KYBERNETIKA — VOLUME 35 (1999), NUMBER 2, PAGES 177–193

OPTIMALITY CONDITIONS FOR A CLASS OF MATHEMATICAL PROGRAMS WITH EQUILIBRIUM CONSTRAINTS: STRONGLY REGULAR CASE[1]

JIŘÍ V. OUTRATA

The paper deals with mathematical programs, where parameter-dependent nonlinear complementarity problems arise as side constraints. Using the generalized differential calculus for nonsmooth and set-valued mappings due to B. Mordukhovich, we compute the so-called coderivative of the map assigning the parameter the (set of) solutions to the respective complementarity problem. This enables, in particular, to derive useful 1st-order necessary optimality conditions, provided the complementarity problem is strongly regular at the solution.

INTRODUCTION

A mathematical program with equilibrium constraints (MPEC) is an optimization problem, where a parameter-dependent variational inequality or, more specifically, a parameter-dependent complementarity problem arises as a side constraint. If this so-called equilibrium constraint is equivalent to a (convex) "lower-level" optimization problem, we get a problem of bilevel programming. Important MPECs arise frequently in natural sciences as well as in economic modelling and so this topic attracts, especially in recent years, an increased attention of many applied mathematicians. Besides the existence and approximation of solutions, the research concentrates on optimality conditions, various numerical approaches and diverse concrete applications. If we reduce our attention just to optimality conditions for finite-dimensional MPECs, we recognize in the recent works the following approaches:

(i) in [7] and [8] the authors compute under so-called "basic constraint qualification" a tangent cone approximating the equilibrium constraint. This leads directly to a primal version of optimality conditions. Via a suitable dualization one gets then a finite family of optimality conditions in the dual, so-called Karush–Kuhn–Tucker (KKT) form.

[1]This research was supported in part by the grant No. 0455/95 from the German-Israeli Foundation and the grant A 1075707 from the Academy of Sciences of the Czech Republic.

(ii) in [21] an error bound is constructed for the equilibrium constraint in a bilevel program using the value-function of the lower-level problem. Under the assumption of so-called partial calmness KKT conditions have been obtained. This idea is further developed and extended to MPECs in [22].

(iii) the works [6, 15] deal only with the strongly regular case, cf. [18]. Then, close to the solution, the equilibrium constraint defines a Lipschitz implicit function assigning the parameters the (unique) solutions of the corresponding variational inequality (complementarity problem). This implicit function is described by means of the generalized Jacobians, cf. [3], and the generalized differential calculus of F. H. Clarke leads then to optimality conditions, again in the KKT form.

(iv) the papers [23, 24] and [20] employ the generalized differentiable calculus of B. Mordukhovich. [24] and [20] deal with bilevel programs, [23] with a general MPEC. In [24] and [23] the equilibrium constraint is augmented to the objective by an exact penalty, whereas in [20] the lower-level problem is replaced by the Mordukhovich's stationarity condition. In all cases the resulting conditions contain some difficult terms so that their verification is not easy in general.

The above list is definitely not exhaustive; further references can be found in [8] or in the collection [1]. For instance the interesting conditions from [4] are related to both approaches (i) and (iii). They are not of the KKT form, but their assumptions are weaker than those of [15].

The main aim of the present paper is to apply the Mordukhovich's generalized differentiable calculus in a new way, close in spirit to the works [6, 15]. Further we intend to derive optimality conditions without any difficult terms so that their verification would not be too complicated. To achieve these goals, we confine ourselves to equilibria described by parameter-dependent nonlinear complementarity problems (NCPs). Such equilibria are met, however, quite frequently e. g. in mechanics, where they describe various (discretized) obstacle and contact problems ([5, 6]). We convert this MPEC into the minimization of a value function for which the optimality conditions are stated in terms of Mordukhovich's subdifferentials and normal cones, cf. [9]. To compute such a subdifferential of the value-function one needs, however, the so-called coderivative of the "equilibrium map" which assigns the solution sets of the considered NCP to the parameters.

The paper is organized as follows. In Section 1 we formulate the problem, give the definitions of the main objects of Mordukhovich's generalized differential calculus and state the crucial result from [10] concerning the subdifferentials of value functions. Section 2 is devoted to the computation of the above mentioned coderivative under a constraint qualification which is ensured in two different ways by verifiable assumptions. In Section 3 we confine ourselves to the strongly regular case and derive the respective optimality conditions. Moreover, we relate them to the conditions from [15] and [8].

The following notation is employed. x^i is the ith component of a vector $x \in \mathbb{R}^n$, A^i is the ith row of a matrix A, E is the unit matrix, $\mathbb{R}^n_+$ is the nonnegative orthant of $\mathbb{R}^n$ and $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ is the extended real line. For an $[m \times n]$ matrix A and index sets $I \subset \{1, 2, \ldots, m\}$, $J \subset \{1, 2, \ldots, n\}$, $A_{I,J}$ denotes the submatrix of

A with rows and columns, specified by I and J, respectively. A_I is the submatrix of A with rows specified by I. Similarly, for a vector $d \in I\!R^n$, d_I is the subvector composed from the components d^i, $i \in I$. Furthermore, $conv\,\Omega$ denotes the convex hull of a set Ω, $Gph\,\Phi$ is the graph of a multifunction Φ, $epi\,f$ is the epigraph of a function f and for a convex set Ω and a point $x \in \Omega$, $N_\Omega(x)$ denotes the standard normal cone to Ω at x in the sense of convex analysis. If D is a cone with vertex at the origin, then D^0 is its negative polar cone. For $x, y \in I\!R^n$ the inequalities $x \geq y$, $x > y$ mean $x^i \geq y^i$ and $x^i > y^i$ for all i, respectively.

1. PROBLEM FORMULATION AND PRELIMINARIES

Let $F[I\!R^n \times I\!R^m \to I\!R^m]$ be a continuously differentiable mapping and consider the parameter-dependent NCP:

For a given parameter $x \in I\!R^n$ find $y \in I\!R^m_+$ such that

$$F(x,y) \geq 0 \quad \text{and} \quad \langle F(x,y),\, y\rangle = 0. \tag{1.1}$$

It is well-known that the NCP (1.1) can be equivalently written down as the *generalized equation* (GE):

$$0 \in F(x,y) + N_{I\!R^m_+}(y), \tag{1.2}$$

where $N_{I\!R^m_+}(y) := \emptyset$ provided $y \notin I\!R^m_+$. The multifunction which assigns x the set of solutions to (1.2) will be termed *equilibrium map* and denoted by S. Assume that $f[I\!R^n \times I\!R^m \to I\!R]$ is a locally Lipschitz objective function and ω is a nonempty and closed set of admissible parameters. The main object of our investigations is the MPEC:

$$\begin{array}{ll} \text{minimize} & f(x,y) \\ \text{subject to} & y \in S(x) \\ & x \in \omega. \end{array} \tag{1.3}$$

Remark. In MPECs with equilibria described by NCPs one has often to do also with "state constraints" of the form $y \in \Omega$, where Ω is a nonempty and closed subset of $I\!R^m$. In what follows, however, we assume that possible constraints of this form have been added to the objective by means of a suitable exact penalty.

Problem (1.3) possesses evidently a solution provided ω is compact and S is single-valued and continuous. Otherwise the existence proofs are more complicated and this topic goes beyond the scope of this paper. The interested reader is referred e. g. to [24], where to this purpose an inf-compactness argument is applied.

Let us introduce the value function

$$\Theta(x) := \inf_{y \in S(x)} f(x,y). \tag{1.4}$$

With the help of it, (1.3) may be written down as a simple mathematical program

$$\begin{array}{ll} \text{minimize} & \Theta(x) \\ \text{subject to} & x \in \omega. \end{array} \tag{1.5}$$

For abstract mathematical programs of the type (1.5), 1st-order necessary optimality conditions have been derived in [9] in terms of the Mordukhovich's subdifferential of the objective and the generalized normal cone to the admissible set. To be able to apply these optimality conditions we need, however, to compute the Mordukhovich's subdifferential of the value function Θ, which involves the computation of the coderivative of S. For the readers convenience we state now the appropriate definitions (cf. [9]) and also an important result from [10] which plays a crucial role in Section 3.

Consider a set $A \subset I\!R^n$.

Definition 1.1. Let $x \in cl\,A$. The nonempty cone

$$T_A(x) := \limsup_{t \downarrow 0} \frac{A - x}{t}$$ [2]

is called the *contingent cone* to A at x. The *generalized normal cone* to A at x, denoted $K_A(x)$, is defined by

$$K_A(x) = \limsup_{u \xrightarrow{A} x} T_A^0(u).$$

If A is convex one has $K_A(x) = T_A^0(x)$. The cone $K_A(x)$ is generally nonconvex, but the multifunction $K_A(\cdot)$ is upper semicontinuous at each point of $cl\,A$ (with respect to $cl\,A$), which is essential in the calculus of Mordukhovich's subdifferentials and coderivatives introduced below.

Definition 1.2. Let $\varphi[I\!R^n \to \overline{I\!R}]$ be an arbitrary extended real-valued function and $x \in dom\,f$. The sets

$$\partial^- \varphi(x) := \{x^* \in I\!R^n \,|\, (x^*, -1) \in K_{\text{epi}\,\varphi}(x, \varphi(x))\}$$

and

$$\partial^\infty \varphi(x) := \{x^* \in I\!R^n \,|\, (x^*, 0) \in K_{\text{epi}\,\varphi}(x, \varphi(x))\}$$

are called the *Mordukhovich's subdifferential* and the *singular subdifferential* of φ at x.

In [9] it has been proved that a lower semicontinuous function φ is Lipschitz near x if and only if $\partial^\infty \varphi(x) = \{0\}$.

[2] The "limsup" in the definitions of $T_A(x)$ and $K_A(x)$ is the upper limit of multifunctions in the sense of Kuratowski–Painlevé, cf. [2].

Definition 1.3. Let $F[\mathbb{R}^n \mapsto \mathbb{R}^m]$ be a multifunction and $(x, y) \in cl\, Gph\, F$. The multifunction $D^* F(x,y)[\mathbb{R}^m \to \mathbb{R}^n]$ defined by

$$D^* F(x,y)(y^*) := \{x^* \in \mathbb{R}^n \,|\, (x^*, -y^*) \in K_{\quad F}(x,y)\}, \quad y^* \in \mathbb{R}^m$$

is called the *coderivative* of F at (x,y). If F is single-valued at x, we write simply $D^* F(x)(y^*)$.

Suitable upper estimates of the subdifferentials of the value function Θ are given in the following important assertion, where

$$M(x) := \{y \in S(x) \,|\, f(x,y) = \Theta(x)\}.$$

Theorem 1.1. Let the images $M(x)$ be nonempty and uniformly bounded sets around an $\overline{x} \in \mathbb{R}^n$. Then one has

$$\partial^- \Theta(\overline{x}) \subset \bigcup \{x_1^* + x_2^* \,|\, x_1^* \in D^* S(\overline{x}, \overline{y})(y^*),\, (x_2^*, y^*) \in \partial^- f(\overline{x}, \overline{y}),\, \overline{y} \in M(\overline{x})\} \tag{1.6}$$

and

$$\partial^\infty \Theta(\overline{x}) \subset \bigcup \{D^* S(\overline{x}, \overline{y})(0) \,|\, \overline{y} \in M(\overline{x})\}. \tag{1.7}$$

Proof. The statement is a direct consequence of [10], Theorem 4.1 since f is locally Lipschitz and $Gph\, S$ is closed. Indeed, under our assumptions, to each pair of sequences $x_i \to x$, $y_i \to y$ such that $y_i \in S(x_i)$ one has $y \in S(x)$ due to the continuity of F. □

2. THE CODERIVATIVE OF THE EQUILIBRIUM MAP

As in [12], we rewrite the GE (1.2) into the form

$$(y, -F(x,y)) \in Gph\, N_{\mathbb{R}^m_+} \tag{2.1}$$

so that

$$S(x) = \{y \in \mathbb{R}^m \,|\, \Phi(x,y) \in \Lambda\}$$

with $\Phi(x,y) = \begin{bmatrix} y \\ -F(x,y) \end{bmatrix}$ and $\Lambda = Gph\, N_{\mathbb{R}^m_+}$. Let us fix a pair $(\overline{x}, \overline{y})$ with $\overline{y} \in S(\overline{x})$. Since F is continuously differentiable, one has

$$D^* \Phi(\overline{x}, \overline{y})(z^*) = \begin{bmatrix} 0 & (-\nabla_x F(\overline{x}, \overline{y}))^{\mathrm{T}} \\ E & (-\nabla_y F(\overline{x}, \overline{y}))^{\mathrm{T}} \end{bmatrix} z^*$$

for all $z^* \in \mathbb{R}^{2m}$. We pose now the following constraint qualification:

(CQ)

$$Ker \begin{bmatrix} 0 & (-\nabla_x F(\overline{x}, \overline{y}))^{\mathrm{T}} \\ E & (-\nabla_y F(\overline{x}, \overline{y}))^{\mathrm{T}} \end{bmatrix} \cap K_{\quad N_{\mathbb{R}^m_+}}(\overline{y}, -F(\overline{x}, \overline{y})) = \{0\}.$$

Under (CQ) it is a direct consequence of [11], Theorem 6.10 that

$$D^* S(\bar{x},\bar{y})(y^*) \subset \Big\{x^* \in \mathbb{R}^n \,|\, x^* + (\nabla_x F(\bar{x},\bar{y}))^{\mathrm{T}} z = 0, \\ -w + (\nabla_y F(\bar{x},\bar{y}))^{\mathrm{T}} z = y^*,\ (w,z) \in K \quad N_{\mathbb{R}^m_+}(\bar{y}, -F(\bar{x},\bar{y}))\Big\} \tag{2.2}$$

for all $y^* \in \mathbb{R}^m$. The above (CQ) as well as inclusion (2.2) will now be simplified using the specific structure of our problem. In the first step we turn our attention to the cone $K \quad N_{\mathbb{R}^m_+}(\bar{y}, -F(\bar{x},\bar{y}))$.

Lemma 2.1. Let $(u,v) \in Gph\, N_{\mathbb{R}^m_+}$, i. e. $u \in \mathbb{R}^m_+$ and $v \in N_{\mathbb{R}^m_+}(u)$. Then

$$(w,z) \in K \quad N_{\mathbb{R}^m_+}(u,v) \ \text{ iff } \ (w^i,z^i) \in K \quad N_{\mathbb{R}_+}(u^i,v^i) \tag{2.3}$$

with $K \quad N_{\mathbb{R}_+}(u^i,v^i)$ given by the following relations:

$$\begin{aligned}
&\text{If } u^i > 0,\ v^i = 0, \text{ then } K \quad N_{\mathbb{R}_+}(u^i,v^i) = \{0\} \times \mathbb{R};\\
&\text{if } u^i = 0,\ v^i < 0, \text{ then } K \quad N_{\mathbb{R}_+}(u^i,v^i) = \mathbb{R} \times \{0\};\\
&\text{if } u^i = 0,\ v^i = 0, \text{ then } K \quad N_{\mathbb{R}_+}(u^i,v^i) = \{(w^i,z^i) \,|\, w^i z^i = 0\} \cup \\
&\qquad \cup \{(w^i,z^i) \,|\, (-w^i,z^i) \in \mathrm{int}\mathbb{R}^2_+\}.
\end{aligned} \tag{2.4}$$

P r o o f. One has evidently

$$Gph\, N_{\mathbb{R}_+} = \{(a,b) \in \mathbb{R}^2 \,|\, a \ge 0,\ b = 0\} \cup \{(a,b) \in \mathbb{R}^2 \,|\, a = 0,\ b \le 0\},$$

cf. Fig. 1.

Relations (2.4) are thus a simple consequence of Def. 1.1. By [2], Table 4.5 (3) $N_{\mathbb{R}^m_+}(u) = \mathsf{X}_{i=1}^m N_{\mathbb{R}_+}(u^i)$ for $u \in \mathbb{R}^m_+$ and so $Gph\, N_{\mathbb{R}^m_+}$ is the Carthesian product of m sets $Gph\, N_{\mathbb{R}_+}$. It remains to apply [9], Proposition 1.6. □

We introduce now the index sets

$$\begin{aligned}
L(\bar{y}) &:= \{i \in \{1,2,\dots,m\} \,|\, \bar{y}^i > 0\}\\
I(\bar{y}) &:= \{i \in \{1,2,\dots,m\} \,|\, \bar{y}^i = 0\}\\
I_+(\bar{x},\bar{y}) &:= \{i \in I(\bar{y}) \,|\, F^i(\bar{x},\bar{y}) > 0\}\\
I_0(\bar{x},\bar{y}) &:= \{i \in I(\bar{y}) \,|\, F^i(\bar{x},\bar{y}) = 0\} = I(\bar{y}) \setminus I_+(\bar{x},\bar{y})
\end{aligned}$$

which will play a crucial role in the whole sequel. The constraints $y^i \ge 0$ for $i \in L(\bar{y})$, $i \in I(\bar{y})$ are termed *nonactive* and *active* at $\bar{y}$, respectively. The constraints $y^i \ge 0$ for $i \in I_+(\bar{x},\bar{y})$ and $i \in I_0(\bar{x},\bar{y})$ are termed *strongly* and *weakly active* at $(\bar{x},\bar{y})$, respectively. For notational simplicity, the arguments at L, I, I_+ and I_0 will sometimes be omitted if it cannot cause a confusion.

The following statement from [18] contains a characterization of the strong regularity for the considered NCP at $(\bar{x},\bar{y})$ in terms of the problem data.

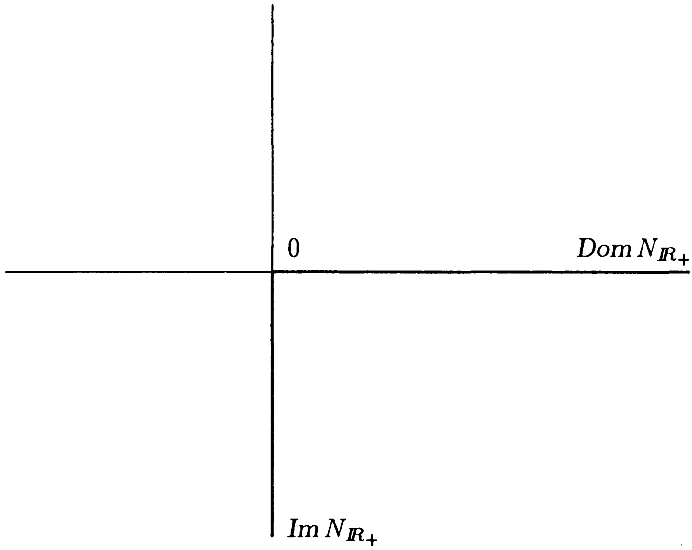

Fig. 1. *Gph* $N_{\mathbb{R}_+}$.

Proposition 2.2. Consider the NCP (1.1) at the reference point $(\bar{x}, \bar{y})$, $\bar{y} \in S(\bar{x})$, and the matrix

$$R(\bar{x}, \bar{y}) = \begin{bmatrix} \nabla_y F_{L,L}(\bar{x}, \bar{y}) & \nabla_y F_{L,I_0}(\bar{x}, \bar{y}) \\ \nabla_y F_{I_0,L}(\bar{x}, \bar{y}) & \nabla_y F_{I_0,I_0}(\bar{x}, \bar{y}) \end{bmatrix}.$$

The NCP (1.1) is strongly regular (in the sense of Robinson) if and only if

(i) $\nabla_y F_{L,L}(\bar{x}, \bar{y})$ is nonsingular, and

(ii) the Schur complement of $\nabla_y F_{L,L}(\bar{x}, \bar{y})$ in $R(\bar{x}, \bar{y})$ is a P-matrix (has positive principal minors).

Under (i), (ii) there exist (possibly closed) neighborhoods $\mathcal{U}$ of $\bar{x}$ and $\mathcal{V}$ of $\bar{y}$ and a Lipschitz operator $\mu[\mathcal{U} \to \mathbb{R}^m]$ such that

$$\mu(\bar{x}) = \bar{y} \quad \text{and} \quad S(x) \cap \mathcal{V} = \mu(x) \quad \text{for all } x \in \mathcal{U}. \tag{2.5}$$

We are now ready to give two verifiable conditions which ensure (CQ).

Proposition 2.3. Let either

(A1) the partial gradients $\nabla_x F^i(\bar{x}, \bar{y})$ for $i \in L(\bar{y}) \cup I_0(\bar{x}, \bar{y})$ be linearly independent,

or

(A2) the conditions (i) and (ii) of Proposition 2.2 be fulfilled.

Then (CQ) holds true at $(\overline{x}, \overline{y})$.

Proof. Consider a pair $(w, z) \in K \quad N_{R^m_+}(\overline{y}, -F(\overline{x}, \overline{y}))$. By Lemma 2.1 for $i \in L(\overline{y})$ one has $w^i = 0$ and for $i \in I_+(\overline{x}, \overline{y})$ one has $z^i = 0$. Therefore, if (w, z) belongs also to $Ker\, D^*\Phi(\overline{x}, \overline{y})$, then

$$w_L = 0, \quad z_{I_+} = 0$$

$$-(\nabla_x F_{L\cup I_0}(\overline{x}, \overline{y}))^{\mathrm{T}} z_{L\cup I_0} = 0 \tag{2.6}$$

$$(R(\overline{x}, \overline{y}))^{\mathrm{T}} z_{L\cup I_0} = w_{L\cup I_0}. \tag{2.7}$$

With respect to equation (2.6) condition (A1) implies that $z_{L\cup I_0} = 0$ and thus the whole vector z is zero. Since $w = (\nabla_y F(\overline{x}, \overline{y}))^{\mathrm{T}} z$, (CQ) is fulfilled.

Let condition (A2) be satisfied. We rewrite equation (2.7) into the form (note that $w_L = 0$)

$$\begin{aligned} (\nabla_y F_{L,L}(\overline{x}, \overline{y}))^{\mathrm{T}} z_L + (\nabla_y F_{I_0,L}(\overline{x}, \overline{y}))^{\mathrm{T}} z_{I_0} &= 0 \\ (\nabla_y F_{L,I_0}(\overline{x}, \overline{y}))^{\mathrm{T}} z_L + (\nabla_y F_{I_0,I_0}(\overline{x}, \overline{y}))^{\mathrm{T}} z_{I_0} &= w_{I_0}. \end{aligned} \tag{2.8}$$

Due to (A2) the matrix $\nabla_y F_{L,L}(\overline{x}, \overline{y})$ is nonsingular and thus

$$z_L = -((\nabla_y F_{L,L}(\overline{x}, \overline{y}))^{\mathrm{T}})^{-1} (\nabla_y F_{I_0,L}(\overline{x}, \overline{y}))^{\mathrm{T}} z_{I_0}. \tag{2.9}$$

Equation (2.9) and the second equation from (2.8) yield

$$\Xi\, z_{I_0} = w_{I_0}, \tag{2.10}$$

where Ξ is the Schur complement of $(\nabla_y F_{L,L}(\overline{x}, \overline{y}))^{\mathrm{T}}$ in $(R(\overline{x}, \overline{y}))^{\mathrm{T}}$ (and thus a P-matrix). By Lemma 2.1 the index set $I_0(\overline{x}, \overline{y})$ splits into three subsets:

$$\begin{aligned} \alpha &:= \{i \in I_0(\overline{x}, \overline{y}) \mid z^i = 0\}, \\ \beta &:= \{i \in I_0(\overline{x}, \overline{y}) \setminus \alpha \mid w^i = 0\}, \\ \gamma &:= \{i \in I_0(\overline{x}, \overline{y}) \mid z^i > 0 \text{ and } w^i < 0\}. \end{aligned} \tag{2.11}$$

Equation (2.10) implies that

$$\Xi_{\beta\cup\gamma,\beta\cup\gamma} z_{\beta\cup\gamma} = w_{\beta\cup\gamma},$$

where $w_\beta = 0$. The matrix $\Xi_{\beta\cup\gamma,\beta\cup\gamma}$ (as a principal submatrix of Ξ) is again a P-matrix. Therefore it has the important "Sign Nonreversal Property", cf. [14], Theorem 3.12, according to which there exists an index i_0 such that

$$z^{i_0}(\Xi_{\beta\cup\gamma,\beta\cup\gamma} z)^{i_0} > 0.$$

This is, however, impossible and therefore $\beta \cup \gamma = \emptyset$. As in the case of condition (A1) the whole vector z is zero and we are done. □

Remark. Condition (A1), (A2) work with the two equations (2.6), (2.7) defining the kernel of $D^*\Phi(\overline{x},\overline{y})$ (for $(w,z) \in K \quad N_{\mathbb{R}^m_+}(\overline{y}, -F(\overline{x},\overline{y}))$), respectively. One could naturally think about a more general condition involving both equations (2.6), (2.7) simultaneously; such a condition has been derived in the forthcoming paper [16].

On the basis of Proposition 2.3 we get now directly the main result of this section.

Proposition 2.4. Consider the reference pair $(\overline{x},\overline{y})$ ($\overline{y} \in S(\overline{x})$) and assume that conditions (A1) or (A2) are fulfilled. Then one has for all $y^* \in \mathbb{R}^m$

$$D^*S(\overline{x},\overline{y})(y^*) \subset \{x^* \in \mathbb{R}^n \mid x^* = -(\nabla_x F_{L\cup I_0}(\overline{x},\overline{y}))^{\mathrm{T}} z_{L\cup I_0}, (R(\overline{x},\overline{y}))^{\mathrm{T}} z_{L\cup I_0}$$
$$= (w+y^*)_{L\cup I_0}, w_L = 0 \text{ and for } i \in I_0(\overline{x},\overline{y}) \text{ either } w^i z^i = 0 \text{ or } w^i < 0 \text{ and } z^i > 0\}. \tag{2.12}$$

Remark. If the matrix $R(\overline{x},\overline{y})$ is positive definite, then the assumption (A2) is satisfied (cf. [18]).

Let us analyze the structure of our estimate of $D^*S(\overline{x},\overline{y})(y^*)$ under (A2). In agreement with the generalized differential calculus of Mordukhovich this set is generally nonconvex. We can embed it, however, in a bounded convex polyhedron.

Assume that $z_{L\cup I_0}$ is admissible with respect to the constraints on the right-hand side of (2.12). We denote

$$b := y^*_{I_0} - (\nabla_y F_{L,I_0}(\overline{x},\overline{y}))^{\mathrm{T}}[(\nabla_y F_{L,L}(\overline{x},\overline{y}))^{\mathrm{T}}]^{-1} y^*_L$$

and observe that the variable z_{I_0} fulfills the equation

$$\Xi z_{I_0} = w_{I_0} + b \tag{2.13}$$

(where Ξ is the Schur complement of $(\nabla_y F_{L,L}(\overline{x},\overline{y}))^{\mathrm{T}}$ in $(R(\overline{x},\overline{y}))^{\mathrm{T}}$). Thus

$$\begin{aligned} (\nabla_y F_{L,L}(\overline{x},\overline{y}))^{\mathrm{T}} z_L + (\nabla_y F_{I_0,L}(\overline{x},\overline{y}))^{\mathrm{T}} z_{I_0} &= y^*_L \\ \Xi z_{I_0} &= w_{I_0} + b \end{aligned} \tag{2.14}$$

and for $i \in I_0(\overline{x},\overline{y})$ either $w^i z^i = 0$ or $w^i < 0$ and $z^i > 0$. As previously the index set $I_0(\overline{x},\overline{y})$ splits into three subsets α, β and γ given by (2.11). This allows to exclude completely the variable w (note that $w_\beta = 0,\ z_\alpha = 0$) and one has

$$\begin{aligned} (\nabla_y F_{L,L}(\overline{x},\overline{y}))^{\mathrm{T}} z_L + (\nabla_y F_{\beta\cup\gamma,L}(\overline{x},\overline{y}))^{\mathrm{T}} z_{\beta\cup\gamma} &= y^*_L, \\ \Xi_{\beta,\,\beta\cup\gamma}\, z_{\beta\cup\gamma} &= b_\beta \\ \Xi_{\gamma,\,\beta\cup\gamma}\, z_{\beta\cup\gamma} &< b_\gamma \\ z_\alpha = 0, \quad z_\gamma &> 0. \end{aligned} \tag{2.15}$$

Indeed, the columns of $(\nabla_y F_{I_0,L}(\bar{x},\bar{y}))^{\mathrm{T}}$ and Ξ for the indices from α may be omitted since $z_\alpha = 0$ and the constraints created by the rows of Ξ for the indices from α may be ignored since the right-hand sides are arbitrary.

Consider now for all subsets σ of γ the linear equations

$$(\nabla_y F_{L\cup\beta\cup\sigma,\, L\cup\beta\cup\sigma}(\bar{x},\bar{y}))^{\mathrm{T}}\; ({}^{\sigma}p) = y^*_{L\cup\beta\cup\sigma}. \tag{2.16}$$

Due to (A2) these equations possess unique solutions ${}^{\sigma}\hat{p}$. By using of them we may state the following useful estimate.

Proposition 2.5. Assume, that $z_{L\cup I_0}$ satisfies the equations and inequalities (2.15). Then one has

$$z_{L\cup I_0} \in conv\, \{{}^{\sigma}\tilde{p} \,|\, \sigma \subset \gamma\}\,, \tag{2.17}$$

where

$${}^{\sigma}\tilde{p}^i = \left\langle \begin{array}{lll} {}^{\sigma}\hat{p}^i & \text{if} & i \in L \cup \beta \cup \sigma \\ 0 & \text{if} & i \in \alpha \cup (\gamma \setminus \sigma). \end{array} \right. \tag{2.18}$$

Proof. Let ℓ be the cardinality of $\beta\cup\gamma$. It is clear that $z_{\beta\cup\gamma}$ belongs to a convex polyhedral set C, where

$$C := \left\{u \in \mathbb{R}^{\ell} \,|\, \Xi_{\beta,\,\beta\cup\gamma} u = b_\beta,\; \Xi_{\gamma,\,\beta\cup\gamma} u \le b_\gamma,\; u_\gamma \ge 0\right\}.$$

We show that C is bounded. Let $u \in C$ and assume, by contradiction that $s \neq 0$ belongs to the recessive cone of C. This implies that

$$u + \lambda s \in C \qquad \text{for all } \lambda \ge 0.$$

Therefore

$$\begin{array}{rcl} \Xi_{\beta,\,\beta\cup\gamma}(u+\lambda s) & = & b_\beta \\ \Xi_{\gamma,\,\beta\cup\gamma}(u+\lambda s) & \le & b_\gamma \\ u_\gamma + \lambda s_\gamma & \ge & 0 \end{array}$$

for all $\lambda \ge 0$. Consequently, it must hold

$$\begin{array}{rcl} \Xi_{\beta,\,\beta\cup\gamma} s & = & 0 \\ \Xi_{\gamma,\,\beta\cup\gamma} s & \le & 0 \\ s_\gamma & \ge & 0. \end{array} \tag{2.19}$$

Since $\Xi_{\beta\cup\gamma,\,\beta\cup\gamma}$ is a P-matrix, by the Sign Nonreversal Property there exists an index i_0 such that

$$s^{i_0}(\Xi_{\beta\cup\gamma,\,\beta\cup\gamma} s)^{i_0} > 0.$$

This is, however, impossible and so $s = 0$. It implies the boundedness of C.

Since each bounded convex polyhedral set is the convex hull of its extreme points, we analyze in the next step the extreme points of C. By [13], Section 3.4 a vector $u \in C$ is an extreme point of C, provided $\sigma \subset \gamma$ and one has

$$\begin{aligned} \Xi_{\beta\cup\sigma,\,\beta\cup\gamma}u &= b_{\beta\cup\sigma}, \\ u_{\gamma\setminus\sigma} &= 0. \end{aligned} \tag{2.20}$$

Indeed, then the number of equations and active inequalities is greater or equal ℓ and it remains to show that the matrix

$$\begin{bmatrix} \Xi_{\beta\cup\sigma,\,\beta\cup\sigma}, & \Xi_{\beta\cup\sigma,\,\gamma\setminus\sigma} \\ 0 & E \end{bmatrix} \tag{2.21}$$

is nonsingular. Since $\Xi_{\beta\cup\sigma,\,\beta\cup\sigma}$ is nonsingular by assumptions, however, the nonsingularity of (2.21) is evident. (Note that not each point satisfying (2.20) belongs necessarily to C.) By the mentioned result from [13] the solutions of (2.20) belonging to C possibly do not exhaust all extreme points of C. In the remaining extreme points there exist indices i such that

$$\langle (\Xi_{\gamma,\beta\cup\gamma})^i, u\rangle < (b_\gamma)^i, \quad (u_\gamma)^i > 0.$$

These points, however, evidently lie in the convex hull of the solutions to (2.20) for all possible choices of σ. We denote the solutions of (2.20) by ${}^\sigma u$ to indicate the dependence on the choice of σ. It follows that

$$\begin{aligned} z_{L\cup I_0} \in \{q \,|\, & q_\alpha = 0,\ q_{\beta\cup\gamma} \in conv\{{}^\sigma u \,|\, \sigma \in \gamma\}, \\ & (\nabla_y F_{L,L}(\overline{x},\overline{y}))^{\mathrm{T}} q_L + (\nabla_y F_{\beta\cup\gamma,L}(\overline{x},\overline{y}))^{\mathrm{T}} q_{\beta\cup\gamma} = y^*_L\} \\ & = \{q \,|\, q_\alpha = 0,\ q_{L\cup\beta\cup\gamma} \in conv\{{}^\sigma v \,|\, \sigma \subset \gamma\}\}, \end{aligned}$$

where the vectors ${}^\sigma v$ are given by

$$(\nabla_y F_{L,\,L\cup\beta\cup\sigma}(\overline{x},\overline{y}))^{\mathrm{T}}({}^\sigma v_L) + (\nabla_y F_{\beta\cup\gamma,\,L\cup\beta\cup\sigma}(\overline{x},\overline{y}))^{\mathrm{T}}({}^\sigma v_{\beta\cup\gamma}) = y^*_{L\cup\beta\cup\sigma}$$
$${}^\sigma v_{\gamma\setminus\sigma} = 0.$$

We note that ${}^\sigma v = ({}^\sigma\tilde{p})_{I_0\setminus\alpha}$ for all $\sigma \subset \gamma$ and the proof is completed. □

On the basis of Proposition 2.5 we get now easily the following estimate for $D^*S(\overline{x},\overline{y})(y^*)$.

Corollary 2.5.1. Consider an arbitrary $y^* \in \mathbb{R}^m$ and let Λ be the corresponding set on the right-hand side of (2.12). Then one has

$$\Lambda \subset -(\nabla_x F_{L\cup I_0}(\overline{x},\overline{y}))^{\mathrm{T}} conv\{{}^\sigma\tilde{p} \,|\, \sigma \subset I_0(\overline{x},\overline{y})\}. \tag{2.22}$$

By [6], Theorem 3.1 the set on the right-hand side of (2.22) is an upper estimate of the transposed generalized Jacobian of S at $\overline{x}$ in the sense of Clarke, cf. [3]. Our estimate of $D^*S(\overline{x},\overline{y})(y^*)$ (i. e. Λ) is thus included in the upper estimate of the transposed generalized Jacobian from [6], which is very important for the "sharpness" of the resulting optimality conditions.

3. OPTIMALITY CONDITIONS

In [9] the 1st-order necessary optimality conditions for problem (1.5) have been proved in the following form:

Proposition 3.1. Let $\hat{x}$ be a local solution of (1.5) and Θ be lower semicontinuous in a neighborhood of $\hat{x}$. Then there exist an element $\hat{x}^* \in \mathbb{R}^n$ and a real $\lambda \geq 0$, not both equal to zero, such that

$$(\hat{x}^*, -\lambda) \in K_{\quad \Theta}(\hat{x}, \Theta(\hat{x})) \quad \text{and} \quad -\hat{x}^* \in K_\omega(\hat{x}). \tag{3.1}$$

Under the additional condition

$$\partial^\infty \Theta(\hat{x}) \cap (-K_\omega(\hat{x})) = \{0\} \tag{3.2}$$

one has $\lambda \neq 0$ and

$$0 \in \partial^- \Theta(\hat{x}) + K_\omega(\hat{x}). \tag{3.3}$$

In what follows $(\hat{x}, \hat{y})$ denotes a local solution of the MPEC (1.3) and we will consider the case, where

$$(\mathrm{A2})^* \begin{cases} \text{(i)} & \text{the matrix } \nabla_y F_{L,L}(\hat{x}, \hat{y}) \text{ is nonsingular, and} \\ \text{(ii)} & \text{the Schur complement of } \nabla_y F_{L,L}(\hat{x}, \hat{y}) \text{ in } R(\hat{x}, \hat{y}) \text{ is a } P\text{-matrix.} \end{cases}$$

By Proposition 2.2 assumption (A2)* is equivalent to the strong regularity of NCP (1.1) at $(\hat{x}, \hat{y})$. Under (A2)* consider, instead of Θ, the "localized" value function $\tilde{\Theta}[\mathcal{U} \to \mathbb{R}]$ defined by

$$\tilde{\Theta}(x) := \inf_{y \in S(x) \cap \mathcal{V}} f(x, y), \tag{3.4}$$

where $\mathcal{U}$, $\mathcal{V}$ are the neighborhoods from Proposition 2.2 (with $(\overline{x}, \overline{y})$ replaced by $(\hat{x}, \hat{y})$). It is clear that if $(\hat{x}, \hat{y})$ is a local solution of (1.3), then $\hat{x}$ is also a local solution of the mathematical program

$$\begin{array}{ll} \text{minimize} & \tilde{\Theta}(x) \\ \text{subject to} & x \in \omega \cap \mathcal{U}. \end{array} \tag{3.5}$$

Theorem 1.1 and Proposition 2.4 lead to the following result.

Theorem 3.2. Assume that the NCP (1.1) is strongly regular at $(\hat{x}, \hat{y})$, which is a local solution of (1.3). Then there exist an index set $\delta \subset I_0(\hat{x}, \hat{y})$, a vector $z_{L\cup\delta}$ and a pair $(x^*, y^*) \in \partial^- f(\hat{x}, \hat{y})$ such that

$$0 \in x^* - (\nabla_x F_{L\cup\delta}(\hat{x}, \hat{y}))^{\mathrm{T}} z_{L\cup\delta} + K_\omega(\hat{x}) \tag{3.6}$$

$$\left.\begin{array}{rcl} (\nabla_y F_{L\cup\delta, L}(\hat{x}, \hat{y}))^{\mathrm{T}} z_{L\cup\delta} & = & y^*_L \\ (\nabla_y F_{L\cup\delta, \delta}(\hat{x}, \hat{y}))^{\mathrm{T}} z_{L\cup\delta} & \leq & y^*_\delta \\ \left(y^{*i} - \langle \nabla_y F_{L\cup\delta, \{i\}}(\hat{x}, \hat{y}), z_{L\cup\delta} \rangle\right) z^i & \geq & 0 \quad \text{for } i \in \delta. \end{array}\right\} \tag{3.7}$$

Proof. By the assumptions for $x \in \mathcal{U}$ one has

$$\tilde{\Theta}(x) = f(x, \mu(x))$$

so that $\tilde{\Theta}$ is Lipschitz near $\hat{x}$. Therefore, $\partial^\infty \tilde{\Theta}(\hat{x}) = \{0\}$ and so, by Proposition 3.1,

$$0 \in \partial^- \tilde{\Theta}(\hat{x}) + K_\omega(\hat{x}). \tag{3.8}$$

Since the mapping $x \rightsquigarrow \{y \in S(x) \cap \mathcal{V} | \, f(x,y) = \tilde{\Theta}(x)\} = \mu(x)$ is single-valued and Lipschitz near $\hat{x}$, Theorem 1.1 implies that

$$\partial^- \tilde{\Theta}(\hat{x}) \subset \bigcup_{y^*} \{x_1^* + x^* | \, x_1^* \in D^*\mu(\hat{x})\,(y^*),\, (x^*, y^*) \in \partial^- f(\hat{x}, \hat{y})\}\,. \tag{3.9}$$

Furthermore, by Proposition 2.4, under (A2)* for all $y^* \in I\!R^m$ one has

$$\begin{aligned} D^*\mu(\hat{x})\,(y^*) = D^*S(\hat{x},\hat{y})(y^*) \subset \Big\{ & x_1^* \in I\!R^n \,|\, x_1^* = -(\nabla_x F_{L\cup I_0}(\hat{x},\hat{y}))^{\mathrm{T}} z_{L\cup I_0}, \\ & (R(\hat{x},\hat{y}))^{\mathrm{T}} z_{L\cup I_0} = (w + y^*)_{L\cup I_0}, \quad w_L = 0 \text{ and for } i \in I_0(\hat{x},\hat{y}) \\ & \text{either } w^i z^i = 0 \text{ or } w^i < 0 \text{ and } z^i > 0 \Big\}. \end{aligned} \tag{3.10}$$

From (3.8), (3.9) and (3.10) we obtain the existence of a pair $(x^*, y^*) \in \partial^- f(\hat{x}, \hat{y})$ and vectors $w,\, z \in I\!R^m$ such that

$$0 \in x^* - (\nabla_x \, F_{L\cup I_0}(\hat{x},\hat{y}))^{\mathrm{T}} \; z_{L\cup I_0} + K_\omega(\hat{x}) \tag{3.11}$$

$$\left.\begin{aligned} (\nabla_y \, F_{L\cup I_0, L}(\hat{x},\hat{y}))^{\mathrm{T}} \; z_{L\cup I_0} &= y_L^* \\ (\nabla_y \, F_{L\cup I_0, I_0}(\hat{x},\hat{y}))^{\mathrm{T}} \; z_{L\cup I_0} &= y_{I_0}^* + w_{I_0} \end{aligned}\right\} \tag{3.12}$$

and for $i \in I_0(\hat{x},\hat{y})$ either $w^i z^i = 0$ or $w^i < 0$ and $z^i > 0$. It remains to show that to (w, z) there exists an index set $\delta \subset I_0(\hat{x},\hat{y})$ such that conditions (3.7) (which do not contain w explicitly) are fulfilled.

Define

$$\delta := \{ i \in I_0(\hat{x},\hat{y}) | \, z^i \neq 0 \} \quad \text{and} \quad \delta_- := \{ i \in \delta | \, w^i < 0 \}\,.$$

Then in both equations (3.12) the columns of the left-hand-side matrices for $i \in I_0(\hat{x},\hat{y}) \setminus \delta$ may be omitted. If we now in the second equation (3.12) ignore the rows for $i \in I_0(\hat{x},\hat{y}) \setminus \delta$, we obtain

$$\begin{aligned} (\nabla_y \, F_{L\cup\delta, L}(\hat{x},\hat{y}))^{\mathrm{T}} z_{L\cup\delta} &= y_L^* \\ (\nabla_y \, F_{L\cup\delta, \delta}(\hat{x},\hat{y}))^{\mathrm{T}} z_{L\cup\delta} &\leq y_\delta^* \end{aligned}$$

because $w_\delta \leq 0$.

Moreover,

$$\begin{aligned} \big(y^{*i} - \langle \nabla_y \, F_{L\cup\delta,\{i\}}(\hat{x},\hat{y}),\, z_{L\cup\delta}\rangle\big)\, z^i &> 0 \quad \text{for} \quad i \in \delta_-, \quad \text{and} \\ \big(y^{*i} - \langle \nabla_y \, F_{L\cup\delta,\{i\}}(\hat{x},\hat{y}),\, z_{L\cup\delta}\rangle\big)\, z^i &= 0 \quad \text{for} \quad i \in \delta \setminus \delta_-, \end{aligned}$$

so that conditions (3.7) hold true. □

Remark. Let a vector $z \in \mathbb{R}^m$ be composed in such a way that its components z^i, $i \in L(\hat{y}) \cup \delta$ with $\delta \subset I_0(\hat{x},\hat{y})$, satisfy the conditions (3.7) and the remaining components are zero. Further consider a vector $w \in \mathbb{R}^m$ given by

$$w^i = \begin{cases} \langle \nabla_y F_{L\cup\delta,\{i\}}(\hat{x},\hat{y}),\, z_{L\cup\delta}\rangle - y^{*i} & \text{if } \; i \in L(\hat{y}) \cup I_0(\hat{x},\hat{y}) \\ \text{an arbitrary real otherwise.} \end{cases}$$

One easily deduces that the pair (w,z) satisfies (3.12) and for $i \in I_0(\hat{x},\hat{y})$ either $w^i z^i = 0$ or $w^i < 0$ and $z^i > 0$.

In [6] necessary optimality conditions have been derived for the MPEC (1.3) under the assumptions that the NCP (1.1) is strongly regular at its local solution $(\hat{x},\hat{y})$ and f is continuously differentiable near $(\hat{x},\hat{y})$.

In our notation the conditions of [6] read

$$0 \in -(\nabla_x F_{L\cup I_0}(\hat{x},\hat{y}))^{\mathrm{T}}\, conv\, \{{}^\sigma\tilde{p} \,|\, \sigma \subset I_0(\hat{x},\hat{y})\} + \nabla_x f(\hat{x},\hat{y}) + \tilde{N}_\omega(\hat{x}), \qquad (3.13)$$

where the vectors ${}^\sigma\tilde{p}$ are given by (2.16), (2.18) and $\tilde{N}_\omega(\hat{x})$ is the Clarke's normal cone to ω at $\hat{x}$. By Cor. 2.5.1 it is, however, immediately clear that if f is continuously differentiable and a pair $(\hat{x},\hat{y})$ fulfills the conditions (3.6), (3.7), then $(\hat{x},\hat{y})$ also fulfills the relation (3.13). Therefore, the conditions of Theorem 3.2 are definitely not less selective (weaker) than the optimality conditions from [6].

We conclude this section with a simple academic MPEC illustrating the conditions of Theorem 3.2.

Example 3.1. Consider the problem of the type (1.3)

$$\begin{array}{ll} \text{minimize} & \frac{3}{4}(y^1)^2 + \frac{1}{2}(y^2-1)^2 \\ \text{subject to} & \\ & 0 \in \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} y^1 \\ y^2 \end{bmatrix} + \begin{bmatrix} -3+x \\ x \end{bmatrix} + N_{\mathbb{R}^2_+}(y) \\ & x \in [0,1]. \end{array} \qquad (3.14)$$

One easily verifies that (3.14) fulfills the assumptions of Theorem 3.2 and $(\hat{x},\hat{y}) = \left(1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ is its (global) solution. One has

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \hat{y}^1 \\ \hat{y}^2 \end{bmatrix} + \begin{bmatrix} -3+\hat{x} \\ \hat{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and so $L(\hat{y}) = \{1\}$, $I_+(\hat{x},\hat{y}) = \emptyset$ and $I_0(\hat{x},\hat{y}) = \{2\}$. Further $\nabla_y f(\hat{x},\hat{y}) = \begin{bmatrix} \frac{3}{2}\hat{y}_1 \\ \hat{y}^2 - 1 \end{bmatrix}$ $= \begin{bmatrix} \frac{3}{2} \\ -1 \end{bmatrix}$. Let us choose $\delta = \{2\}$. Then the conditions (3.7) reduce to the system of equations and inequalities

$$\begin{aligned} 2z^1 - z^2 &= \tfrac{3}{2} \\ -z^1 + z^2 &\leq -1 \\ (-1 + z^1 - z^2)z^2 &\geq 0. \end{aligned} \qquad (3.15)$$

It is clear that only the point $\hat{z} = (\frac{1}{2}, -\frac{1}{2})$ is admissible with respect to (3.15). For $\delta = \emptyset$ conditions (3.7) give only the point $\tilde{z} = (\frac{3}{4}, 0)$. So in this example we have to test whether either $\hat{z}$ or $\tilde{z}$ fulfill relation (3.6) (which both of them do). When using the conditions from [6], we had to look for a suitable point among all points from the line segment $[\hat{z}, \tilde{z}]$. This might be generally (not in this example) much more demanding.

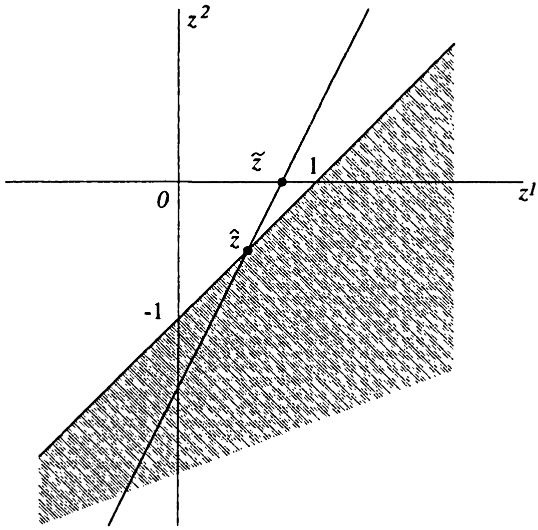

Fig. 2. The set $\Lambda = \{\hat{z}, \tilde{z}\}$ in Example 3.1.

Let us now replace the objective in (3.14) by

$$\frac{1}{2}\left(y^1 - \frac{4}{3}\right)^2 + \frac{1}{2}\left(y^2 + \frac{1}{3}\right)^2$$

and ω by the interval $[0, 2]$. Then the corresponding MPEC has the same solution, but $\nabla_y f(\hat{x}, \hat{y}) = \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$. Let us again choose $\delta = \{2\}$. Then the conditions (3.7) reduce to the system of equations and inequalities

$$\begin{aligned} 2z^1 - z^2 &= -\tfrac{1}{3} \\ -z^1 + z^2 &\le \tfrac{1}{3} \\ \left(\tfrac{1}{3} + z^1 - z^2\right) z^2 &\ge 0. \end{aligned} \tag{3.16}$$

For $\delta = \emptyset$ conditions (3.7) give only the point $\left(-\frac{1}{6}, 0\right)$, but the whole line segment, specified by the points $\left(-\frac{1}{6}, 0\right)$ and $\left(0, \frac{1}{3}\right)$, is admissible with respect to (3.16). The point $z^1 = -\frac{1}{9}$, $z^2 = \frac{1}{9}$ belongs to this line segment and one has

$$0 \in -[1, 1] \begin{bmatrix} -\frac{1}{9} \\ \frac{1}{9} \end{bmatrix}$$

so that relation (3.6) is satisfied. In this case the optimality conditions (3.6),(3.7) coincide with the conditions of [6].

We conclude by a brief comparison of the optimality condition in Theorem 3.2 with the corresponding conditions in [8] (Theorem 3.3.6). In our conditions we look at a local solution of (1.3) for one KKT vector z, whereas in [8], Theorem 3.3.6 a finite family of KKT vectors has to be computed in general. The verification of stationarity is, however, not always easier when using Theorem 3.2. It is also important to note that the stationarity in the Mordukhovich sense does not necessarily means that there does not exist a first-order descent direction.

4. CONCLUSION

It is a consequence of [17] that in the strongly regular case the equilibrium map S is piecewise continuously differentiable (PC^1) in a neighborhood of $(\hat{x}, \hat{y})$. For such maps it is not difficult to get an upper estimate of the generalized Jacobian, cf. [19], but it is, according to our opinion, a complicated task to compute the Mordukhovich's coderivative. In this paper we have computed an upper estimate of this coderivative indirectly, using [11], Theorem 6.10 and the characterization of the generalized normal cone to $Gph\, N_{\mathbb{R}^m_+}$, provided by Lemma 2.1. This way suggests how one could proceed in the case of equilibria, described by variational inequalities.

ACKNOWLEDGEMENT

The author thanks to B. Mordukhovich for numerous consultations and a persistent encouragement to this research.

(Received April 21, 1998.)

REFERENCES

[1] G. Anandalingam and T. Friesz (eds.): Hierarchical optimization. Ann. Oper. Res. *34* (1992).

[2] J.-P. Aubin and H. Frankowska: Set-Valued Analysis. Birkhäuser, Boston 1990.

[3] F. H. Clarke: Optimization and Nonsmooth Analysis. Wiley, New York 1983.

[4] S. Dempe: A necessary and sufficient optimality condition for bilevel programming problems. Optimization *25* (1992), 341–354.

[5] M. Kočvara and J. V. Outrata: On the solution of optimum design problems with variational inequalities. In: Recent Advances in Nonsmooth Optimization (D. Du, L. Qi and R. Womersley, eds.), World Scientific, Singapore 1995, pp. 172–192.

[6] M. Kočvara and J. V. Outrata: A nonsmooth approach to optimization problems with equilibrium constraints. In: Proc. of the ICCP (M. Ferris and J.-S. Pang, eds.), SIAM 1997, pp. 148–164.

[7] Z.-Q. Luo, J.-S. Pang, D. Ralph and S.-Q. Wu: Exact penalization and stationary conditions of mathematical programs with equilibrium constraints. Math. Programming *75* (1996), 19–76.

[8] Z.-Q. Luo, J.-S. Pang and D. Ralph: Mathematical Programs with Equilibrium Constraints. Cambridge University Press, Cambridge 1996.

[9] B. S. Mordukhovich: Approximation Methods in Problems of Optimization and Control. Nauka, Moscow 1988. (In Russian; English translation to appear in Wiley–Interscience).

[10] B. S. Mordukhovich: Sensitivity analysis in nonsmooth optimization. In: Theoretical Aspects of Industrial Design (D. A. Field, and V. Komkov, eds.), SIAM Publications, Philadelphia 1992, pp. 32–46.

[11] B. S. Mordukhovich: Generalized differential calculus for nonsmooth and set-valued mappings. J. Math. Anal. Appl. *183* (1994), 250–288.

[12] B. S. Mordukhovich: Lipschitzian stability of constraint systems and generalized equations. Nonlinear Analysis, Theory, Methods & Applications *22* (1994), 173–206.

[13] K. G. Murty: Linear Programming. Wiley, New York 1983.

[14] K. G. Murty: Linear Complementarity, Linear and Nonlinear Programming. Heldermann, Berlin 1988.

[15] J. V. Outrata: On optimization problems with variational inequality constraints. SIAM J. Optimization *4* (1994), 340–357.

[16] J. V. Outrata: Optimality conditions for a class of mathematical programs with equilibrium constraints, to appear.

[17] J.-S. Pang and D. Ralph: Piecewise smoothness, local invertibility, and parametric analysis of normal maps. Math. Oper. Res. *21* (1996), 401–426.

[18] S. M. Robinson: Strongly regular generalized equations. Math. Oper. Res. *5* (1980), 43–62.

[19] S. Scholtes: Introduction to Piecewise Differentiable Equations. Habil. Thesis, University of Karlsruhe, 1994.

[20] J. S. Treiman: General optimality conditions for bi-level optimization problems. Preprint.

[21] J. J. Ye and D. L. Zhu: Optimality conditions for bilevel programming problems. Optimization *33* (1995), 9–27.

[22] J. J. Ye, D. L. Zhu and Q. J. Zhu: Exact penalization and necessary optimality conditions for generalized bilevel programming problems. SIAM J. Optimization 7 (1997), 481–507.

[23] J. J. Ye and X. Y. Ye: Necessary optimality conditions for optimization problems with variational inequality constraints. Math. Oper. Res. *22* (1997), 977–997.

[24] R. Zhang: Problems of hierarchical optimization in finite dimensions. SIAM J. Optimization *4* (1994), 521–536.

Doc. Ing. Jiří V. Outrata, DrSc., Institute of Information Theory and Automation – Academy of Sciences of the Czech Republic, Pod vodárenskou věží 4, 182 08 Praha 8. Czech Republic.

e-mail: outrata@utia.cas.cz