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# DELAY-DEPENDENT ROBUST STABILITY CONDITIONS AND DECAY ESTIMATES FOR SYSTEMS WITH INPUT DELAYS

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The robust stabilization of uncertain systems with delays in the manipulated variables is considered in this paper. Sufficient conditions are derived that guarantee closed-loop stability under state-feedback control in the presence of nonlinear and/or time-varying perturbations. The stability conditions are given in terms of scalar inequalities and do not require the solution of Lyapunov or Riccati equations. Instead, induced norms and matrix measures are used to yield some easy to test robust stability criteria. The problem of constrained control is also discussed, and alternative stability tests for the case of saturation nonlinearities are presented. Estimates of the transient behavior of the controlled system are also obtained. Finally, an example illustrates the results.

## 1. INTRODUCTION

It has been known that engineering systems have dynamic behavior that is often significantly affected by time delays. The transport of reactants across membranes or the transmission of signals by the circulation of hormones, are examples of events that can induce a delayed outcome on the regulation of reaction paths in biochemical processes (Shell and Ross [16]). Other delays are imposed by process design demands as is the case of long transmission lines in hydraulic or electric networks (delayed feedback). The description of time-delay systems leads to differential-difference equations, the solutions of which require knowledge of past values of the system variables. The response of a system with a time delay can be quite complex. For example studies of isothermal reactions indicate that delayed feedback may stabilize unstable stationary states, or destabilize an otherwise stable steady state (Inamdar et al [10]). It is then evident that the existence of time delays may cause major difficulties in the design and implementation of control that may result in significant performance deterioration.

Several methods have been developed to determine the stability of delay systems. The most common techniques were to use Lyapunov theory or to analyze the delay system from an algebraic point of view (Bourlès [2]; Hale and Lunel [7] and the list of references therein; Bourlès and Kosmidou [3]). While both techniques provide a powerful theoretical framework for analysis there are some disadvantages: 1. general systematic procedures to construct appropriate Lyapunov functions are not available yet, 2. solving the resulting Lyapunov Matrix or Riccati equations can be troublesome and often nontrivial and 3. the results are sometimes difficult to verify. Another approach, makes use of differential inequalities and the stability conditions are given in terms of matrix measure. These techniques have some design features, and have been used to analyze ordinary (Hrissagis and Crisalle [8]) as well as state-delayed systems (Mori [15]). The use of matrix measures in the analysis of delay equations has the advantages of being both an algorithmic procedure and computationally simple (Mori [15]; Hrissagis and Crisalle [9]; Vidyasagar [18]). Two kinds of criteria exist: conditions that are independent of the size of time delay, and delay-dependent stability criteria (Chen et al [4]; Hale and Lunel [7]).

In the present work differential inequalities and matrix measures are used to derive simple stability conditions for uncertain systems with input delays. Some of the first approaches that concern the stabilization of *exactly known* input delay systems include the work of Kwon and Pearson [13], Shen and Kung [17], and more recently the $H_\infty$ approach of Lee et al [14] and Choi and Chung [5]. However, most of these works do not consider the commonly occuring problem of input saturation. In practical systems the actuators have physical limitations that may cause saturation in the course of operation. If such nonlinearities are not taken into account during control design, integral wind-up or limit cycles may occur (Krikelis and Barkas [12]). The stability of linear systems with saturating actuators has been studied extensively (Bernstein and Michel [1]); however, there are few results available on the *robust stabilization* of state-delay systems with saturation nonlinearities and none to our knowledge for input-delay systems.

In this article uncertain input delay models are employed where nonlinearities appear in two different terms: the first term includes perturbations, and the second term represents saturation-type nonlinearities in the input channel. The system uncertainty: a) may be time-varying and possibly nonlinear b) no statistical characterization of uncertainty is needed, and c) only the bounds of a "magnitude" of the uncertainty are assumed available. Also the present method provides estimates on the controlled system's transient behavior by examining the decay-rate of its stable solutions.

## 2. MAIN RESULTS - UNCONSTRAINED CONTROL

It is assumed in this section that no input-saturation constraints are present. In what follows the concept of the matrix measure is defined and the most important properties of it are given. For a more extensive discussion on matrix norms and measures the reader is referred to Vidyasagar [18].

**Definition 1.** The matrix measure is a function $\mu : \mathbf{R}^{n\times n} \to \mathbf{R}$

$$\mu(A) = \lim_{\epsilon \to 0+} \frac{\|I + \epsilon A\| - 1}{\epsilon} \tag{1}$$

where $\|\cdot\|$ is an induced matrix-norm on $\boldsymbol{R}^{n\times n}$. The matrix measure is also known in the litterature as the logarithmic norm. Some of the properties that are relevant to this work are listed below:

i) $\mu(\cdot)$ is a convex function

ii) $\mu(\delta A) = \delta\mu(A)$, and

iii) $Re\,\lambda(A) \le \mu(A)$, where $\lambda$ is any eigenvalue of matrix $A$.

It should be noted here that for a Hurwitz matrix $A$: $\mu(A) < 0$. Consider now the class of uncertain dynamic systems with a single time-delay in the input variables represented by the equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + B_d u(t-h) + g(x(t),t) \quad (2)\\ x(\theta) &= \phi(\theta)\\ y(t) &= Cx(t) \quad (3) \end{aligned}$$

where $\theta \in [-h, 0]$; $x(t) \in \boldsymbol{R}^n$ is the state vector with initial state $x(0) = x_0$; $u(t) \in \boldsymbol{R}^m$ is the input vector; $y(t) \in \boldsymbol{R}^p$ is the output vector; $A$, $B$, $B_d$, and $C$ are constant matrices of appropriate dimensions; $\phi(t)$ is a continuous vector-valued initial function and $h > 0$ is the input delay. The vector function $g(x(t),t) \in \boldsymbol{R}^n$ represents nonlinear modeling perturbations that depend on the state $x(t)$. No statistical information is required for the uncertainty $g$; it is only assumed that

$$\|g(x(t),t)\| \le k\|x(t)\| \quad (4)$$

where $k$ is *a priori* known positive-real constant.

A memoryless state-feedback control law of the form

$$u(t) = Fx(t) \quad (5)$$

where $F$ is a constant matrix, is used. The objective is to *find conditions that the feedback $F$ must satisfy in order to asymptotically stabilize the closed-loop* (2), (3) *and* (5) *for all modeling uncertainties consistent with* (4). Any matrix $F$ that stabilizes the uncertain delay system is said to be *robustly stabilizing*.

**Theorem 1.** Suppose that the plant uncertainty satisfies condition (4) and the following inequality holds

$$-\mu(\bar{A} + \bar{B}_d) - k - hM > 0 \quad (6)$$

where $\bar{A} = A + BF$, $\bar{B}_d = B_d F$ and $M$ is a scalar given by $M = \|\bar{B}_d\bar{A}\| + k\|\bar{B}_d\| + \|\bar{B}_d\bar{B}_d\|$. Then the uncertain time delayed system (2)–(3) is asymptotically stable using state-feedback control given by (5).

The proof of the theorem is given in the Appendix. When the time delay $h$ is uncertain, Theorem 1 can be restated to find an upper bound on $h$: Let the feedback

(5) be implemented where $F$ is a known matrix. Then the closed-loop system (2)-(3) is asymptotically stable if the input delay $h$ is bounded by

$$0 < h < \bar{h} = \frac{-\mu(\bar{A}+\bar{B}_d) - k}{M} \tag{7}$$

which is simply a rearrangement of inequality (6). Some important remarks concerning the derived stability conditions are in order:

**Remark 1.** An advantage of the present method is that it also gives information about the transient response of the system by examining the decay rate of the solution of (2). As shown in the Appendix, $\|x(t)\| \leq Ke^{-\gamma(t-h)}$ where $\gamma$ is the decay rate and can always be found for stable closed-loop systems. To this end, define a function of $\gamma$ as

$$f(\gamma) = \mu(\bar{A}+\bar{B}_d) + k + hMe^{\gamma h} + \gamma.$$

Since $f(\gamma)$ is monotone increasing with $f(+\infty) = +\infty$ and $f(0) = \mu(\bar{A}+\bar{B}_d) + k + hM < 0$ from (6), then there exists a unique $\gamma_0$ such that $f(\gamma_0) = 0$ or

$$\gamma_0 = -\mu(\bar{A}+\bar{B}_d) - k - hMe^{\gamma_0 h}$$

thus the decay rate can be obtained by solving a simple transcendental equation.

**Remark 2.** The tightness of the upper bound in (7) varies with the chosen norm and the corresponding matrix measure (Vidyasagar [18]). It is possible to determine stability with a given norm and matrix measure while with other choices no conclusions can be drawn. The largest bound computed for the usual 1, 2, or infinity norms should be selected.

**Remark 3.** When checking the asymptotic stability of a given delay system one should try the 1 or infinity vector norms first, dispensing the more involved singular value computations associated with the 2-norm. The choice of a suitable norm and matrix measure aiming to improve the stability bounds resembles that of constructing an appropriate Lyapunov function candidate in a widely used Lyapunov approach for determining stability.

**Remark 4.** For the nominal case, that is when the uncertainty is negligible (i.e. $g$ is zero) and also $h = 0$, inequality (6) of Theorem 1 reduces to $\mu(\bar{A}+\bar{B}_d) < 0$ which implies that $\bar{A}+\bar{B}_d$ is asymptotically stable, since $Re\,\lambda(\bar{A}+\bar{B}_d) \leq \mu(\bar{A}+\bar{B}_d) < 0$. When $h \neq 0$, condition (6) simply means that $\bar{A}+\bar{B}_d$ should be stable enough to overcome the difficulty posed by the time delay in the system. It is thus evident that time delay can destabilize an otherwise stable closed-loop.

**Remark 5.** It has been known (Mori [15]) that delay independent criteria are conservative due to lack of information on the delay, especially when delays are small. It is then reasonable when checking the stability of a specific uncertain delay system to start with delay independent criteria and if those fail to turn to delay-dependent ones.

**Remark 6.** Besides the well known 1, 2, and infinity norms, other induced norms and matrix measures involving weighting parameters may be utilized in the stability conditions. As an example consider the following matrix norm and corresponding measure:

$$\begin{aligned} \|A\|_w &= \max_i \sum_j \frac{w_j}{w_i} |a_{ij}| \\ \mu_w(A) &= \max_i \left\{ a_{ij} + \sum_{i \neq j} \frac{w_j}{w_i} |a_{ij}| \right\}. \end{aligned}$$

Other weighted norms can be defined similarly. A simple optimization problem with respect to the arbitrary weighting factors is likely to yield less conservative robust stability bounds. This is a topic that merits further investigation and is not pursued here.

## 3. ROBUST STABILITY IN THE PRESENCE OF INPUT NONLINEARITIES

In this section a stability analysis is given for time delay systems affected by nonlinearities in the input channel. Prior to the discussion of robust stability some useful concepts are depicted.

**Definition 2.** For a continuous nonlinear mapping $N : \boldsymbol{R}^m \rightarrow \boldsymbol{R}^m$, and for two real numbers $p$ and $q$ such that $1 \geq q > p \geq 0$, $N$ is said to lie inside the sector $[p, q]$ if $N$ satisfies the following two properties: i) $N(0) = 0$ and ii)

$$\left\| N(u(t)) - \frac{p+q}{2} u(t) \right\| \leq \frac{q-p}{2} \|u(t)\| \tag{8}$$

where $(p+q)/2$ is the center of the sector and $(q-p)/2$ is its radius. This means that the graph of the nonlinearity lies between two straight lines that pass through the origin with slopes $p$ and $q$, respectively.

A general saturating actuator function is defined by

$$N(u(t)) = [N(u_1(t)),\ N(u_2(t)), \ldots, N(u_m(t))]^T \tag{9}$$

where the operation range of the nonlinear actuator $N(u_i(t))$ is considered to be inside the sector $[p, q]$ and saturates at $\underline{u}_i$ and $\overline{u}_i$ which represent lower and upper saturation limits, respectively.

In the presence of a nonlinear saturating actuator the uncertain time delay system is rendered by the equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + BN(u(t)) + B_d N(u(t-h)) + g(x(t), t) \\ x(\theta) &= \phi(\theta) \\ y(t) &= Cx(t) \end{aligned} \tag{10}$$

where $\theta \in [-h, 0]$ and where all relevant quantities are as previously defined in Section 2. The following theorem provides sufficient conditions for the asymptotic stability of uncertain saturating systems with input delays.

**Theorem 2.** Suppose that the plant uncertainties satisfy condition (4) and the following inequality holds:

$$-\mu(\bar{A} + \bar{B}_d) - \frac{q-p}{2}(\|BF\| + \|B_dF\|) - k - hM > 0 \tag{11}$$

where $\bar{A} = A + \frac{p+q}{2}BF$, $\bar{B}_d = \frac{p+q}{2}B_dF$ and the scalar $M$ is $M = \|\bar{B}_d\bar{A}\| + \|\bar{B}_d\|[\frac{q-p}{2}(\|BF\| + \|B_dF\|) + k] + \|\bar{B}_d\bar{B}_d\|$. Then the uncertain input delayed saturating system (10) is asymptotically stable using the feedback law (5).

The proof of Theorem 2 is sketched in the Appendix. When the time delay $h$ is not exactly known, Theorem 2 can be stated in the following way which yields an upper bound on $h$: Let the feedback (5) be implemented, where $F$ is a known matrix; then the closed-loop system (10), (5) is asymptotically stable if the delay $h$ is bounded by

$$0 < h < \bar{h} := \frac{-\mu(\bar{A} + \bar{B}_d) - \frac{q-p}{2}(\|BF\| + \|B_dF\|) - k}{M}. \tag{12}$$

The conditions of Theorem 2 can readily be specialized to the usual case of the standard saturation function, using $[p, q] = [0, 1]$ with inequalities (11) or (12) that remain applicable.

## 4. DESIGN PROCEDURE

The previous analysis is used as a guide here to propose an iterative procedure for selecting a matrix $F$ to satisfy the robust stability conditions.

*Step 1.* Given the norm bounds of the plant uncertainty, select distinct negative eigenvalues $\lambda_i$, $i = 1, \ldots, n$ for the matrix $\bar{A}$.

*Step 2.* When the input nonlinearities are known to lie in sector $[p, q]$ find the control matrix $F$ using a standard pole-placement technique. Check whether inequality (11) is satisfied. If so stop; a robust matrix $F$ has been obtained. Otherwise continue to Step 3.

*Step 3.* Shift the system eigenvalues to the left using $\lambda_i = \lambda_i - \Delta\lambda_i$, $i = 1, \ldots, n$, where $\Delta\lambda_i > 0$; then go back to Step 2.

From the inverse point of view, one may estimate the sector where the input nonlinearities must lie so that the system with a given structure remains asymptotically stable. In this case the first step of the procedure remains the same. In Step 2, condition (11) should be checked as if the nonlinearities were not present. If inequality (11) is satisfied go to Step 4. Otherwise continue with Step 3.

**Step 4.** From inequality (11) and the equality $\frac{p+q}{2} = 1$ chose variables $p$ and $q$ such that the input nonlinearities lie in the sector $[p, q]$. The uncertain time delay feedback system is then guaranteed to be stable.

**Remark 7.** As pointed out earlier, the present work can accomodate uncertain time delays. If this is the case inequality (12) should be used instead of (11) in Step 2.

**Remark 8.** In Step 2 pole-placement is proposed as a method to find the robust state-feedback matrix $F$. Other techniques that include eigenstructure assignment, receding horizon, or output feedback may be used in the present framework to derive robust stability conditions for uncertain systems with input delays. This is currently under investigation.

**Example.** This section demonstrates the applicability of the robustness conditions to the stabilization of an uncertain input delay system defined as in (10) with dynamics described by the matrices

$$A = \begin{bmatrix} -1.6 & 0.25 \\ 0.25 & 0.15 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.15 \\ 1.2 \end{bmatrix}$$

$$B_d = \begin{bmatrix} 0.25 \\ 0.5 \end{bmatrix}$$

and nonlinear uncertainty bounded as in (4) where $k = 0.25$. Let the operational range of the saturating actuator lie in the sector $[0.3, 0.8]$ and therefore use inequality (11) as prescribed by Theorem 2. The input variable has upper and lower saturation limits given by $||u|| \leq 1$. The time delay $h$ is not exactly known and an upper bound is sought for it, as discussed in Section 4. Notice that the open-loop system is unstable since matrix $A$ has one positive eigenvalue.

With a standard pole-placement technique (i.e. using MATLAB), choose the eigenvalues of $\bar{A}$ to be $\{-1.63, -0.89\}$, and find the feedback matrix $F = -0.9 * [0.2\ 1.8]$. Then condition (12) gives the following upper bounds on the decay $h$:

(i) for the 1-norm $\bar{h} = 0.095$,

(ii) for the 2-norm $\bar{h} = 0.335$, and for the $\infty$-norm $\bar{h} = 0.23$. Therefore according to Theorem 2 when the delay $h$ is smaller than 0.335 the saturating system is guaranteed to be stable.

Suppose now that the input nonlinearities lie in the sector $[0.3, 0.8]$ and the time delay is known to be $h = 0.1$. Using the same feedback matrix, for the 2 and infinity norms and matrix measures we find that condition (11) is satisfied – which implies stability of the system – while using the 1-norm inequality (11) is violated; therefore no conclusion can be drawn about the stability of the system (cf., Remark 2).

If there were no saturation nonlinearities in the input channel, proceeding in a similar manner, a matrix $F = -0.49 * [0.2\ 1.8]$ is found to give the same spectrum of $\bar{A}$ as above. Further, use of (6) yields (i) $\bar{h} = 0.94$ for the 1-norm, (ii) $\bar{h} = 1.38$ for the 2-norm, and (iii) $\bar{h} = 1.33$ for the infinity norm. Therefore according to Theorem 1, for input delays $h$ smaller than 1.38 the asymptotic stability of the system is established. The larger delay bounds found for the nonsaturating system is a further testimony to the observation that time-delay with saturation nonlinearities may destabilize the closed-loop system.

## 5. CONCLUSIONS

Easily calculable tests for robust stabilization of input delay systems via state-feedback are derived. The conditions are expressed in terms of succinct scalar inequalities of matrix norms and measures and do not require the solution of Riccati or Lyapunov equations. Additionally, the decay rate of stable solutions of the system, can be assessed by solving a transcendental equation. A central requirement is that the logarithmic norm of the closed-loop matrix can be made negative enough by appropriate feedback to overcome the effect of perturbations and time delays. A trial-and-error procedure is proposed to find the feedback matrix and an example illustrates the simplicity of the derived conditions, showing that the method is not restricted to systems with strictly Hurwitz matrix $A$.

## APPENDIX

**Lemma 1.** Let a scalar function $f(t)$ satisfy the inequality $\dot{f}(t) \leq -\alpha f(t) + \beta \sup_{t-h \leq s \leq t} f(s)$, $t \geq t_0$, where $\alpha, \beta$ are real constants such that $\alpha > \beta \geq 0$. Then there exist scalars $\gamma > 0$ and $K > 0$ such that $f(t) \leq K\exp(-\gamma(t - t_0))$ for $t \geq t_0$ (Kolmanovskii and Nosov [11]).

*Proof of Theorem 1.* Consider $t_0 = 0$ and let $x(t)$ be the solution of (2) for $t \geq 0$. Since $x(t)$ is continuously differentiable for $t \geq 0$, write

$$x(t)-x(t-h) = \int_{t-h}^{t} \dot{x}(s)\,\mathrm{d}s = \int_{t-h}^{t} [Ax(s)+Bu(s)+B_d u(t-h)+g(x(s),s)]\,\mathrm{d}s. \quad (13)$$

Now, define $\bar{B}_d = B_d F$ and substitute for $u(t-h)$ in (2) using (5) and (13) to obtain

$$\dot{x}(t) = (\bar{A}+\bar{B}_d)x(t) + g(x(t),t) - \bar{B}_d \int_{t-h}^{t} \{\bar{A}x(s) + \bar{B}_d x(s-h) + g(x(s),s)\}\,\mathrm{d}s \quad (14)$$

where matrix $\bar{A}$ is defined as $\bar{A} = A + BF$. The solution to (14) for $t \geq 0$ may be expressed as the integral equation (through the variation-of-parameters formula)

$$x(t) \;=\; e^{(\bar{A}+\bar{B}_d)t}x(0) + \int_0^t e^{(\bar{A}+\bar{B}_d)(t-s)}\,\{(-\bar{B}_d) \quad (15)$$

$$\cdot \int_{t-h} [\bar{A}x(\theta) + \bar{B}_d x(\theta - h) + (g(x(\theta), \theta))] \, d\theta + g(x(s), s) \Big\} \, ds.$$

An upper bound on the norm of the solution of (15) is found after taking the norm of both sides in (15), using known norm properties, inequality (6) and $\|e^{At}\| \leq e^{\mu(A)t}$, $t \geq 0$ (Vidyasagar [18])

$$\|x(t)\| \leq \sup_{-h \leq t \leq h} \|x(t)\| e^{\mu(\bar{A}+\bar{B}_d)t} + \int_0^t e^{\mu(\bar{A}+\bar{B}_d)(t-s)} \left\{ \int_{t-h}^t [\|\bar{B}_d\bar{A}\| \|x(\theta)\| \right.$$

$$\left. + \|\bar{B}_d\bar{B}_d\| \|x(\theta - h)\| + \|\bar{B}_d\| \, \|g(x(\theta), \theta)\|] \, d\theta + \|g(x(s), s)\| \right\} ds \tag{16}$$

Now use the plant uncertainty bounds (9) in (16) and define $\chi := \sup_{-h \leq t \leq h} \|x(t)\|$ to obtain

$$\|x(t)\| \leq \chi e^{\mu(\bar{A}+\bar{B}_d)t} + \int_0^t e^{\mu(\bar{A}+\bar{B}_d)(t-s)} \tag{17}$$

$$\left\{ \int_{t-h}^t [(\|\bar{B}_d\bar{A}\| + k\|\bar{B}_d\|)\|x(\theta)\| + \|\bar{B}_d\bar{B}_d\| \|x(\theta - h)\|] \, d\theta + k\|x(s)\| \right\} ds.$$

After carrying out the inside integration, inequality (17) becomes

$$\|x(t)\| \leq \chi e^{\mu(\bar{A}+\bar{B}_d)t} + \int_0^t e^{\mu(\bar{A}+\bar{B}_d)(t-s)} \left\{ hM \sup_{s-2h \leq \theta \leq s} \|x(\theta)\| + k\|x(s)\| \right\} ds \tag{18}$$

where $M = \|\bar{B}_d\bar{A}\| + k\|\bar{B}_d\| + \|\bar{B}_d\bar{B}_d\|$. Let $z(t) \in \boldsymbol{R}$ be a trajectory such that

$$z(t) = \chi e^{\mu(\bar{A}+\bar{B}_d)t} + \int_0^t e^{\mu(\bar{A}+\bar{B}_d)(t-s)} \left\{ hM \sup_{s-2h \leq \theta \leq s} \|x(\theta)\| + k\|x(s)\| \right\} ds. \tag{19}$$

Then

$$\dot{z}(t) = \mu(\bar{A} + \bar{B}_d)z(t) + k\|x(t)\| + hM \sup_{t-2h \leq \theta \leq t} \|x(\theta)\| \tag{20}$$

From equations (18) and (19) it is obvious that $\|x(t)\| \leq z(t)$ for $t \geq 0$ (cf. also the comparison theorem, Kolmanovskii and Nosov [11]). Hence

$$\begin{aligned} \sup_{t-2h \leq \theta \leq t} \|x(\theta)\| &\leq \sup_{t-2h \leq \theta \leq t} z(\theta) \\ \|x(t-h)\| \leq z(t-h) &\leq \sup_{t-2h \leq \theta \leq t} z(\theta). \end{aligned} \tag{21}$$

After substituting (21) in (20) the following differential inequality holds:

$$\dot{z}(t) \leq -(-\mu(\bar{A} + \bar{B}_d) - k)z(t) + hM \sup_{t-2h \leq \theta \leq t} z(\theta). \tag{22}$$

Finally, invoking Lemma 1 one has $z(t) \leq Ke^{-\gamma(t-h)}$, i.e. $z(t)$ and similarly $\|x(t)\|$ is asymptotically stable if

$$(-\mu(\bar{A} + \bar{B}_d) - k) > hM > 0. \tag{23}$$

Exploiting Lemma 1, it is easy to show (Halanay [6]) that the decay rate $\gamma$ satisfies the transcendental equation

$$\gamma = -\mu(\bar{A} + \bar{B}_d) - k - hMe^{\gamma h} \tag{24}$$

and is a measure of how fast $x(t)$ converges to zero. Note that the decay rate $\gamma$ depends on the magnitude of the input delay $h$, i.e. it is also delay-dependent like the derived stability conditions.

Proof of Theorem 2. Adding and subtracting the terms $\frac{p+q}{2}Bu(t)$ and $\frac{p+q}{2}B_d u(t-h)$ system (10) takes the form

$$\begin{aligned} \dot{x}(t) &= \bar{A}x(t) + B[N(u(t)) - \frac{p+q}{2}u(t)] + \frac{p+q}{2}B_d u(t-h) \\ &\quad + B_d[N(u(t-h)) - \frac{p+q}{2}u(t-h)] + g(x(t),t) \end{aligned} \tag{25}$$

where matrix $\bar{A}$ is defined as $\bar{A} = A + \frac{p+q}{2}BF$. Denote $\bar{B}_d = \frac{p+q}{2}B_d F$ and let $x(t)$ be the solution of (25) for $t \geq 0$. Taking now similar steps as in the proof of Theorem 1 it can be readily shown that system (10) is asymptotically stable if

$$-\mu(\bar{A} + \bar{B}_d) - \frac{q-p}{2}||BF|| - k > hM + \frac{q-p}{2}||B_d F|| > 0 \tag{26}$$

where the scalar quantity $M$ is given by

$$M = ||\bar{B}_d \bar{A}|| + ||\bar{B}_d||[\frac{q-p}{2}(||BF|| + ||B_d F||) + k] + ||\bar{B}_d \bar{B}_d||. \tag{27}$$

## Matrix Measure Computation

For the usual 1,2 and infinity induced norms the matrix measure is given by the simple formulas below. The induced norms are also included for completeness.

$$\begin{aligned} ||A||_\infty &= \max_i \sum_j |\alpha_{ij}|, & \mu_\infty(A) &= \max_i(\alpha_{ii} + \sum_{j \neq i} |\alpha_{ij}|) \\ ||A||_1 &= \max_j \sum_i |\alpha_{ij}|, & \mu_1(A) &= \max_j(\alpha_{jj} + \sum_{i \neq j} |\alpha_{ij}|) \\ &\lambda_{\max}(A^T A)^{1/2}, & \mu_2(A) &= \lambda_{\max}(\frac{A^T + A}{2}) \end{aligned}$$

where $A^T$ is the transpose of matrix $A$, and $\lambda_{\max}$ denotes the maximum eigenvalue.

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