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DESIGN OF PREDICTIVE LQ CONTROLLER

MIROSLAV FIKAR, SEBASTIAN ENGELL AND PETR DOSTÁL

A single variable controller is developed in the predictive control framework based upon minimisation of the LQ criterion with infinite output and control horizons. The infinite version of the predictive cost function results in better stability properties of the controller and still enables to incorporate constraints into the control design. The constrained controller consists of two parts: time-invariant nominal LQ controller and time-variant part given by Youla–Kučera parametrisation of all stabilising controllers.

1. INTRODUCTION

Predictive control (MBPC) originally invented by Clarke and co-workers [1, 2, 3] has been shown to be a very effective method of controlling SISO discrete-time systems and has received much attention even in industry. This because the method offers good performance as well as being easy to understand and tune. Moreover, state and input constraints can directly be incorporated.

As predictive control is based on receding finite horizon minimisation, it inherently suffers from stability problems. Stability results have been reported that use either terminal constraints (Clarke and Scattolini [3], Mosca and Zhang [11]) where it is required that the system converges to origin at the end of the output horizon or optimise future reference sequence (Kouvaritakis and Rossiter [14, 15]). This can also be thought as the terminal constraint approach as the setpoint is required to be at the desired place at the end of the horizon. Another approach follow Fikar and Engell [4] and choose the predictive controller from the set of controllers given by Youla–Kučera (YK) parametrisation. It has been shown that terminal constraints are automatically satisfied and need not explicitly be given.

All these approaches suffer from minimisation of finite horizon criterion which invariantly leads to stability problems if the various horizons are set incorrectly, mainly too small. Therefore the researchers focused the attention on specification of minimal horizons and weight settings that assure the stability of the closed-loop system. However if the horizons are large, increased demand on computer speed results. Moreover, the introduction of horizons leads to new user parameters and it is then not quite clear how to set all of them. Therefore, several procedures have been developed when all but one parameter are set to some value and the remaining parameter is used for the determination of the speed of the controller (see for example McIntosh et al [9], where 3 different procedures are discussed).

The second approach how to stabilise receding horizon controllers is to use infinite output horizon. Rawlings and Muske [12] have introduced linear time-invariant controller with infinite output horizon and finite control horizon in state-space formulation and discuss the choice of the control horizon that stabilises the unconstrained closed-loop system. In the constrained case feasibility of optimisation is required. Scokaert and Clarke [16] have formulated GPC$^\infty$, but only for purposes of proofs and not as a new method, in practice they set this horizon as a very large number. An optimisation based algorithm is given in Scokaert and Rawlings [17] where the length of the control horizon is searched iteratively until corresponding constrained control problem if feasible. These methods use finite control horizon in order to keep the number of optimised variables finite. Thus the control horizon achieves feasibility and the output horizon global stability. However the result of [17] only states the existence of finite control horizon and thus makes the results of truncation theoretical. Rossiter et al [13] discuss the use of both control and output infinite horizons and develop and efficient technique that avoids the need for solving a Lyapunov equation.

In this article we use infinite horizon for both output and control predictions but still keep the number of optimised variables finite. We follow the same approach as Fikar and Engell [4] and parametrise the YK set of stabilising controllers. Two YK transfer functions are given as polynomials. One shapes the response to reference changes, the other to disturbances. In the unconstrained case the controller is reduced to nominal time-invariant LQ controller. If constraints are active time-varying piece-wise linear controller results as in the case of finite horizons. The closed-loop expressions describing behaviour of the constrained controller are derived and thus make it possible to study the system properties with methods dealing with time-varying systems.

The two-degree-of freedom controller is based on algebraic approach developed by Kučera. The control design is performed in input-output formulation leading to Diophantine and spectral factorisation equations. The algebraic approach and transformation of original polynomial to matrix-polynomial equations make it possible to transform original infinite terms into finite expressions without loss of generality. The controller incorporates an integral part. The nominal controller minimises modified LQ criterion and the constrained predictive controller is searched in the subspace of stable controllers with the poles given by spectral factorisation equation.

The article is organised as follows. Section 2 presents a design of a LQ controller based on Youla–Kučera parametrisation of all stabilising controllers. Section 3 contains derivation of predictions, objective function, and proof of equality of nominal and predictive controller is given. In Section 4 constrained controller is discussed and the control algorithm is summarised. Section 5 presents the conclusions of the article. Some definitions and properties needed for the proofs are given in the Appendix.

1.1. Notation

All systems in this work are assumed to be SISO, linear, time-invariant, and discrete-time. The systems are described by means of fractions of polynomials in an indeterminate z^{-1}, used in Z-transform and normally interpreted as delay operator. The reader is referred to Kučera [7] whose notations are adopted hereafter as much as possible.

For simplicity, the arguments of polynomials are omitted whenever possible – a polynomial $X(z^{-1})$ is denoted by X. We define adjoint of polynomial $X^*(z) = X(z^{-1})$. Further, for any polynomial or sequence X, we define $\langle X \rangle$ as the coefficient of z^0, i.e. the constant term. The causal part of expression x denoted by $\langle x \rangle_{cp}$ denotes only the terms with z^{-i}, $i \geq 0$.

2. STANDARD POLYNOMIAL LQ DESIGN

2.1. System description

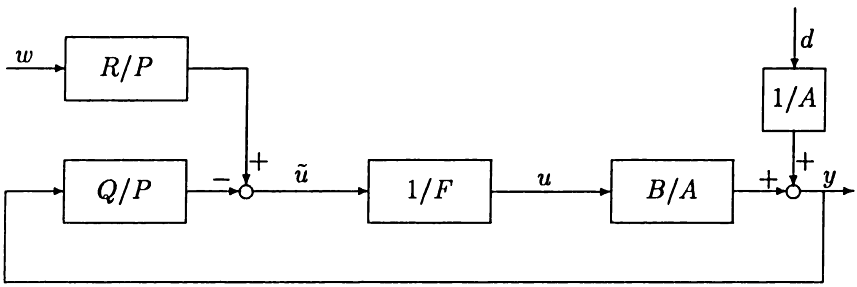

Fig. 1. Block diagram of the closed-loop system.

Consider the closed-loop system illustrated in Figure 1. A discrete-time linear time-invariant input-output representation of the plant to be controlled is considered

$$A(q^{-1})\,y(t) = B(q^{-1})\,u(t) + d(t) \tag{1}$$

$(qy(t) \equiv y(t+1))$ or, after taking the Z-transform (see Property 3 in Appendix 2)

$$A(z^{-1})\,y = B(z^{-1})\,u + H_y(z^{-1}) + d \tag{2}$$

where y, u, d are process output, controller output, and measurable disturbance sequences, H_y is a polynomial describing initial conditions of the controlled system. $A(0)$ is assumed to be nonzero, and $B(0) = 0$. A and B are assumed coprime and describe the input-output properties of the plant.

We assume that the reference w and the disturbance d are constant sequences with step changes and are generated via

$$Fw = G_w \tag{3}$$

$$Fd = G_d \tag{4}$$

where $F = 1 - z^{-1}$, G_w, G_d are zero order polynomials (constants).

The controller is described by the equations

$$P\tilde{u} = Rw - Qy + H_{\tilde{u}} \tag{5}$$

$$Fu = \tilde{u} + H_u \tag{6}$$

where P, Q, R are controller polynomials that are coprime, $P(0)$ is nonzero, and $H_{\tilde{u}}$ and H_u represent the initial conditions of the controller. The second equation assures that the controller has an integral part and achieves zero tracking error for the above class of references w and disturbances d. From this equation also follows that the sequence $\tilde{u}$ represents sequence of control increments, i.e. $\tilde{u}(t) = u(t) - u(t-1)$.

2.2. Controller design

Consider the quadratic cost criterion that is the infinite version of standard cost function used in predictive control approach

$$J = \sum_{t=0}^{\infty}(\varphi\tilde{u}(t)^2 + \psi e(t)^2) \tag{7}$$

where $e(t) = w(t) - y(t)$ is the tracking error.

The solution to the standard LQ problem is summarised in the following theorem

Theorem 1. (LQ Controller design) Define stable polynomial M resulting from spectral factorisation

$$A^*F^*\varphi AF + B^*\psi B = M^*M \tag{8}$$

then internal stability and solution of the deterministic LQ problem (7) is given by the controller polynomials P, Q, R calculated from two pairs of Diophantine equations. The solution exists if AF and B have no unstable common factors and is unique.

The feedback part of the controller results as a solution of the coupled bilateral Diophantine equations:

$$\begin{aligned} M^*Q - Z_b^*AF &= B^*\psi \\ M^*P + Z_b^*B &= A^*F^*\varphi \end{aligned} \tag{9}$$

and $\langle Z_b \rangle = 0$.

The feedforward part is a solution of another coupled bilateral Diophantine equations:

$$\begin{aligned} M^*R - Z_f^*F &= B^*\psi \\ M^*S + Z_f^*B &= A^*F^*A\varphi \end{aligned} \tag{10}$$

and $\langle Z_f \rangle = 0$. The polynomial S is involved in the Diophantine equations only and does not influence the controller.

Proof. Straightforward modification of the proofs given in Kučera [8] and Šebek [18]. □

Corollary 1. If polynomials AF and B are coprime then the two pairs of Diophantine equations (9), (10) are reduced to two implied Diophantine equations

$$\begin{aligned} AFP + BQ &= M \\ FS + BR &= M. \end{aligned} \tag{11}$$

Proof. See Kučera [8], Hunt and Šebek [6]. □

There are infinitely many solutions of (11). The minimum degree solution (P_0, Q_0, R_0) that minimises the degrees of Q, R is the LQ optimal controller. All other solutions have in common the same closed-loop polynomial M. It is possible to search among general solutions to satisfy additional requirements on the controller.

Then the minimum degree controller P_0, Q_0, R_0 only serves as a basis to find an expression for the set of all stabilising controllers. In our case, all such controllers with the closed-loop polynomial M are given by the following theorem:

Theorem 2. A controller (P, Q, R) gives rise to the closed-loop polynomial M if and only if it can be expressed as

$$P = P_0 + BX_b, \quad Q = Q_0 - AFX_b \tag{12}$$

$$R = R_0 + FX_f \tag{13}$$

where X_b, X_f are any proper stable transfer functions and (P_0, Q_0, R_0) is a particular solution of (11).

Proof. See Middleton and Goodwin [10]. □

Remark 1. In the Theorem 2, X_b, X_f are any proper stable transfer functions that will constitute extra degree of freedom and will be used to handle constraints in predictive controller. However, from the computational point of view, optimising over their parameters would in general lead to non-convex optimisation problem. Therefore, X_b, X_f will be in the sequel assumed to be polynomials with degrees d_b, d_f. This assumption will keep number of optimised variables finite despite infinite horizon formulation. As any polynomial in z^{-1} is proper stable transfer function in z, this will assure the conditions needed in Theorem 2.

Compared to the whole set of all stabilising controllers, two assumptions have been made that result only in search within some subset. Firstly, only the controllers that lead to the closed-loop poles M are considered. Secondly, the YK parameters are constrained to be polynomials. This compromise has been taken in order to have the optimisation problem convex.

3. DERIVATION OF PREDICTIVE CONTROLLER

In this section, the predictive controller which minimises the cost function (7) will be developed. The absence of the finite horizons will result in stable predictive time-invariant controller with only a few user defined parameters, but still able to incorporate constraints on process signals.

From the criterion it follows that we need predictions of the signals $\tilde{u}$ and e. These can be obtained using equations (2)–(6), (11) as

$$\begin{aligned} \tilde{u} &= \frac{1}{M}[A(RG_w + FH_{\tilde{u}}) - Q(BH_u + FH_y + G_d)] && (14) \\ e &= \frac{1}{M}[SG_w - P(FH_y + BH_u + G_d) - BH_{\tilde{u}}]. && (15) \end{aligned}$$

For the purpose of constrained control, the signal u or other signals can also be needed and can be constructed in a similar manner.

The signal predictions consist of two parts, *free* responses which depend only on past data and represent the behaviour of the basic minimum degree controller (P_0, Q_0, R_0), and *forced* responses depending on the polynomials X_b, X_f which can be manipulated to satisfy the control objectives.

The construction of signal predictions $\tilde{u} = \tilde{u}_0 + \tilde{u}_f$, $e = e_0 + e_f$ is given in the next theorem.

Theorem 3. (Construction of predictions) The free and forced responses for signals $\tilde{u}, e$ can be written as

$$\begin{aligned} \tilde{u}_0 &= \frac{1}{M}[A(R_0G_w + FH_{\tilde{u}0}) - Q_0(BH_u + FH_y + G_d)] = \frac{U_0}{M} && (16) \\ e_0 &= \frac{1}{M}[S_0G_w - P_0(FH_y + BH_u + G_d) - BH_{\tilde{u}0}] = \frac{E_0}{M} && (17) \\ \tilde{u}_f &= \frac{1}{M}FA\boldsymbol{T}_x\boldsymbol{x} = \frac{1}{M}\boldsymbol{U}_f\boldsymbol{x} && (18) \\ e_f &= -\frac{1}{M}B\boldsymbol{T}_x\boldsymbol{x} = -\frac{1}{M}\boldsymbol{E}_f\boldsymbol{x} && (19) \end{aligned}$$

where $H_{\tilde{u}0}$ is that part of $H_{\tilde{u}}$ that does not depend on X_b, X_f. $\boldsymbol{T}_x$ is a row polynomial vector depending on past values of sequences w, d and $\boldsymbol{x}$ is a column vector containing all coefficients of unknown polynomials X_b, X_f

$$\begin{aligned} \boldsymbol{T}_x &= \left(\sum_{j=0}^{d_f} \langle \boldsymbol{P}_{d_f} z^j \rangle_{cp} \Delta w_{t-j} \;,\; \sum_{j=0}^{d_b} \langle \boldsymbol{P}_{d_b} z^j \rangle_{cp} \Delta d_{t-j} \right) && (20) \\ \boldsymbol{x} &= (\mathrm{coef}(X_f)^T, \mathrm{coef}(X_b)^T)^T. && (21) \end{aligned}$$

Polynomial vector $\boldsymbol{P}_{(\cdot)}$ and coef($\cdot$) are defined in Appendix 1.

Proof. See Fikar and Engell [4]. □

The expressions for predictions (16)–(19) and Diophantine equations (9), (10) enable to rewrite infinite horizon cost function (7) into standard quadratic function with finite number of variables – coefficients of the vector $\boldsymbol{x}$.

Theorem 4. (Cost function) The objective function (7) for the predictive controller is given as

$$J = c + \boldsymbol{x}^T \boldsymbol{H} \boldsymbol{x} \tag{22}$$

where c is the minimum cost value associated with nominal LQ controller (P_0, Q_0, R_0) and Hessian matrix $\boldsymbol{H}$ is given as

$$\boldsymbol{H} = \begin{pmatrix} \boldsymbol{H}_{ww} & \boldsymbol{H}_{wd} \\ \boldsymbol{H}_{wd}^T & \boldsymbol{H}_{dd} \end{pmatrix} \tag{23}$$

where $\boldsymbol{H}_{ww} \in \mathcal{R}_{d_f+1,d_f+1}$, $\boldsymbol{H}_{wd} \in \mathcal{R}_{d_f+1,d_b+1}$, $\boldsymbol{H}_{dd} \in \mathcal{R}_{d_b+1,d_b+1}$ and

$$\boldsymbol{H}_{ww} = \sum_{j=0,k=0}^{d_f} \Delta w_{t-j} \Delta w_{t-k} (\boldsymbol{B}_f^j)^T \boldsymbol{B}_f^k \tag{24}$$

$$\boldsymbol{H}_{wd} = \sum_{j=0}^{d_f} \sum_{k=0}^{d_b} \Delta w_{t-j} \Delta d_{t-k} (\boldsymbol{B}_f^j)^T \boldsymbol{I}_{d_f+1,d_b+1} \boldsymbol{B}_b^k \tag{25}$$

$$\boldsymbol{H}_{dd} = \sum_{j=0,k=0}^{d_b} \Delta d_{t-j} \Delta d_{t-k} (\boldsymbol{B}_b^j)^T \boldsymbol{B}_b^k \tag{26}$$

where $\boldsymbol{I}_{d_f+1,d_b+1} \in \mathcal{R}_{d_f+1,d_b+1}$ contains ones on main diagonal. Shift matrices $\boldsymbol{B}_f \in \mathcal{R}_{d_f+1,d_f+1}$, $\boldsymbol{B}_b \in \mathcal{R}_{d_b+1,d_b+1}$ are given by Property 2.

Proof. In the first part the equation (7) will be transformed into matrix form It can be rewritten as

$$J = \varphi \langle \tilde{u}^* \tilde{u} \rangle + \psi \langle e^* e \rangle .$$

After substituting equations (16) – (19) follows

$$\begin{aligned} J &= \left\langle \tfrac{1}{M^*M} (\varphi U_0^* U_0 + \psi E_0^* E_0) \right\rangle - 2\boldsymbol{x}^T \left\langle \tfrac{1}{M^*M} (\varphi \boldsymbol{U}_f^* U_0 + \psi \boldsymbol{E}_f^* E_0) \right\rangle + \\ &\quad + \boldsymbol{x}^T \left\langle \tfrac{1}{M^*M} (\varphi \boldsymbol{U}_f^* \boldsymbol{U}_f + \psi \boldsymbol{E}_f^* \boldsymbol{E}_f) \right\rangle \boldsymbol{x} \\ &= c - 2\boldsymbol{x}^T \boldsymbol{g} + \boldsymbol{x}^T \boldsymbol{H} \boldsymbol{x}. \end{aligned}$$

The Hessian matrix $\boldsymbol{H}$ is given as

$$\boldsymbol{H} = \left\langle \tfrac{1}{M^*M} (\varphi \boldsymbol{U}_f^* \boldsymbol{U}_f + \psi \boldsymbol{E}_f^* \boldsymbol{E}_f) \right\rangle .$$

Substituting for $\boldsymbol{E}_f, \boldsymbol{U}_f$ from Theorem 3 follows

$$\boldsymbol{H} = \left\langle \boldsymbol{T}_x^* \tfrac{1}{M^*M} (\varphi A^* F^* A F + \psi B^* B) \boldsymbol{T}_x \right\rangle = \langle \boldsymbol{T}_x^* \boldsymbol{T}_x \rangle .$$

As $\boldsymbol{T}_x$ is a polynomial vector consisting of two parts (cf. (20)), equation (23) follows. The derivation of equations (24) – (26) is straightforward by using Property 2.

Next, consider the gradient vector $\boldsymbol{g}$ that is given as

$$\boldsymbol{g} = \left\langle \tfrac{1}{M^*M} (\varphi \boldsymbol{U}_f^* U_0 + \psi \boldsymbol{E}_f^* E_0) \right\rangle = \langle \boldsymbol{g}_0 \rangle .$$

We will show that it is equal to zero. For g_0 follows

$$\begin{aligned} g_0 &= \frac{1}{M^*M}(\varphi U_f^* U_0 + \psi E_f^* E_0) \\ &= \frac{1}{M^*M} T_x^* \Big\{ \frac{1}{F}[\varphi A^* F^* A F R_0 - \psi B^* F S_0] G_w \\ &\quad + \frac{1}{AF}[-\varphi A^* F^* A F Q_0 + \psi B^* A F P_0] G_d \\ &\quad + \frac{1}{AF}[-\varphi A^* F^* A F Q_0 + \psi B^* A F P_0](F H_y + B H_u) \\ &\quad + [\varphi A^* F^* A F + \psi B^* B] H_{\tilde{u}0} \Big\}. \end{aligned}$$

Substituting (9)–(11) into g_0 yields

$$\begin{aligned} g_0 &= T_x^* \frac{Z_f^*}{M^*} G_w - T_x^* \frac{Z_b^*}{M^*} G_d + T_x^* \frac{1}{M^*}[M^* H_{\tilde{u}0} - Z_b^*(F H_y + B H_u)] \\ &= g_w - g_d + g_h. \end{aligned}$$

As $\langle Z_b \rangle = \langle Z_f \rangle = 0$ and G_w, G_d are constants then $\langle g_w \rangle = \langle g_d \rangle = 0$. The proof of the time-varying part $\langle g_h \rangle = 0$ is based on assumption that

$$M^* H_{\tilde{u}0} - Z_b^*(F H_y + B H_u) = Z_h^*, \quad \langle Z_h \rangle = 0.$$

From Appendix 2, equation (41) for Z_h^* follows

$$\begin{aligned} Z_h^* &= M^*[-\tilde{U}^* P_0 + Z_{P,\tilde{u}}^* - Y^* Q_0 + Z_{Q,y}^*] \\ &\quad - Z_b^*[F(-Y^* A + Z_{A,y}^* + U^* B - Z_{B,u}^*) + B u_{t-1}] \\ &= Y^*(-Z_b^* A F + M^* Q_0) + (Z_b^* F B U^* + Z_b^* B u_{t-1} + M^* P_0 \tilde{U}^*) \\ &\quad + (Z_b^* Z_{A,y}^* F + Z_b^* Z_{B,u}^* F + M^* Z_{P,\tilde{u}}^* + M^* Z_{Q,y}^*). \end{aligned}$$

Using equations (42), (9), (11) and considering the fact that $\mathrm{ord}(Z_b^* Z_{A,y}^* F) > 0$, $\mathrm{ord}(Z_b^* Z_{B,u}^* F) > 0$, after some algebraic manipulations follows

$$Z_h^* = Y^* B^* \psi + \tilde{U}^* A^* F^* \varphi + Z_t^*, \quad \langle Z_t \rangle = 0$$

hence $\langle Z_h \rangle$ and also g are equal to zero. The cost function is therefore of the form (22). The c term is given as $\langle \varphi \tilde{u}_0^* \tilde{u}_0 + \psi e_0^* e_0 \rangle$ and it is the minimum value of the cost function (7) of the nominal LQ controller P_0, Q_0, R_0. □

Corollary 1. Unconstrained predictive controller based on infinite horizon cost function is equal to nominal LQ controller.

Proof. Minimisation of the cost (22) in the absence of constraints gives $x = 0$ and hence the forced part of the predictive controller vanishes. □

The corollary only confirms the fact that has been expected for unconstrained predictive controller. The nominal controller is proven to be optimal for the formulation of the cost (7). As the predictive controller tries to minimise the same criterion then both controller must coincide.

However, the expressions for predictions make it possible to incorporate constraints in the controller design. Then the forced parts are nonzero and can be optimised subject to (22) and constraints.

4. PREDICTIVE CONSTRAINED CONTROL

We assume for simplicity in this paper, that only variable e is constrained. The results however apply equally when inequality constraints are specified on an array of variables. The free and the forced predictions can be obtained by means of the polynomial division from equations (17), (19) $M^{-1}T$, $T \in \{E_0, \boldsymbol{E}_x\}$ as quotients with degree N corresponding to N steps into future.

$$e = \overline{E}_0 + \overline{\boldsymbol{E}}_x \boldsymbol{x}. \tag{27}$$

It follows from the properties of the Z-transform, that the coefficients of polynomials $\overline{E}_0, \overline{\boldsymbol{E}}_x \boldsymbol{x}$ are the free and forced predictions of e within the horizon $j = 0, \dots, N$. However, we note that the free responses can also be computed recursively from the difference equations (2)-(6) in the same manner as in GPC.

The constrained predictive control problem can be written as an optimisation problem

$$\min_{\boldsymbol{x}} \boldsymbol{x}^T \boldsymbol{H} \boldsymbol{x} \quad \text{subject to} \quad e_{\min} \leq \overline{E}_0 + \overline{\boldsymbol{E}}_x \boldsymbol{x} \leq e_{\max}. \tag{28}$$

The Hessian matrix $\boldsymbol{H}$ can be shown to be symmetric and positive semidefinite as it can contain zero rows and columns. That follows from autoswitching properties of the controller – it is only switched on if some change on w, d occurs and switched off after d_f steps (the X_f part is blocked) and d_b steps (the X_b part is blocked). The coefficients of $\boldsymbol{x}$ are accordingly gradually excluded from optimisation and hence the forced predictions and $\boldsymbol{H}$ contain at corresponding positions zeros. See Fikar and Engell [4] for details. The zero rows and columns can be deleted and thus the complexity of optimisation reduced.

This can also easily be seen from the structure of the Hessian matrix given by equations (24) – (26). It depends entirely on history of external signals differences Δw, Δd and not on the parameters of the controlled system.

The autoswitching property is used mainly for X_f part as setpoint changes are usually less frequent as disturbance changes. If the disturbance is not measurable, then it must be in each sampling time be reconstructed from (1) as

$$d(t-i) = A(q^{-1})\, y(t-i) - B(q^{-1})\, u(t-i) \quad i = 0, \dots, d_b + 1. \tag{29}$$

4.1. User specifications

There are only a few variables that user must specify in the implementation of the algorithm. The situation is simple if no constraints are specified. Then only weighting coefficients φ, ψ or their ratio as the only one parameter must be chosen. If $\psi = 0$ then the dead-beat controller results and the control actions are very excessive. The opposite situation when $\varphi \rightarrow 0$ penalises control actions and cautious controller results. We note that φ must not be set to zero, in this case the controller remains inactive. This is clear from equations (8), (11).

If the constraints on signals are defined and are active, three new parameters appear. These are the degrees d_f, d_b of polynomials X_b, X_f and the choice of horizon length N for constraints.

At first N is chosen conservatively as the largest number of steps when signals remain at constraints. This is mainly influenced by the distance of steady-state from constraints and also by system poles and zeros. The smaller the distance, the larger the constraint horizon.

For further issues on output constraints, softening or relaxation of constraints see [12, 17, 19, 20] as the problems are not specific for particular constrained predictive method but have to do with constraints.

The function of d_f, d_b has been described in Fikar and Engell [4]. Polynomial X_b shapes response to disturbances and X_f to reference changes. The value of their degrees depend on references and disturbances and can be set independently. For the stability requirements it has been shown in [4] that following should hold

$$\max(d_b, d_f) = N - \max(\deg(AF), \deg(B)). \tag{30}$$

This is based on fact that the degrees and horizon N cannot be chosen independently as the predictive part operates only finite number of steps. On the other hand it assures that some important inner sequences satisfy zero terminal constraints for any N [4].

The values of N, d_b, d_f are given as maximal values because if disturbances or setpoint changes occur frequently, then forced part of the controller remains activated, constraints handling is enabled and the degrees and N can be set smaller.

If activation of the constraints results mainly from setpoint changes then d_f should satisfy (30) and d_b can be smaller. Maximum value of d_b can be set if mainly regulation is performed and constraints are hit due to disturbances.

Stabilisation of the controlled system in the presence of constraints can be checked as in other constrained algorithms – by checking the feasibility of the quadratic program. This again is more property of constrained control as of a particular algorithm.

4.2. Control algorithm

The control algorithm given below summarises the steps necessary for the computation of the constrained predictive control.

For given controlled system specified by polynomials A, B and weighting coefficients φ, ψ, calculate spectral factorisation (8) and nominal controller P_0, Q_0, R_0. Further specify degrees d_f, d_b and horizon for constraints N. Then in the each sampling time repeat:

1. Construct predictions of $\tilde{u}$, e (Theorem 3), objective function and constraints. Solve QP (28).
2. Construct controller P, Q, R from equations (12), (13).
3. Implement the control law according to equations (5), (6).

We note that usually in the second step a reduction of complexity of Quadratic Programming results due to the autoswitching property of the predictive controller and only some coefficients of X_b, X_f are to be found.

5. CONCLUSIONS

In the article predictive controller minimising LQ criterion has been presented. The derivation of the controller is based on algebraic approach of Kučera.

The design differs from other predictive control approaches in two aspects. At first, it calculates predictions and minimises the cost function only in the subspace of stable controllers. The second difference is utilisation of infinite output and control horizons. Both properties result in clearer behaviour with respect to stability of the closed-loop system. The proof of equality of the predictive controller with the nominal controller in the unconstrained case is given.

If no constraints are given or are inactive, then the controller reduces to nominal LQ controller and only one user parameter must be specified. If the constraints are active, time-varying part given by YK parametrisation of stabilising controllers is enabled and Quadratic Programming results as in other predictive control methods, however with different structure and with variable number of unknown parameters. In this case the controller is time-varying until constraints are again inactive or the controller is automatically switched off when no change of input signals w, d occurs for certain period of time.

It is the structure of YK parameters that makes the number of optimised variables finite and resulting controller time-varying. In this article we have used FIR structure. However, any other stable transfer function structures can be utilised and it is thus open to further research to design other YK parameters with different properties of the constrained controller. As all expressions are derived after closing the feedback loop, methods dealing with time-varying systems can be used to study the properties of the controller.

The introduction of two YK parameters X_b, X_f leads to separation between effects of references and disturbances. In many cases such detailed decomposition is not needed and only one polynomial X can be used to handle constraints.

The extensions to adaptive control by including an appropriate RLS method for the identification of process parameters are straightforward.

The subject of the further investigations is focused on stochastic nature of the input signals and therefore minimisation of LQG criterion and on MIMO version of the algorithm.

APPENDIX

A.1 Transformations

Let us consider a polynomial $A(z^{-1})$ with degree da:

$$A(z^{-1}) = a_0 + a_1 z^{-1} + \cdots + a_{da} z^{-da}. \tag{31}$$

Definition 1. Let $A(z^{-1})$ be a polynomial of the form (31). Then coef(A) is defined by stacking all coefficients of A into one column vector of dimension $[da+1\times 1]$

$$\mathrm{coef}(A) = (a_0, a_1, \dots, a_{da})^T. \tag{32}$$

Property 1. (Polynomial Transformation) Let $A(z^{-1})$ be a polynomial of the form (31). Then A is related to coef(A) by the equation

$$A(z^{-1}) = \boldsymbol{P}_{da}(z^{-1})\,\mathrm{coef}(A) \tag{33}$$

where $\boldsymbol{P}_{da}(z^{-1}) \in \mathcal{R}_{1,da+1}$ is row polynomial polynomial vector

$$\boldsymbol{P}_{da}(z^{-1}) = (1, z^{-1}, \dots, z^{-da}). \tag{34}$$

Property 2. (Shift) Let $\boldsymbol{P}_d(z^{-1}) \in \mathcal{R}_{1,d+1}$ is row polynomial polynomial vector

$$\boldsymbol{P}_d(z^{-1}) = (1, z^{-1}, \dots, z^{-d}). \tag{35}$$

Then the following identity holds

$$\langle \boldsymbol{P}_d z^j \rangle_{cp} = \boldsymbol{P}_d \boldsymbol{B}_d^j \tag{36}$$

where $\boldsymbol{B}_d \in \mathcal{R}_{d+1,d+1}$ is "backward shift" Toeplitz matrix (Horn and Johnson [5]) of the form

$$\boldsymbol{B} = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ & & \ddots & 1 \\ 0 & \dots & \dots & 0 \end{pmatrix}. \tag{37}$$

A.2 Initial conditions

The initial conditions of both the controller and the controlled system were assumed to be nonzero. This is the result of the receding horizon control strategy, where at each time sample new predictions are calculated.

As the result of the Z-transform properties the time shift of a discrete function can be expressed as

$$Z\{q^{-1} f_k\} = z^{-1}[Z\{f_k\} + z f_0] \tag{38}$$

where q is the forward shift operator $q f_k \equiv f_{k+1}$.

The following property generalises (38) for the case of the dynamical system described by a difference equation.

Property 3. (Initial conditions for discrete systems) Let the process be described by the difference equation

$$y_t = b_0 u_t + b_1 u_{t-1} + \dots + b_n u_{t-n} = B(q^{-1})\, u_t \tag{39}$$

where $B \in \mathcal{R}[q^{-1}]$. The Z-transform of this equation is then given by

$$y(z) = B(z^{-1})\,u(z) + H(z^{-1})$$

where the polynomial $H(z^{-1})$ has degree $n-1$ and can be expressed as

$$H(z^{-1}) = \sum_{j=1}^{n} \langle B(z^{-1})\,u_{t-j}z^{j}\rangle_{cp}$$

where $\langle\cdot\rangle_{cp}$ denotes the causal part of the given expression, i.e. only the terms z^{-i}, $i \geq 0$.

An alternate form for $H(z^{-1})$ can be given as

$$H(z^{-1}) = \langle U^* B\rangle_{cp} = U^* B - Z^*_{B,u} \tag{40}$$

where $\langle Z_{B,u}\rangle = 0$, $Z^*_{B,u}$ is the noncausal part of the expression U^*B, and

$$U(z^{-1}) = u_{t-1}z^{-1} + u_{t-2}z^{-2} + \cdots$$

is a sequence of past values of the signal $u(t)$.

Property 3 allows us to express the initial conditions H_u, H_y, $H_{\tilde{u}0}$ needed for the optimal controller which are

$$\begin{aligned} H_u &= u_{t-1}, \quad \deg(H_u) = 0 \\ H_y &= -Y^*A + Z^*_{A,y} + U^*B - Z^*_{B,u} \\ H_{\tilde{u}0} &= -\tilde{U}^*P_0 + Z^*_{P,\tilde{u}} - Y^*Q_0 + Z^*_{Q,y}. \end{aligned} \tag{41}$$

The relation between sequences $\tilde{U}$ and U is given as

$$\begin{aligned} \tilde{U}^* &= \tilde{u}_{t-1}z + \tilde{u}_{t-2}z^2 + \cdots \\ &= (u_{t-1} - u_{t-2})z + (u_{t-2} - u_{t-3})z^2 + \cdots \\ &= U^* + u_{t-1} - z^{-1}U^* \\ &= U^*F + u_{t-1}. \end{aligned} \tag{42}$$

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Dr. Ing. Miroslav Fikar, Department of Process Control, Faculty of Chemical Technology, Slovak University of Technology, Radlinského 9, SK-812 37 Bratislava. Slovakia.
e-mail: fikar@cvt.stuba.sk

Prof. Dr.-Ing. Sebastian Engell, Process Control Group, Department of Chemical Engineering, University of Dortmund, D-44221 Dortmund. Germany.
e-mail: ast@ast.chemietechnik.uni-dortmund.de

Doc. Ing. Petr Dostál, CSc., Department of Automatic Control, Faculty of Technology TU, nám. T. G. Masaryka 275, CZ-762 72 Zlín. Czech Republic.
e-mail: dostalp@zlin.vutbr.cz