are the generators of $R(B)^{\perp} \cap P$ is

$$Q = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

(because $R(B) + P = R^n$) and hence $S_{T-1} = R^n$ and $E_{T-1} = D_{T-1}$. If we perform another step, we have $E_{T-2} = D_{T-2}$, and generalizing $E_t = D_t$, $t \geq 0$.

Therefore the set of all admissible solutions is given by the solutions of the closed loop system:

$$x(t+1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t)$$

$$-\begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t) \leq -t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x(t)$$

$$x(0) \in P.$$

The provided state feedback control law is non stationary and it does not depend on $x(0)$. The question is if it exists a state feedback non stationary linear control law, not depending on the initial state, solving the same feasibility problem: i. e. the question is if matrices F_t exist such that the solutions of the closed loop system

$$x(t+1) = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} F_t \right) x(t)$$

are admissible, starting from any nonnegative initial state. We can immediately say that such matrices do not exist, because if $x(0) = 0$, it follows that $x(1) = 0$, for every value of F_0, and therefore the constraint $x(1) \in D_1$, cannot be satisfied.

3.2. Example 2

Let us consider the time invariant discrete time system

$$x(t+1) = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.8 \end{pmatrix} x(t) + \begin{pmatrix} 5 \\ 4 \end{pmatrix} u(t)$$

with time variant state constraints:

$$x(t) \in D_t, \qquad t = 0, 1, 2$$

where the sets D_t are described as follows:

$$D_0 = \{x : \begin{pmatrix} -1 & -1 \end{pmatrix} x \le -1\}$$

$$D_t = \left\{ x : \begin{pmatrix} 0.5 & -1.0 \\ -0.4 & 0.8 \\ -1.0 & -1.0 \end{pmatrix} x \le t \begin{pmatrix} 0.75 \\ 1.20 \\ -2.00 \end{pmatrix} \right\}, \qquad t \ge 1.$$

Let us consider the constraint at time $t = 2$. Substituting for $x(2)$, we have

$$\begin{pmatrix} 0.5 & -1.0 \\ -0.4 & 0.8 \\ -1.0 & -1.0 \end{pmatrix} (Ax(1) + Bu(1)) \le 2 \begin{pmatrix} 0.75 \\ 1.20 \\ -2.00 \end{pmatrix}$$

and hence:

$$\begin{pmatrix} -1.5 \\ 1.2 \\ -9.0 \end{pmatrix} u(1) \le 2 \begin{pmatrix} 0.75 \\ 1.20 \\ -2.00 \end{pmatrix} - \begin{pmatrix} 0.25 & -0.80 \\ -0.20 & 0.64 \\ -0.50 & -0.80 \end{pmatrix} x(1).$$

The matrix Q whose rows are the generators of $R \begin{pmatrix} -1.5 \\ 1.2 \\ -9.0 \end{pmatrix}^{\perp} \cap P$ is

$$Q = \begin{pmatrix} 1.0 & 1.25 & 0.00000 \\ 0.0 & 1.00 & 0.13333 \end{pmatrix}$$

and hence

$$S_1 = \{x : \begin{pmatrix} -1 & 2 \end{pmatrix} x \le 7\}, \quad E_1 = D_1.$$

Performing another backward step, we obtain that

$$S_0 = \{x : \begin{pmatrix} -1 & 2 \end{pmatrix} x \le 3.5\}$$

$$E_0 = \left\{ x : \begin{pmatrix} -1 & 2 \\ -1 & -1 \end{pmatrix} x \le \begin{pmatrix} 3.5 \\ -1.0 \end{pmatrix} \right\}.$$

From the above computations, we see that the admissible control law is

$$\begin{pmatrix} -1.5 \\ 1.2 \\ -9.0 \end{pmatrix} u(t) \le (t+1) \begin{pmatrix} 0.75 \\ 1.20 \\ -2.00 \end{pmatrix} - \begin{pmatrix} .25 & -.8 \\ -.2 & .64 \\ -.5 & -.8 \end{pmatrix} x(t)$$

$$x(0) \in E_0, \qquad t = 0, 1.$$

The question is now if matrices F_0, F_1 exist such that

$$\begin{aligned} (A + BF_0)\, x \in E_1 \qquad & \forall\, x \in E_0 \\ (A + BF_1)\, x \in E_2 \qquad & \forall\, x \in E_1. \end{aligned}$$

Let us now assume that a matrix $F_0 = (f_1\ f_2)$ of the above sort exists. The vector $v = \begin{pmatrix} -0.5 + 2\alpha \\ 1.5 + \alpha \end{pmatrix}$ belongs to E_0, $\forall\, \alpha \geq 0$.

We have that $(A + BF_0)v \in E_1$ if and only if, for some value of the parameters f_1 and f_2, the following inequalities are satisfied for every α greater or equal to zero

$$\begin{pmatrix} 0.5 & -1.0 \\ -0.4 & 0.8 \\ -1.0 & -1.0 \end{pmatrix} (A + BF_0)v \leq \begin{pmatrix} 0.75 \\ 1.20 \\ -2.00 \end{pmatrix}$$

or also, substituting for A, B and F_0

$$\begin{pmatrix} -.3\alpha + (.75 - 3\alpha)\, f_1 - (2.25 + 1.5\alpha)\, f_2 \\ .24\alpha - (.6 - 2.4\alpha)\, f_1 + (1.8 + 1.2\alpha)\, f_2 \\ -1.8\alpha + (4.5 - 18\alpha)\, f_1 - (13.5 + 9\alpha)\, f_2 \end{pmatrix} \leq \begin{pmatrix} 2.075 \\ .14 \\ -1.05 \end{pmatrix}.$$

We see that the above inequalities cannot be satisfied for every nonnegative α. Therefore we can conclude that our assumption is false, and hence we can say that in general does not exist a linear state feedback control law solving a finite horizon feasibility problem.

3.3. Example 3

Let us consider the time invariant discrete time system

$$x(t+1) = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.8 \end{pmatrix} x(t) + \begin{pmatrix} 5 \\ 4 \end{pmatrix} u(t)$$

with time invariant state constraints:

$$x(t) \in D \qquad t \geq 0$$

$$D = \left\{ x : \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} x \leq \begin{pmatrix} 2 \\ 2 \\ -1 \\ -1 \end{pmatrix} \right\}.$$

The set D is invariant controllable. In fact, if we consider the constraint

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} x(T) \leq \begin{pmatrix} 2 \\ 2 \\ -1 \\ -1 \end{pmatrix}$$

for some $T > 2$, substituting for $x(T)$, we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} (Ax(T-1) + Bu(T-1)) \leq \begin{pmatrix} 2 \\ 2 \\ -1 \\ -1 \end{pmatrix}$$

or also

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} Bu(T-1)) \leq \begin{pmatrix} 2 \\ 2 \\ -1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} Ax(T-1)$$

and substituting for A, B:

$$\begin{pmatrix} 5 \\ 4 \\ -5 \\ -4 \end{pmatrix} u(T-1)) \leq \begin{pmatrix} 2 \\ 2 \\ -1 \\ -1 \end{pmatrix} - \begin{pmatrix} 0.5 & 0.0 \\ 0.0 & 0.8 \\ -0.5 & 0 \\ 0.0 & -0.8 \end{pmatrix} x(T-1).$$

The matrix Q whose rows are the generators of $R\begin{pmatrix} 5 \\ 4 \\ -5 \\ -4 \end{pmatrix}^{\perp} \cap P$ is:

$$Q = \begin{pmatrix} 1.00 & 0.00 & 1.00 & 0.00 \\ 1.00 & 0.00 & 0.00 & 1.25 \\ 0.00 & 1.00 & 0.80 & 0.00 \\ 0.00 & 1.00 & 0.00 & 1.00 \end{pmatrix}$$

and hence

$$S_{T-1} = \left\{ x : \begin{pmatrix} 0.5 & -1.0 \\ -0.4 & 0.8 \end{pmatrix} x \leq \begin{pmatrix} 0.75 \\ 1.20 \end{pmatrix} \right\}.$$

We can verify that $D_{T-1} \subset S_{T-1}$, and $E_{T-1} = D_{T-1}$. Therefore $E_t = D \ \forall t \geq 0$.

The set of all admissible solutions is given by the solutions of the closed loop system:

$$x(t+1) = \begin{pmatrix} 0.5 & 0.0 \\ 0.0 & 0.8 \end{pmatrix} x(t) + \begin{pmatrix} 5 \\ 4 \end{pmatrix} u(t)$$

$$\begin{pmatrix} 5 \\ 4 \\ -5 \\ -4 \end{pmatrix} u(t) \leq \begin{pmatrix} 2 \\ 2 \\ -1 \\ -1 \end{pmatrix} - \begin{pmatrix} 0.5 & 0.0 \\ 0.0 & 0.8 \\ -0.5 & 0.0 \\ 0.0 & -0.8 \end{pmatrix} x(t)$$

$$x(0) \in D.$$

It is trivial to see that a matrix F such that the solutions of the closed loop system

$$x(t+1) = (A + BF)\, x(t), \quad x(0) \in D$$

satisfy the given state constraints does not exist. In fact, because of the structure of the constraining set, it should be true that:

i) $A + BF$ is stable

ii) $\forall \lambda \in \sigma(A + BF)$, λ is real and $|\lambda| = 1$.

Therefore necessarily $(A + BF) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. But we can immediately see that a matrix F such that this last condition is true does not exist.

It is easy to prove that in this example the problem of making invariant the given polytope can be solved applying an affine state feedback control law, i.e. $u(t) = Fx(t) + \overline{u}$, but this is not true in general, as we have shown in [4].

(Received December 11, 1998.)

REFERENCES

[1] F. Blanchini: Ultimate boundedness control for uncertain discrete-time systems via set-induced Lyapunov functions. IEEE Trans. Automat. Control *AC-39* (1994), 428–433.

[2] P. d'Alessandro and E. De Santis: General closed loop optimal solutions for linear dynamic systems with linear constraints. J. Math. Systems, Estimation and Control *6* (1996), 2, 1–14.

[3] P. d'Alessandro: A Conical Approach to Linear Programming, Scalar and Vector Optimization Problems. Gordon and Breach Science Publishers, 1997.

[4] P. d'Alessandro and E. De Santis: Controlled Invariance and Feedback Laws. Research Report no. R.99-31, Dept. of Electrical Engineering, University of L'Aquila, 1999 (submitted).

[5] J. C. Hennet and C. E. T. Dorea: Invariant regulators for linear systems under combined input and state constraints. In: Proc. of 33rd IEEE Conference on Decision and Control, Lake Buena Vista 1994, pp. 1030–1035.

Prof. Dr. Paolo d'Alessandro, Department of Mathematics, Third University of Rome, L.go S. Leonardo Murialdo 1, 00146 Rome. Italy.
e-mail: dalex@mat.uniroma3.it

Prof. Dr. Elena De Santis, Department of Electrical Engineering, University of L'Aquila, Monteluco di Roio, 67040 L'Aquila. Italy.
e-mail: desantis@ing.univaq.it