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DISCRETE-TIME STATE DESCRIPTION OF PURE DEADTIME PROCESSES[1]

VÁCLAV SOUKUP

This contribution deals with the discrete-time linear state models of pure deadtime multi-input, multi-output dynamic processes. A straightforward way is presented to obtain minimum-dimensional state realizations of these processes.

1. INTRODUCTION

Time delays (deadtimes) occur frequently in industrial technologies, transportation, communication or transmission processes, robotics and other branches. Mathematical models of the deadtime dynamic processes have long been treated and used in system and control theory. Especially, the input-output models in the time or frequency domain are usual in modeling, simulation and control.

Unlike a continuous-time state description, discrete-time state models of deadtime processes are finite-dimensional, and therefore preferred and widely introduced. The fundamental relations concerning the discrete-time state models of deadtime processes can be found, e. g., in [1] or [2]. Above all, single-input, single-output models are considered here.

This work deals with the discrete-time description of pure deadtime multi-input, multi-output (MIMO) linear processes, when time delays dominate and other dynamics are missing or negligible. Pure deadtime processes can well approximate various systems and subsystems in the technologies above. State space models of these processes are considered and the way is presented to find the minimum-dimensional state realizations.

The work is organized into several parts. Following Introduction several preliminary relations are introduced in Section 2. The problem of the minimum state realization is then solved in Section 3 and briefly adapted to an intersample behaviour of process-oriented models in Section 4. Two illustrative examples are given in the last section.

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2. PRELIMINARY RELATIONS AND PROBLEM FORMULATION

A primary continuous-time, pure deadtime process is considered and described by

$$\boldsymbol{Y}(s) = \boldsymbol{G}(s)\,\boldsymbol{U}(s), \tag{1}$$

where $\boldsymbol{Y}(s)$ and $\boldsymbol{U}(s)$ are the Laplace transforms of p-dimensional output vector $\boldsymbol{y}(t)$ and r-dimensional input vector $\boldsymbol{u}(t)$, respectively, and $\boldsymbol{G}(s)$ is $(p \times r)$ transfer function matrix (TFM) with the entries

$$G_{ij}(s) = \alpha_{ij0} + \alpha_{ij1}\exp(-s\tau_{ij1}) + \ldots + \alpha_{ij\nu(i,j)}\exp(-s\tau_{ij\nu(i,j)}); \tag{2}$$

$$i = 1, \ldots, p \quad \text{and} \quad j = 1, \ldots, r.$$

The corresponding time domain equations have the form

$$y_i(t) = \sum_{j=1}^{r}[\alpha_{ij0}u_j(t) + \alpha_{ij1}u_j(t-\tau_{ij1}) + \ldots + \alpha_{ij\nu(i,j)}u_j(t-\tau_{ij\nu(i,j)})], \quad i = 1, \ldots, p.$$

For computer simulation and control a discrete-time model of the process (1), (2) is usually required. It is assumed that $y_i(t)$ is observed at times kT only and $u_j(t)$ sampled uniformly at kT and reconstructed subsequently by zero-order hold; T is a sampling time and $k = 0, 1, 2, \ldots$

Let us put (without subscripts)

$$\tau = mT + \mu T; \quad \text{integer} \quad m \geq 0; \quad 0 \leq \mu < 1 \tag{3}$$

for any time delay $\tau = \tau_{ijl}$ which stands in (2).

Any τ in (3) will be replaced in discrete-time form by

$$qT = (m+1)T \quad \text{if}\ \mu > 0 \quad \text{and} \quad qT = mT \quad \text{if} \quad \mu = 0. \tag{4}$$

Using Z-transform we can write

$$\boldsymbol{Y}(z) = \boldsymbol{G}(z)\,\boldsymbol{U}(z),$$

where $\boldsymbol{G}(z)$ is a $(p \times r)$ polynomial discrete-time TFM. Its general entry

$$G_{ij}(z) = \alpha_{ij0} + a_{ij}(z^{-1}), \tag{5}$$

where $a_{ij}(z^{-1})$ is a noncausal polynomial (z^{-1} divides a_{ij})

$$a_{ij}(z^{-1}) = \alpha_{ij1}z^{-q_{ij1}} + \ldots + \alpha_{ij\nu(i,j)}z^{-q_{ij\nu(i,j)}} \quad \text{with} \quad \deg a_{ij} = q_{ij\nu(i,j)}. \tag{6}$$

An example is introduced here for illustration. A two-input, two-output pure deadtime process is originally described by

$$y_1(t) = -u_1(t-0.3) + 2u_1(t-2) + 0.5u_2(t) + u_2(t-1.4)$$

and

$$y_2(t) = u_1(t-1) + 0.5u_2(t-0.6)$$

and therefore

$$G(s) = \begin{bmatrix} -e^{-0.3s} + 2e^{-2s} & 0.5 + e^{-1.4s} \\ e^{-s} & 0.5e^{-0.6s} \end{bmatrix}.$$

For the discrete-time model with the sampling time $T = 0.6$ sec time delays are gradually referred to T as:

$0.3 : m = 0,\ \mu = 0.5$ $\quad 2 : m = 3,\ \mu = 0.33$
$1.4 : m = 2,\ \mu = 0.33$ $\quad 1 : m = 1,\ \mu = 0.67$ $\quad 0.6 : m = 1,\ \mu = 0.$

The discrete-time equations (in dimensionless time k) are

$$y_1(k) = -u_1(k-1) + 2u_1(k-4) + 0.5\,u_2(k) + u_2(k-3) \quad \text{and}$$

$$y_2(k) = u_1(k-2) + 0.5u_2(k-1),$$

$$G(z) = \begin{bmatrix} -z^{-1} + 2z^{-4} & 0.5 + z^{-3} \\ z^{-2} & 0.5z^{-1} \end{bmatrix} = \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -z^{-1} + 2z^{-4} & z^{-3} \\ z^{-2} & 0.5z^{-1} \end{bmatrix},$$

and in accordance with (6) the degrees of a_{ij} are

$$q_{11\nu(1,1)} = 4, \quad q_{12\nu(1,2)} = 3, \quad q_{21\nu(2,1)} = 2 \quad \text{and} \quad q_{22\nu(2,2)} = 1.$$

Consider now state equations of a discrete-time system

$$\boldsymbol{x}(k+1) = \boldsymbol{F}\boldsymbol{x}(k) + \boldsymbol{H}\boldsymbol{u}(k) \qquad \text{and} \qquad \boldsymbol{y}(k) = \boldsymbol{C}\boldsymbol{x}(k) + \boldsymbol{D}\boldsymbol{u}(k), \tag{7}$$

where $\boldsymbol{x}(k)$ is n-dimensional state vector, and $\boldsymbol{F}, \boldsymbol{H}, \boldsymbol{C}$ and $\boldsymbol{D}$ matrices of the dimensions $(n \times n), (n \times r), (p \times n)$ and $(p \times r)$, respectively.

Then TFM $\boldsymbol{G}(z)$ with the entries (5) is

$$\boldsymbol{G}(z) = \boldsymbol{G}_0 + \boldsymbol{G}_1(z),$$

where the following relations hold:

$$\boldsymbol{G}_0 = [\alpha_{ij0}] = \boldsymbol{D} \tag{8}$$

and

$$\boldsymbol{G}_1(z) = \left[a_{ij}(z^{-1})\right] = \boldsymbol{C}(z\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{H} = \boldsymbol{C}(\boldsymbol{I} - \boldsymbol{F}z^{-1})\boldsymbol{H}z^{-1}. \tag{9}$$

A way to find simply the minimal (i.e., reachable as well as observable) state realization (7) of a pure deadtime process will be shown in the next section. Since $\boldsymbol{D} = \boldsymbol{G}_0$ does not change in various state space forms, hereafter only $\boldsymbol{G}_1(z)$ will be considered.

To check system properties the following theorem will be useful.

Theorem. [3] A system realization (7) is

i) reachable if and only if there exists no left eigenvector $\gamma^T \neq \mathbf{0}$ of $\boldsymbol{F}$, which satisfies

$$\gamma^T(\boldsymbol{F}-\lambda\boldsymbol{I})=\mathbf{0} \qquad \text{and} \qquad \gamma^T\boldsymbol{H}=\mathbf{0}, \tag{10}$$

ii) observable if and only if there exists no right eigenvector $\boldsymbol{\rho} \neq \mathbf{0}$ of $\boldsymbol{F}$, such that

$$(\boldsymbol{F}-\lambda\boldsymbol{I})\boldsymbol{\rho}=\mathbf{0} \qquad \text{and} \qquad \boldsymbol{C}\boldsymbol{\rho} = \mathbf{0}, \tag{11}$$

where λ are eigenvalues of $\boldsymbol{F}$.

Hereafter, the leading column degrees in $\boldsymbol{G}_1(z)$ are denoted by

$$q_j = \max_i q_{ij\nu(i,j)}; \quad j=1,\dots,r, \tag{12}$$

and the notation

$$\boldsymbol{J}_{\nu+1} = \begin{bmatrix} 0 & & & \\ \cdot & & I_\nu & \\ \cdot & & & \\ 0 & \cdot & \cdot & \cdot\ 0 \end{bmatrix}$$

where $\boldsymbol{I}_\nu$ is the $(\nu \times \nu)$ identity matrix, will be used.

3. DISCRETE-TIME MINIMUM STATE MODEL

In general it is not easy to find directly the minimum model dimension and a corresponding minimum state realization of a MIMO dynamic process. Either Hankel matrices of system Markov parameters or irreducible TFM fractions have to be determined, ([3]). Minimal diagonalizable Gilbert realization can be determined for systems with all distinct eigenvalues only; unfortunately it is just not the case for deadtime models. Also, there is a technique of successive steps to search for MSR. It includes

i) finding a reachable realization R;

ii) checking the observability of R;

iii) arranging R into an observable, i. e., minimum state realization form, if R is found not to be observable in ii).

The present work aims to show that this general approach becomes very simple in the case of pure deadtime processes, when state variables are selected to be the delayed inputs.

Generally, the dual way starting with an observable realization is also possible. Nevertheless it is without a natural and transparent character in our case of pure deadtime processes and therefore not considered here.

Thus, having a nonsingular TFM $\boldsymbol{G}_1(z)$ with no zero columns (i. e., all inputs u_j are delayed and $q_j > 0$ for any j), the initial reachable realization may be directly written in the controller-canonical form ([3]) as

$$\boldsymbol{x}(k) = [u_1(k-q_1)\dots u_1(k-1);\dots;\ u_r(k-q_r)\dots\ u_r(k-1)]\,, \tag{13}$$

$$F = \text{block diag } J_{q_j} = \begin{bmatrix} J_{q_1} & & & \\ & \cdot & 0 & \\ & & \cdot & \\ & 0 & \cdot & \\ & & & J_{q_r} \end{bmatrix}, \quad H = \begin{bmatrix} H_1 \\ \cdot \\ \cdot \\ \cdot \\ H_r \end{bmatrix}, \tag{14}$$

where H_j is the $(q_j \times r)$ matrix the entries of which are zeros except the unit at the jth column position in the last row,

$$C = \begin{bmatrix} c_{11} & \cdot & \cdot & \cdot & c_{1r} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ c_{p1} & \cdot & \cdot & \cdot & c_{pr} \end{bmatrix}, \tag{15}$$

where c_{ij} is the $(1 \times q_j)$ row vector constructed from the corresponding coefficients of $a_{ij}(z^{-1})$, zeros included.

Using (9) the reader can simply verify that (13) to (15) gives the TFM; at the same time all eigenvalues of F are found to be $\lambda = 0$. Therefore the realization (13) to (15) is always reachable since the condition (10) has the form $\gamma^T\ [F\ H] = 0$ which has the only solution $\gamma = 0$.

Secondly, observability of the realization (13) to (15) is checked using (11). Let us consider a right eigenvector of F to be $\rho = [\rho_1^T; \ldots; \rho_r^T]^T$ with $\rho_j = [\rho_{j1} \ldots \rho_{jq_j}]^T$. Then the first condition (11) resulting into $F\rho = 0$ yields $\rho_{j2} = \ldots = \rho_{jq_j} = 0$ for any j and the first terms ρ_{j1} in ρ_j can only be nonzero.

Hence the second condition (11) is reduced into

$$C_1[\rho_{11} \ldots \rho_{r1}]^T = 0, \tag{16}$$

where C_1 is a $(p \times r)$ column leading matrix formed by r columns of C which are at the same position as zero columns in F.

In view of (16) the condition

$$\text{rank } C_1 = r \tag{17}$$

is found to be necessary as well as sufficient for (13) to (15) to be observable and therefore minimal.

If (17) fails and rank $C_1 = \alpha < r$, the minimal model dimension is $n_1 = n - r + \alpha$; a minimum state realization is given by the new n_1-dimensional state vector

$$\overline{x}(k) = \begin{bmatrix} C_{1\alpha} v(k) \\ w(k) \end{bmatrix} = Kx(k) = \begin{bmatrix} C_{1\alpha} & 0 \\ 0 & I_{n-r} \end{bmatrix} Qx(k),$$

where $v(k) = [u_1(k - q_1) \ldots\ u_r(k - q_r)]^T$ and
$w(k) = [u_1(k - q_1 + 1) \ldots\ u_1(k - 1);\ \ldots;\ u_r(k - q_r + 1) \ldots\ u_r(k - 1)]^T$,
$C_{1\alpha}$ is the $(\alpha \times r)$ matrix containing α independent rows of C_1 and Q is the $(n \times n)$ nonsingular matrix of units and zeros, by means of which $x(k)$ is reordered into $Qx(k) = [v^T(k); w^T(k)]^T$, and the new system matrices $\overline{F}, \overline{H}$ and $\overline{C}$ follow from the relations

$$\overline{F}K = KF, \quad \overline{H} = KH \quad \text{and} \quad \overline{C}K = C.$$

Note that r has to be replaced by $r_1 < r$ if there are $r - r_1$ undelayed inputs u_j with $q_j = 0$.

4. PROCESS-ORIENTED MINIMUM STATE MODEL

The presented method is applicable unchanged for the process description which takes the intersample behaviour into account.

Considering a continuous time $t = kT + \epsilon T$, $0 \le \epsilon < 1$, then

$$qT = (m+1)T \quad \text{if } \epsilon < \mu \quad \text{and} \quad qT = mT \quad \text{if } \epsilon \ge \mu$$

is true instead of (4).

Thus, for a given ϵ the minimum state realization can be found in the same way as described above. Of course, a minimum dimension of the state realization will be different within various ranges of ϵ.

5. EXAMPLES

Example 1. Assume a discrete-time TFM is given by

$$G(z) = \begin{bmatrix} z^{-1} + 2z^{-2} & -1 + 3z^{-2} \\ 2 & 2z^{-1} \\ z^{-1} & 2 - 3z^{-1} \end{bmatrix}$$

and a minimum state realization should be found.

First,

$$G_0 = \begin{bmatrix} 0 & -1 \\ 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad G_1(z) = \begin{bmatrix} z^{-1} + 2z^{-2} & 3z^{-2} \\ 0 & 2z^{-1} \\ z^{-1} & -3z^{-1} \end{bmatrix},$$

$n = 4$, $q_1 = 2$ and $q_2 = 2$ are determined.

Then the initial reachable state realization defined by (13) is

$$x(k) = [u_1(k-2)\ u_1(k-1)\ u_2(k-2)\ u_2(k-1)]^T,$$

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ H = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \ C = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \end{bmatrix}., \text{ and } D = G_0.$$

$$\text{Then } C_1 = \begin{bmatrix} 2 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{with rank } C_1 = 1.$$

Hence $n_1 = 3$ and

$$v(k) = [u_1(k-2)\ u_2(k-2)]^T, \quad w(k) = [u_1(k-1)\ u_2(k-1)]^T, \quad C_{1\alpha} = [2 \ \ 3],$$

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 2 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then the minimum state realization is described by

$$\overline{x}(k) = [2u_1(k-2) + 3u_2(k-2) \;\; u_1(k-1) \;\; u_2(k-1)]^T,$$

$$\overline{F} = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; \overline{H} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \; \overline{C} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix} \; \text{and} \; \overline{D} = D = G_0.$$

Example 2. A pure deadtime process given by

$$y_1(t) = u_1(t-1.5) - u_2(t-0.7)$$

and

$$y_2(t) = 2u_1(t-0.2) + u_2(t-2.2)$$

should be described in the discrete-time process-oriented state-space form provided it is sampled with $T = 1$.

We have

a) for $0 \leq \epsilon < 0.2$

$$y_1(k+\epsilon) = u_1(k-2) - u_2(k-1),$$

$$y_2(k+\epsilon) = 2u_1(k-1) + u_2(k-3),$$

$q_1 = 2, \;\; q_2 = 3, \;\; n = 5$ and the realization

$$x(k) = [u_1(k-2); u_1(k-1); u_2(k-3); u_2(k-2); u_2(k-1)]^T,$$

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, H = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 2 & 1 & 0 & 0 \end{bmatrix} \; \text{and} \; D = O;$$

b) for $0.2 \leq \epsilon < 0.5$

$$y_1(k+\epsilon) = u_1(k-2) - u_2(k-1),$$

$$y_2(k+\epsilon) = 2u_1(k) + u_2(k-2),$$

$q_1 = 2, q_2 = 2, n = 4$ and the realization

$$x(k) = [u_1(k-2); u_1(k-1); u_2(k-2); u_2(k-1)]^T,$$

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, H = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix};$$

c) for $0.5 \le \epsilon < 0.7$

$$y_1(k+\epsilon) = u_1(k-1) - u_2(k-1),$$
$$y_2(k+\epsilon) = 2u_1(k) + u_2(k-2),$$

$q_1 = 1, q_2 = 2, n = 3$ and

$$x(k) = [u_1(k-1); u_2(k-2); u_2(k-1)]^T, \tag{18}$$

$$F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}; \tag{19}$$

d) for $0.7 \le \epsilon < 1$

$$y_1(k+\epsilon) = u_1(k-1) - u_2(k),$$
$$y_2(k+\epsilon) = 2u_1(k) + u_2(k-2),$$

$q_1 = 1$, $q_2 = 2$, $n = 3$, $x(k)$ stands in (18), F and H in (19) and

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}.$$

Since

$$C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \operatorname{rank} C_1 = 2 = r$$

in all four cases, the presented realizations have the minimum dimensions.

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Doc. Ing. Václav Soukup, CSc., Department of Automatic Control, Faculty of Electrical Engineering – Czech Technical University, Karlovo nám. 13, 121 35 Praha 2. Czech Republic.
e-mail: soukup@control.felk.cvut.cz