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THE TRACKING AND REGULATION PROBLEM FOR A CLASS OF GENERALIZED SYSTEMS[1]

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The tracking and regulation problem is considered for a class of generalized systems, in case of exponential reference signals and of disturbance functions. First, the notions of steady-state response and of blocking zero, which are classical for linear time-invariant systems, are given for generalized systems. Then, the tracking and regulation problem is stated and solved for the class of generalized systems under consideration, giving a general design procedure. As a corollary of the effectiveness proof of the design procedure, an algebraic version of the internal model principle is stated for generalized systems.

1. INTRODUCTION

Generalized systems are largely studied in the existing literature because of the large number of physical processes that can be modeled by a set of coupled differential and algebraic equations [23] (in case of continuous-time processes) or by a set of coupled difference and algebraic equations [6] (in case of discrete-time processes).

The generalized systems [1, 3, 10, 28] are also called singular systems [2, 7, 8, 11, 12, 19, 21, 22, 24, 26, 30], descriptor systems [9, 15, 16, 17, 18, 20, 27, 29], or implicit systems [4, 5].

Several control problems have been solved for generalized systems: analysis of the controllability and observability properties [1, 23, 29], pole assignment [1, 23, 29], eigenstructure assignment [15, 22], observer design [12, 24, 27], and disturbance decoupling [5]. For a complete survey, the reader is referred to [19].

Purpose of this paper is to study in a general framework the tracking and regulation problem for a class of generalized systems, in case of exponential reference signals and disturbance functions.

The outline of the paper is as follows. Section 2 is devoted to the analysis of generalized systems: Subsection 2.1 contains some background material, while Subsection 2.2 extends in case of generalized systems some notions classical for linear time-invariant systems, such as the notions of steady-state response and of blocking zero. In Section 3, the tracking and regulation problem is stated and solved for a

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class of generalized systems, in case of exponential reference signals and disturbance functions; a general design procedure is given and tested by a simple example. Section 4 proves that the proposed design procedure can be actually completed under some necessary and sufficient conditions and that the compensator thus obtained is actually a solution of the tracking and regulation problem under consideration; the latter proof is done by giving an algebraic version of the internal model principle for generalized systems.

2. ANALYSIS OF GENERALIZED SYSTEMS

Basic definitions, and classical and preliminary results are given with respect to generalized systems described by the following equations:

$$E\,\dot{x}(t) = A\,x(t) + B\,u(t) + M\,d(t), \quad t \in \mathbb{R},\ t \geq 0, \tag{1a}$$

$$y(t) = C\,x(t) + D\,u(t) + N\,d(t), \quad t \in \mathbb{R},\ t \geq 0, \tag{1b}$$

where $x(t) \in \mathbb{R}^n$ is the pseudo-state, $u(t) \in \mathbb{R}^p$ is the control input, $d(t) \in \mathbb{R}^m$ is the disturbance input, $y(t) \in \mathbb{R}^q$ is the measured output, E, A, B, C, D, M, N are real matrices of proper dimensions.

The following Assumption 1 will be supposed to hold throughout this section, without mentioning it explicitly.

Assumption 1. The characteristic polynomial of (1a),

$$p_{E,A}(\lambda) := \det(A - \lambda\,E), \quad \lambda \in \mathbb{C}, \tag{2}$$

is different from 0 for some complex λ, and his degree ν as polynomial function of λ is greater than zero.

2.1. Background material

The following basic definitions and classical results are briefly recalled.

(i) The roots λ_i, $i = 1, 2, \ldots, \nu$, of $p_{E,A}(\lambda) = 0$ are called the *eigenvalues* (at finite) of (1a).

(ii) If, for some $\gamma \in \mathbb{R}$, $\gamma < 0$, all the eigenvalues λ_i of (1a) have real part $\mathrm{re}(\lambda_i) < \gamma$, then the output and pseudo-state free responses of (1) exponentially go to zero faster than $e^{\gamma t}$.

(iii) System (1) has an impulsive pseudo-state free response $x(t) = v\,\delta(t)$, with $v \in \mathbb{R}^n$, $v \neq 0$, and $\delta(t)$ being the Dirac function, if there exists an initial condition $x(0^-) = x_0$, $x_0 \neq 0$, such that

$$E\,v = 0, \tag{3a}$$

$$A\,v + E\,x_0 = 0. \tag{3b}$$

(iv) System (1) has no impulsive pseudo-state free response (briefly, it is *impulse-free*) if and only if the following relation holds:

$$\operatorname{rank}\left(\begin{bmatrix} E & 0 \\ A & E \end{bmatrix}\right) = n + \operatorname{rank}(E), \tag{4}$$

where n is the dimension of A.

(v) For some $\gamma \in \mathbb{R}$, $\gamma < 0$, system (1) is *γ-stabilisable* if there exists a real matrix $K \in \mathbb{R}^{p \times n}$ such that all the roots $\widehat{\lambda}_i$ of

$$\det(A + BK - \widehat{\lambda} E) = 0$$

have real part $\operatorname{re}(\widehat{\lambda}_i) < \gamma$. For some $\gamma \in \mathbb{R}$, $\gamma < 0$, system (1) is γ-stabilisable if and only if the following condition

$$\operatorname{rank}([A - \lambda E \quad B]) = n \tag{5}$$

holds for all the eigenvalues $\lambda = \lambda_i$ of (1a) having real part $\operatorname{re}(\lambda_i) \geq \gamma$.

(vi) For some $\gamma \in \mathbb{R}$, $\gamma < 0$, system (1) is *γ-detectable* if there exists a real matrix $H \in \mathbb{R}^{n \times q}$ such that all the roots $\widetilde{\lambda}_i$ of

$$\det(A + HC - \widetilde{\lambda} E) = 0$$

have real part $\operatorname{re}(\widetilde{\lambda}_i) < \gamma$. For some $\gamma \in \mathbb{R}$, $\gamma < 0$, system (1) is γ-detectable if and only if the following condition

$$\operatorname{rank}\left(\begin{bmatrix} A - \lambda E \\ C \end{bmatrix}\right) = n \tag{6}$$

holds for all the eigenvalues $\lambda = \lambda_i$ of (1a) having real part $\operatorname{re}(\lambda_i) \geq \gamma$.

2.2. Preliminary results

The following assumption will be useful in the remainder of this section.

Assumption 2. The complex scalar α is such that $p_{E,A}(\alpha) \neq 0$, i.e. α is not eigenvalue of (1a).

The following definitions and lemmas allow the notions of blocking zero and of steady-state response to be introduced for generalized systems.

Definition 1. Under Assumption 2, let system (1) be subject to the following input functions

$$u(t) = w_u \, e^{\alpha t}, \quad t \in \mathbb{R}, t \geq 0, \tag{7a}$$

$$d(t) = w_d \, e^{\alpha t}, \quad t \in \mathbb{R}, t \geq 0, \tag{7b}$$

where $w_u \in \mathbb{C}^p$, $w_d \in \mathbb{C}^m$; if the vector function

$$x_{ss}(t) = z\,e^{\alpha t}, \quad t \in \mathbb{R}, t \geq 0, \tag{8}$$

where $z \in \mathbb{C}^n$, is solution of (1a),(7), then (8) is termed a *pseudo-steady-state* response in the pseudo-state of (1) to the input functions (7); the corresponding output function $y_{ss}(t)$ is called a *pseudo-steady-state* response in the output of (1) to the input functions (7).

The proof of the following Lemma 1 can be easily derived by virtue of the non-singularity of matrix $[\alpha E - A]$, which is yielded by Assumption 2.

Lemma 1. Under Assumption 2, for each $w_u \in \mathbb{C}^p$, $w_d \in \mathbb{C}^m$, the pseudo-steady-state response in the pseudo-state of (1) to the input functions (7) exists and is uniquely determined by (8) with

$$z = [\alpha E - A]^{-1}(B\,w_u + M\,w_d). \tag{9}$$

By Lemma 1, under Assumption 2, for each $w_u \in \mathbb{C}^p$, $w_d \in \mathbb{C}^m$, the pseudo-steady-state response in the output of (1) to the input functions (7) exists and is uniquely determined too:

$$y_{ss}(t) = (C\,z + D\,w_u + N\,w_d)e^{\alpha t}, \quad t \in \mathbb{R},\ t \geq 0.$$

The notion of blocking zero introduced in the following Definition 2, which extends the classical definition for linear time-invariant systems in state-space form, will be useful for giving an algebraic version of the internal model principle for generalized systems.

Definition 2. Under Assumption 2, the complex α is a *blocking zero* of system (1) from the input $d(t)$ to the output $y(t)$ if the pseudo-steady-state response in the output of (1) to the input functions (7) is constant and equal to zero for $w_u = 0$ and for all $w_d \in \mathbb{C}^m$.

Similar definition applies for a *blocking zero* of system (1) from the input $u(t)$ to the output $y(t)$.

The following Lemma 2 gives a necessary and sufficient algebraic condition for a complex α to be blocking zero for a generalized system.

Lemma 2. Under Assumption 2, the complex α is a blocking zero of system (1) from the input $d(t)$ to the output $y(t)$ if and only if the following condition holds:

$$\operatorname{Im}\begin{bmatrix} M \\ N \end{bmatrix} \subseteq \operatorname{Im}\begin{bmatrix} A - \alpha E \\ C \end{bmatrix}. \tag{10}$$

P r o o f . Let $x_{ss}(t)$ given in (8) be the pseudo-steady-state response in the pseudo-state of (1) to the input functions (7) for $w_u = 0$ and for some $w_d \in \mathbb{C}^m$; let $y_{ss}(t)$ be the corresponding pseudo-steady-state response in the output of (1). Then, the following relations hold for all $t \in \mathbb{R},\ t \geq 0$:

$$\alpha\, E\, z\, \mathrm{e}^{\alpha t} = A\, z\, \mathrm{e}^{\alpha t} + M\, w_d\, \mathrm{e}^{\alpha t}, \tag{11a}$$

$$y_{ss}(t) = C\, z\, \mathrm{e}^{\alpha t} + N\, w_d\, \mathrm{e}^{\alpha t}. \tag{11b}$$

If α is a blocking zero of (1) from $d(t)$ to $y(t)$, then $y_{ss}(t) = 0$ in (11b) for all $t \in \mathbb{R},\ t \geq 0$; consequently, taking into account that $\mathrm{e}^{\alpha t} \neq 0$ for all $t \in \mathbb{R}, t \geq 0$, relations (11) become:

$$[A - \alpha\, E]\,(-z) = M\, w_d, \tag{12a}$$

$$C\,(-z) = N\, w_d, \tag{12b}$$

which imply (10) for the arbitrariness of $w_d \in \mathbb{C}^m$, thus proving the "only if" part of the lemma.

If (10) holds, then for each $w_d \in \mathbb{C}^m$ there exists a $z \in \mathbb{C}^n$ solution of (12); for such w_d and z, relations (11) hold with $y_{ss}(t) = 0$ for all $t \in \mathbb{R}, t \geq 0$. Complex α is then a blocking zero from $d(t)$ to $y(t)$, as to be proved for the "if" part of the lemma. □

The proof of the following Lemma 3 is a direct consequence of items (ii) and (iv) of Subsection 2.1.

Lemma 3. Under Assumption 2, let (8), (9) be the pseudo-steady-state response in the pseudo-state of (1) to the input functions (7), and let $x(t)$ be the pseudo-state response of (1) to the input functions (7) from the initial condition $x(0^-) = x_0$, for some $x_0 \in \mathbb{C}^n$; then, for some $\gamma \in \mathbb{R}, \gamma < 0$, function $\widetilde{x}(t) := x(t) - x_{ss}(t)$ exponentially goes to zero faster than $\mathrm{e}^{\gamma t}$ for all $x_0 \in \mathbb{C}^n$ and is impulse-free if and only if:

(i) all the eigenvalues λ_i of (1a) have real part $\mathrm{re}(\lambda_i) < \gamma$;

(ii) condition (4) holds.

By Lemmas 1, 3, the following definition extends the notion of steady-state response, which is classical for linear time-invariant systems in state-space form, in case of generalized systems.

Definition 3. If $\mathrm{re}[\alpha] \geq 0$ and all the eigenvalues of (1a) have negative real part, then function (8), (9), which by Lemma 1 exists and is uniquely determined and by Lemma 3 is attractive with an impulse-free transient behavior, is called the *state steady-state* response of (1) to the input functions (7); the corresponding output function is called the *output steady-state* response of (1) to the input functions (7).

3. PROBLEM FORMULATION AND SOLUTION

Consider the following class of *generalized continuous-time time-invariant linear systems* $\mathcal{S}$ (briefly, *generalized systems*)

$$E\,\dot{x}(t) = A\,x(t) + B\,u(t) + \sum_{i=1}^{\mu} M_i\,d_i(t), \quad t \in \mathbb{R}, t \geq 0, \tag{13a}$$

$$y(t) = C\,x(t) + D\,u(t) + \sum_{i=1}^{\mu} N_i\,d_i(t), \quad t \in \mathbb{R}, t \geq 0, \tag{13b}$$

where $x(t) \in \mathbb{R}^n$ is the pseudo-state, $u(t) \in \mathbb{R}^p$ is the control input, $y(t) \in \mathbb{R}^q$ is the output to be controlled (which is assumed to be measured), $d_i(t) \in \mathbb{R}^{m_i}, i = 1, 2, \ldots, \mu$, are the unmeasurable and unknown disturbance inputs, and E, A, B, C, D, M_i, N_i, $i = 1, 2, \ldots, \mu$, are real matrices of proper dimensions. Assumption 1 rewritten for system $\mathcal{S}$ will be supposed to hold throughout the remainder of the paper, without mentioning it explicitly.

It is assumed that each one of the first q_0, with $q_0 \leq q$, components of $y(t)$ must track the corresponding component of the reference vector $r(t) \in \mathbb{R}^{q_0}$, whereas the other $q - q_0$ components of $y(t)$ must merely be regulated to zero. Hence, the error signal $e(t) \in \mathbb{R}^q$ is defined by:

$$e(t) := V\,r(t) - y(t), \quad t \in \mathbb{R}, t \geq 0, \tag{14a}$$

$$V := \begin{bmatrix} I \\ 0 \end{bmatrix}. \tag{14b}$$

It is also assumed that the class $\mathcal{R}$ of the reference signals $r(\cdot)$ to be exponentially tracked and the classes $\mathcal{D}_i$ of the disturbance functions $d_i(t)$, $i = 1, 2, \ldots, \mu$, to be exponentially rejected are of the following type:

$$\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \ldots \oplus \mathcal{R}_{\mu_0}, \tag{15a}$$

$$\mathcal{R}_i := \{r(\cdot) : r(t) = \widehat{w}\,e^{\alpha_i t} + \widehat{w}^*\,e^{\alpha_i^* t},\ \forall t \in \mathbb{R},\ t \geq 0,\ \widehat{w} \in \mathbb{C}^{q_0}\},\ i = 1, 2, \ldots, \mu_0, \tag{15b}$$

$$\mathcal{D}_i := \{d_i(\cdot) : d_i(t) = \widetilde{w}\,e^{\alpha_i t} + \widetilde{w}^*\,e^{\alpha_i^* t},\ \forall t \in \mathbb{R},\ t \geq 0,\ \widetilde{w} \in \mathbb{C}^{m_i}\},\ i = 1, 2, \ldots, \mu, \tag{15c}$$

for some non-negative integer $\mu_0 \leq \mu$ and some $\alpha_i \in \mathbb{C}$, $i = 1, 2, \ldots, \mu$, which are assumed to be all distinct and to satisfy $\mathrm{re}(\alpha_i) \geq 0$, $\mathrm{im}(\alpha_i) \geq 0$, $i = 1, 2, \ldots, \mu$, where $*$ means complex conjugate, and $\mathrm{re}(\cdot)$ and $\mathrm{im}(\cdot)$ denote, respectively, real and imaginary part of the complex number at argument. The pair (M_i, N_i) is assumed to be non-zero for each $i \in \{\mu_0 + 1, \ldots, \mu\}$.

Notice that, for any set of $\mu + 1$ classes of form (15), it can be easily found an exosystem that generates such classes as its output free responses. The description (15) of disturbance functions and of reference signals is preferred to the one based on an exosystem, since the description (15) allows the design of the compensator to be straightforward for system (13), with the error signal defined by (14).

The *tracking and regulation problem* under consideration can be formally stated as the following Problem 1.

Problem 1. For a given $\gamma \in \mathbb{R}$, $\gamma < 0$, find (if any) a non-singular continuous-time time-invariant linear dynamic compensator $\mathcal{K}$, having $y(t)$ and $r(t)$ as inputs and $u(t)$ as output, such that the over-all control system $\widetilde{\mathcal{S}}$, which is obtained by the feedback connection of $\mathcal{S}$ and $\mathcal{K}$ and is described by equations of the following form

$$\widetilde{E}\,\dot{\widetilde{x}}(t) = \widetilde{A}\,\widetilde{x}(t) + \widetilde{B}\,r(t) + \sum_{i=1}^{\mu} \widetilde{M}_i\,d_i(t), \quad t \in \mathbb{R},\ t \geq 0, \tag{16a}$$

$$e(t) = \widetilde{C}\,\widetilde{x}(t) + \widetilde{D}\,r(t) + \sum_{i=1}^{\mu} \widetilde{N}_i\,d_i(t), \quad t \in \mathbb{R},\ t \geq 0, \tag{16b}$$

satisfies the following requirements:

(a) $\widetilde{\mathcal{S}}$ is impulse-free;

(b) all the eigenvalues $\widetilde{\lambda}_i$ of $\widetilde{\mathcal{S}}$ have real part $\mathrm{re}(\widetilde{\lambda}_i) < \gamma$;

(c) the error response $e(t)$ of $\widetilde{\mathcal{S}}$ exponentially goes to zero for all the disturbance functions $d_i(\cdot) \in \mathcal{D}_i$, $i = 1, 2, \ldots, \mu$, for all the reference signals $r(\cdot) \in \mathcal{R}$, and for all the initial conditions.

It is stressed that a control system satisfying (a) and (b) has the property that all its state free responses converge to zero, without impulsive behavior, faster than $e^{\gamma t}$, where γ is a design parameter. A control system satisfying (c), in addition to (a) and (b), has the additional property that all the error responses $e(t)$ converge to zero, without impulsive behavior, faster than $e^{\gamma t}$.

The notion of blocking zero introduced in the previous section allows one to simply prove the following proposition, which is useful from the analysis point of view.

Proposition 1. Let the compensator $\mathcal{K}$ be designed so that the over-all control system $\widetilde{\mathcal{S}}$ satisfies requirements (a) and (b) of Problem 1. Then, such a compensator $\mathcal{K}$ is a solution of Problem 1 (i.e., it satisfies the additional requirement (c)) if and only if α_i is a blocking zero of $\widetilde{\mathcal{S}}$ from $d_i(t)$ to $e(t)$ for $i = 1, 2, \ldots, \mu$, and α_i is a blocking zero of $\widetilde{\mathcal{S}}$ from $r(t)$ to $e(t)$ for $i = 1, 2, \ldots, \mu_0$, i.e. if and only if the following algebraic conditions hold:

$$\mathrm{Im}\begin{bmatrix} \widetilde{M}_i \\ \widetilde{N}_i \end{bmatrix} \subseteq \mathrm{Im}\begin{bmatrix} \widetilde{A} - \alpha_i\,\widetilde{E} \\ \widetilde{C} \end{bmatrix}, \quad i = 1, 2, \ldots, \mu, \tag{17a}$$

$$\operatorname{Im}\begin{bmatrix}\widetilde{B}\\ \widetilde{D}\end{bmatrix} \subseteq \operatorname{Im}\begin{bmatrix}\widetilde{A}-\alpha_i\widetilde{E}\\ \widetilde{C}\end{bmatrix}, \quad i=1,2,\ldots,\mu_0. \tag{17b}$$

The following conditions on the original system $\mathcal{S}$ will be useful for stating a theorem giving a solution to Problem 1.

(A) System $\mathcal{S}$ is impulse-free.

(B) For some $\gamma \in \mathbb{R}, \gamma < 0$, system $\mathcal{S}$ is γ-stabilisable and γ-detectable.

(C) The following relations hold for system $\mathcal{S}$:

$$\operatorname{Im}\begin{bmatrix}M_i\\ N_i\end{bmatrix} \subseteq \operatorname{Im}\begin{bmatrix}A-\alpha_i E & B\\ C & D\end{bmatrix}, \quad i=1,2,\ldots,\mu, \tag{18a}$$

$$\operatorname{Im}\begin{bmatrix}0\\ V\end{bmatrix} \subseteq \operatorname{Im}\begin{bmatrix}A-\alpha_i E & B\\ C & D\end{bmatrix}, \quad i=1,2,\ldots,\mu_0. \tag{18b}$$

The following theorem (which will be proved in the subsequent Section 4) gives the necessary and sufficient conditions for the existence of a solution of Problem 1, under the assumption that Condition (A) holds. Under this assumption and these conditions, the design procedure of a compensator $\mathcal{K}$ that is solution of Problem 1 will be detailed just after the theorem.

Theorem 1. Under the assumption that Condition (A) holds, Problem 1 is solvable if and only if Conditions (B) and (C) hold.

It is stressed that Condition (C) is implied by the stronger condition

$$\operatorname{rank}\begin{bmatrix}A-\alpha_i E & B\\ C & D\end{bmatrix} = n+q, \quad i=1,2,\ldots,\mu, \quad \text{(i.e., full row-rank)} \tag{19}$$

which extends the Davison condition [13, 14], in case of generalized systems. As a matter of fact, condition (19) cannot be satisfied when $q > p$, whereas Condition (C) is compatible with the case $q > p$, since it takes into account the hypothesis that the disturbance functions and/or the reference signals enter the plant $\mathcal{S}$ only partially. Actually, for $q_0 = q$ and $\mu_0 = \mu$, the γ-stabilisability condition and condition (18b) imply (19).

The aformentioned design procedure of compensator $\mathcal{K}$ will now be given with reference to the specific control system structure depicted in Figure 1, where $\mathcal{K}_a$ and $\mathcal{K}_b$ are two dynamic sub-compensators. The role of $\mathcal{K}_b$ is to enlarge, as much as consistent with the possibility of meeting requirement (b) (while preserving the assumed requirement (a)) of Problem 1, the (partial) internal model of disturbance functions and of reference signals that is possibly contained in $\mathcal{S}$, with the purpose of satisfying requirement (c). The role of $\mathcal{K}_a$, which is to be designed on the basis of the series connection $\widehat{\mathcal{S}}$ of $\mathcal{K}_b$ and $\mathcal{S}$, is to fulfill requirement (b) of Problem 1, while preserving requirement (a), which has been assumed for the original system $\mathcal{S}$. The proof that the following design procedure can be actually completed and that it gives actually a solution of Problem 1, will be a part of the proof of Theorem 1, given in Section 4.

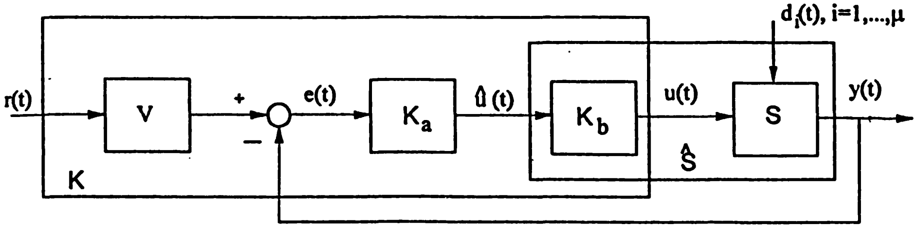

Fig. 1. The over-all control system $\tilde{\mathcal{S}}$.

Procedure 1. (which is valid under Conditions (A), (B), (C))

Step 1. For each $i = 1, 2, \ldots, \mu$, define:

$$\eta_i := \operatorname{rank} \begin{bmatrix} A - \alpha_i E & B \\ C & D \end{bmatrix}, \tag{20a}$$

$$\rho_i := \eta_i - n. \tag{20b}$$

Notice that, by Condition (B), numbers ρ_i, $i = 1, 2, \ldots, \mu$, are non-negative.

Step 2. For each $i = 1, 2, \ldots, \mu$, compute two real matrices $H_{i,1}$ and $H_{i,2}$, of respective dimensions $p \times \rho_i$ and $p \times (p - \rho_i)$, such that (matrices $H_{i,1}$ and $H_{i,2}$ are to be used for the selection of the columns of matrix B, whence they can always taken with real entries)

$$\operatorname{rank} \begin{bmatrix} A - \alpha_i E & B\,H_{i,1} \\ C & D\,H_{i,1} \end{bmatrix} = n + \rho_i = \eta_i$$

(i.e., maximum and full column-rank), (21a)

$$\det [\,H_{i,1} \quad H_{i,2}\,] \neq 0. \tag{21b}$$

Step 3. For each $i = 1, 2, \ldots, \mu$, if $\alpha_i \in \mathbb{R}$, then define a sub-compensator $\mathcal{K}_{b,i}$ (of dimension ρ_i) described by:

$$\dot{w}_{b,i}(t) = \alpha_i\, w_{b,i}(t) + [\,0 \quad I\,]\; \hat{u}_i(t), \tag{22a}$$

$$u_i(t) = H_{i,1}\, w_{b,i}(t) + [\,H_{i,2} \quad 0\,]\; \hat{u}_i(t), \tag{22b}$$

where $w_{b,i}(t) \in \mathbb{R}^{\rho_i}$, $\hat{u}_i(t) \in \mathbb{R}^p$, $u_i(t) \in \mathbb{R}^p$, while if $\alpha_i \notin \mathbb{R}$, then define a sub-compensator $\mathcal{K}_{b,i}$ (of dimension $2\rho_i$) described by:

$$\dot{w}^1_{b,i}(t) = \mathrm{re}(\alpha_i)\, w^1_{b,i}(t) - \mathrm{im}(\alpha_i)\, w^2_{b,i}(t) + 2\,[0 \quad I]\, \hat{u}^1_i(t), \tag{23a}$$

$$\dot{w}^2_{b,i}(t) = \mathrm{im}(\alpha_i)\, w^1_{b,i}(t) + \mathrm{re}(\alpha_i)\, w^2_{b,i}(t) + 2\,[0 \quad I]\, \hat{u}^2_i(t), \tag{23b}$$

$$\begin{aligned} u_i(t) &= \mathrm{re}(H_{i,1})\, w^1_{b,i}(t) - \mathrm{im}(H_{i,1})\, w^2_{b,i}(t) \\ &\quad +2\,[\mathrm{re}(H_{i,2}) \quad 0]\, \hat{u}^1_i(t) - 2\,[\mathrm{im}(H_{i,2}) \quad 0]\, \hat{u}^2_i(t), \end{aligned} \tag{23c}$$

where $w^1_{b,i}(t), w^2_{b,i}(t) \in \mathbb{R}^{\rho_i}$, $\hat{u}^1_i(t), \hat{u}^2_i(t) \in \mathbb{R}^p$, $u_i(t) \in \mathbb{R}^p$, and define $\hat{u}_i(t) := [(\hat{u}^1_i(t))^T \quad (\hat{u}^2_i(t))^T]^T$ and $w_{b,i}(t) := [(w^1_{b,i}(t))^T \quad (w^2_{b,i}(t))^T]^T$. It is stressed that if $\rho_i = 0$ for some $i \in \{1, 2, \dots, \mu\}$, then the sub-compensator $\mathcal{K}_{b,i}$ reduces to a memory-less connection.

Step 4. Define the sub-compensator $\mathcal{K}_b$ as the parallel connection of the μ sub-compensators $\mathcal{K}_{b,i}$, $i = 1, 2, \dots, \mu$, so that it has $\hat{u}(t) := [\hat{u}^T_1(t) \quad \cdots \quad \hat{u}^T_\mu(t)]^T$ as input and $u(t) := \sum_{i=1}^{\mu} u_i(t)$ as output, as in Figure 2.

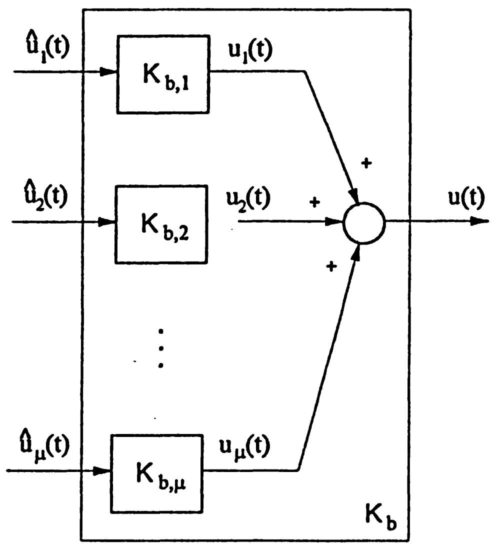

Fig. 2. The sub-compensator $\mathcal{K}_b$ as the parallel connection of the μ sub-compensators $\mathcal{K}_{b,i}$, $i = 1, 2, \dots, \mu$.

Step 5. Let $\hat{\mathcal{S}}$ be the series connection of $\mathcal{K}_b$ and $\mathcal{S}$ and be described by:

$$\hat{E}\,\dot{\hat{x}}(t) = \hat{A}\,\hat{x}(t) + \hat{B}\,\hat{u}(t) + \sum_{i=1}^{\mu} \widehat{M}_i\, d_i(t), \quad t \in \mathbb{R},\ t \geq 0, \tag{24a}$$

$$y(t) = \widehat{C}\,\widehat{x}(t) + \widehat{D}\,\widehat{u}(t) + \sum_{i=1}^{\mu} \widehat{N}_i\, d_i(t), \quad t \in I\!R,\ t \geq 0. \tag{24b}$$

Making use of a suitable design procedure, e. g. the one reported in [25], design the sub-compensator $\mathcal{K}_a$ (on the basis of $\widehat{\mathcal{S}}$) so that all the eigenvalues of the over-all control system $\widetilde{\mathcal{S}}$ have real part less than γ and the impulse-free property of $\widehat{\mathcal{S}}$, which is yielded by the impulse-free property of $\mathcal{S}$ (see the assumed Condition (A) and the proof of Theorem 1), is preserved.

The following very simple example is useful for better clarifying (before the proof of Theorem 1) how Conditions (A), (B) and (C) allows Procedure 1 to be completed and how the so obtained controller $\mathcal{K}$ is a solution to Problem 1.

Example 1. Let system $\mathcal{S}$ be a one-input two-output two-dimensional generalized linear system (i. e., $p = 1, q = 2, n = 2$), whose outputs must merely be regulated to zero (i. e., $q_0 = 0, \mu_0 = 0$), and whose description is affected by two scalar disturbances (i. e., $\mu = 2, m_1 = m_2 = 1$) characterized by $\alpha_1 = \jmath$ and $\alpha_2 = 1$.

In addition, let $\mathcal{S}$ be characterized by the following matrices:

$$E = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since

$$[A - \lambda\, E] = \begin{bmatrix} -\lambda & 1 \\ -\lambda & 0 \end{bmatrix},$$

the system $\mathcal{S}$ thus characterized has only one eigenvalue $\lambda_1 = 0$.

The matrices appearing in the relation (4) of Condition (A), in the relations (5), (6) of Condition (B) and in the relations (18) of Condition (C) take the following form:

$$\begin{bmatrix} E & 0 \\ A & E \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right], \tag{25a}$$

$$[A - \lambda_1 E \quad B] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & -1 \end{array}\right], \quad \begin{bmatrix} A - \lambda_1 E \\ C \end{bmatrix} = \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right], \tag{25b}$$

$$\begin{bmatrix} 0 \\ V \end{bmatrix} = \left[\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right], \quad \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} = \left[\begin{array}{c} 1 \\ -1 \\ \hline 1 \\ 2 \end{array}\right], \quad \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} = \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \\ 1 \end{array}\right], \tag{25c}$$

$$\begin{bmatrix} A-\alpha_1 E & B \\ C & D \end{bmatrix} = \left[\begin{array}{cc|c} -\jmath & 1 & 0 \\ -\jmath & 0 & -1 \\ \hline 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right], \tag{25d}$$

$$\begin{bmatrix} A-\alpha_2 E & B \\ C & D \end{bmatrix} = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & 0 & -1 \\ \hline 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]. \tag{25e}$$

Since the rank of matrix (25e) is equal to 3, Condition (A) holds. Since the ranks of the two matrices (25b) are both equal to 2, Condition (B) holds for all $\gamma \in I\!R, \gamma < 0$. By (25c), (25d), (25e), it is easy to see that relations (18) hold, whence that Condition (C) holds too.

Now, the application of Steps 1–4 of Procedure 1 to system $\mathcal{S}$ is detailed as follows.

Step 1. From matrices (25d), (25e), it is easy to compute:

$$\eta_1 = 3, \quad \rho_1 = 1, \quad \eta_2 = 2, \quad \rho_2 = 0.$$

Step 2. Since $\rho_1 = 1$, H_{11} can be taken equal to the scalar 1, while H_{12} vanishes. Since $\rho_2 = 0$, H_{21} vanishes, while H_{22} can be taken equal to the scalar 1.

Step 3. Since α_1 is complex, compensator $\mathcal{K}_{b,1}$ takes the form (23) and is two-dimensional because $\rho = 1$:

$$\dot{w}^1_{b,1}(t) = -w^2_{b,1}(t) + 2\,\widehat{u}^1_1(t), \tag{26a}$$

$$\dot{w}^2_{b,1}(t) = w^1_{b,1}(t) + 2\,\widehat{u}^2_1(t), \tag{26b}$$

$$u_1(t) = w^1_{b,1}(t). \tag{26c}$$

Since α_2 is real, compensator $\mathcal{K}_{b,2}$ takes the form (22) and reduces to a memory-less connection because $\rho_2 = 0$:

$$u_2(t) = \widehat{u}_2(t). \tag{27}$$

Step 4. The compensator $\mathcal{K}_b$ is the parallel connection of $\mathcal{K}_{b,1}$ and $\mathcal{K}_{b,2}$:

$$\dot{w}^1_{b,1}(t) = -w^2_{b,1}(t) + 2\,\widehat{u}^1_1(t), \tag{28a}$$

$$\dot{w}^2_{b,1}(t) = w^1_{b,1}(t) + 2\,\widehat{u}^2_1(t), \tag{28b}$$

$$u(t) = w^1_{b,1}(t) + \widehat{u}_2(t). \tag{28c}$$

As for Step 5 of Procedure 1, it is left to the reader the verification that the series connection $\widehat{\mathcal{S}}$ of $\mathcal{K}_b$ and $\mathcal{S}$ is impulse-free, γ-stabilisable and γ-detectable for all $\gamma \in I\!R, \gamma < 0$. Therefore, by using the procedure given in [25], compensator $\mathcal{K}_a$ can be easily designed so to complete the design of $\mathcal{K}$. □

4. PROOF OF THEOREM 1

Necessity.

The necessity of Condition (B) is trivial.

In order to prove the necessity of Condition (C), assume that Problem 1 admits a solution $\mathcal{K}$. For an arbitrary $j \in \{1,2,\ldots,\mu\}$, consider the over-all control system $\widetilde{\mathcal{S}}$ under a null reference signal $r(t)=0$, under null disturbance functions $d_i(t)=0$ for $i \in \{1,2,\ldots,\mu\}$, $i \neq j$, and under $d_j(t)=d_j(0)\mathrm{e}^{\alpha_j t}$, with $d_j(0) \in \mathbb{R}^{m_j}$. Since $\widetilde{\mathcal{S}}$ satisfies requirements (b) and (c), it is noted that system $\widetilde{\mathcal{S}}$ has α_j as blocking zero from $d_j(t)$ to $e(t)$. Assume that $\alpha_j \in \mathbb{R}$: the case $\alpha_j \notin \mathbb{R}$ admits a similar proof. The existence and the uniqueness in all the variables of $\widetilde{\mathcal{S}}$ of the steady-state response of the same exponential form as $d_j(t)$, imply the existence of $x(0) \in \mathbb{R}^n$ and $u(0) \in \mathbb{R}^p$ such that

$$\alpha_j\, E\, x(0)\, \mathrm{e}^{\alpha_j t} = A\, x(0)\, \mathrm{e}^{\alpha_j t} + B\, u(0)\, \mathrm{e}^{\alpha_j t} + M_j\, d_j(0)\, \mathrm{e}^{\alpha_j t}, \quad t \in \mathbb{R},\ t \geq 0, \quad (29a)$$

$$0 = C\, x(0)\, \mathrm{e}^{\alpha_j t} + D\, u(0)\, \mathrm{e}^{\alpha_j t} + N_j\, d_j(0)\, \mathrm{e}^{\alpha_j t}, \quad t \in \mathbb{R},\ t \geq 0. \quad (29b)$$

Since $\mathrm{e}^{\alpha_j t} \neq 0$, relations (29) yield

$$\begin{bmatrix} A-\alpha_j E & B \\ C & D \end{bmatrix} \begin{bmatrix} -x(0) \\ -u(0) \end{bmatrix} = \begin{bmatrix} M_j \\ N_j \end{bmatrix} d_j(0).$$

This relation, by virtue of the arbitrariness of $d_j(0) \in \mathbb{R}^{m_j}$, implies (18a) restricted to $i=j$. The arbitrariness of $j \in \{1,2,\ldots,\mu\}$ proves (18a). Relation (18b) follows in a similar way, thus completing the necessity proof.

Sufficiency.

It will be shown that: (i) Procedure 1 for the design of the compensator $\mathcal{K}$ having the structure depicted in Figure 1 can be actually completed; and that: (ii) a compensator $\mathcal{K}$ designed according to Procedure 1 is actually a solution of Problem 1.

Consider the above reported item (i).

As for Step 1, by virtue of the γ-detectability assumption, the first block-column of the matrix appearing in (20a) has full-column rank equal to n, thus implying that numbers ρ_i are non-negative for all $i \in \{1,2,\ldots,\mu\}$.

As for Step 2, since ρ_i is non-negative, matrices $H_{i,1}$ and $H_{i,2}$ can be easily computed so that relations (21) hold for $i=1,2,\ldots,\mu$.

The possibility of completing Steps 3 and 4 is trivially seen.

As for Step 5, it is sufficient to prove that the series connection $\widehat{\mathcal{S}}$ of $\mathcal{K}_b$ and $\mathcal{S}$ satisfies Conditions (A) and (B) rewritten for $\widehat{\mathcal{S}}$ instead of $\mathcal{S}$, since in this case the procedures reported in [25] allows a compensator $\mathcal{K}_a$ to be designed so that $\widetilde{\mathcal{S}}$ has all the eigenvalues with real part less than γ and is impulse-free.

For this demonstration, for each $j \in \{1,2,\ldots,\mu\}$ call $\widehat{\mathcal{S}}_j$ the series connection of $\mathcal{K}_{b,j}$ and $\mathcal{S}$ obtained by setting $u(t)=u_j(t)$. It will now be shown that if, for each $j \in \{1,2,\ldots,\mu\}$, system $\widehat{\mathcal{S}}_j$ is γ-stabilisable, γ-detectable and impulse-free, then also system $\widehat{\mathcal{S}}$ is γ-stabilisable, γ-detectable and impulse-free. Afterwards, it will be proved that, for each $j \in \{1, 2, \ldots, \mu\}$, system $\widehat{\mathcal{S}}_j$ is γ-stabilisable, γ-detectable and impulse-free.

First, consider the γ-stabilisability and the γ-detectability conditions, which are reported at items (vii) and (viii) of Subsection 2.1, rewritten with the matrices of $\widehat{\mathcal{S}}_j$. Such conditions imply the existence of a linear dynamic output feedback for $\widehat{\mathcal{S}}_j$ such that all the pseudo-state free responses of the closed-loop thus obtained go to zero faster than $e^{\gamma t}$. This trivially implies the property that, for each initial condition $x(0), w_{b,j}(0)$ of $\widehat{\mathcal{S}}_j$, there exists an input function $\widehat{u}_j(j, x(0), w_{b,j}(0); t)$ of $\widehat{\mathcal{S}}_j$ such that its pseudo-state response $x(t), w_{b,j}(t)$ to such a $\widehat{u}_j(j, x(0), w_{b,j}(0); t)$ from the initial condition $x(0), w_{b,j}(0)$ goes to zero faster than $e^{\gamma t}$. For any initial condition $x(0), w_{b,1}(0), \ldots, w_{b,\mu}(0)$ of $\widehat{\mathcal{S}}$, decompose $x(0)$ as the sum of μ initial conditions x_0^i, $i = 1, 2, \ldots, \mu$ (i.e., $x(0) = \sum_{i=1}^{\mu} x_0^i$); consider the input function

$$\widehat{u}(x(0), w_{b,1}(0), \ldots, w_{b,\mu}(0); t) = [\,\widehat{u}_1^T(1, x_0^1, w_{b,1}(0); t) \quad \ldots \quad \widehat{u}_\mu^T(\mu, x_0^\mu, w_{b,\mu}(0); t)\,]^T,$$

and, for each $j = 1, 2, \ldots, \mu$, denote by $x^j(t), w_{b,j}^j(t)$ the pseudo-state response of $\widehat{\mathcal{S}}_j$ to the input function $\widehat{u}_j(j, x_0^j, w_{b,j}(0); t)$ from the initial condition $x_0^j, w_{b,j}(0)$. Then, by the linearity of $\widehat{\mathcal{S}}$, the pseudo-state response $x(t), w_{b,1}(t), \ldots, w_{b,\mu}(t)$ of $\widehat{\mathcal{S}}$ to $\widehat{u}(x(0), w_{b,1}(0), \ldots, w_{b,\mu}(0); t)$ from the initial condition $x(0), w_{b,1}(0), \ldots, w_{b,\mu}(0)$ is such that

$$\begin{aligned} x(t) &= \sum_{j=1}^{\mu} x^j(t), \\ w_{b,j}(t) &= w_{b,j}^j(t), \quad j = 1, 2, \ldots, \mu, \end{aligned}$$

and therefore converges to zero faster than $e^{\gamma t}$. This implies the γ-stabilisability and the γ-detectability of $\widehat{\mathcal{S}}$, because if $\widehat{\mathcal{S}}$ were not γ-stabilisable and γ-detectable, then it would exist an initial condition of $\widehat{\mathcal{S}}$ such that, for any input function $\widehat{u}(t)$ of $\widehat{\mathcal{S}}$, its pseudo-state response should not converge to zero faster than $e^{\gamma t}$.

Now, consider the impulse-free condition, which is reported at item (iv) of Subsection 2.1, rewritten with the matrices of $\widehat{\mathcal{S}}_j$. Such a condition implies the property that, for each initial condition $x(0^-), w_{b,j}(0^-)$ of $\widehat{\mathcal{S}}_j$, the pseudo-state free response of $\widehat{\mathcal{S}}_j$ is impulse-free. For any initial condition $x(0^-), w_{b,1}(0^-), \ldots, w_{b,\mu}(0^-)$ of $\widehat{\mathcal{S}}$, decompose $x(0^-)$ as the sum of μ initial conditions x_0^i, $i = 1, 2, \ldots, \mu$ (i.e., $x(0^-) = \sum_{i=1}^{\mu} x_0^i$); for each $j = 1, 2, \ldots, \mu$, denote by $x^j(t), w_{b,j}^j(t)$ the pseudo-state free response of $\widehat{\mathcal{S}}_j$ from $x_0^j, w_{b,j}(0^-)$. Then, by the linearity of $\widehat{\mathcal{S}}$, the pseudo-state free response $x(t), w_{b,1}(t), w_{b,\mu}(t)$ of $\widehat{\mathcal{S}}$ from the initial condition $x(0^-), w_{b,1}(0^-), \ldots$ $\ldots, w_{b,\mu}(0^-)$ is such that:

$$\begin{aligned} x(t) &= \sum_{j=1}^{\mu} x^j(t), \\ w_{b,j}(t) &= w_{b,j}^j(t), \quad j = 1, 2, \ldots, \mu, \end{aligned}$$

and therefore is impulse-free.

Now, it will be proved algebraically that $\widehat{\mathcal{S}}_j$ is γ-stabilisable, γ-detectable and impulse-free, thus completing the proof of item (i).

As for the γ-stabilisability of $\widehat{\mathcal{S}}_j$, taking into account that $\widehat{\mathcal{S}}_j$ is the series connection of $\mathcal{K}_{b,j}$ and $\mathcal{S}$, in the case $\alpha_j \in \mathbb{R}$, the matrix appearing in condition (5) rewritten for $\widehat{\mathcal{S}}_j$ becomes

$$\left[\begin{array}{cc|cc} A-\lambda E & B\,H_{j,1} & B\,H_{j,2} & 0 \\ 0 & (\alpha_j-\lambda)\,I & 0 & I \end{array}\right], \tag{30}$$

which, by grouping together its second and third block column, can be recast as follows:

$$\begin{bmatrix} A-\lambda E & B[H_{j,1} \;\; H_{j,2}] & 0 \\ 0 & [(\alpha_j-\lambda)\,I \;\; 0] & I \end{bmatrix}. \tag{31}$$

It is stressed that the eigenvalues $\widehat{\lambda}_i$ of $\widehat{\mathcal{S}}_j$ coincide with the eigenvalues λ_i of $\mathcal{S}$ and with α_j. It is evident from (31) that matrix (30) has full row-rank for all $\lambda = \widehat{\lambda}_i$: this is yielded by the γ-stabilisability of $\mathcal{S}$ and by the non-singularity of $[H_{j,1} \;\; H_{j,2}]$ (see (21b)). This proves the γ-stabilisability of $\widehat{\mathcal{S}}_j$ in the case $\alpha_j \in \mathbb{R}$. A similar proof holds for the case $\alpha_j \notin \mathbb{R}$.

As for the γ-detectability of $\widehat{\mathcal{S}}_j$, taking into account that $\widehat{\mathcal{S}}_j$ is the series connection of $\mathcal{K}_{b,j}$ and $\mathcal{S}$, in the case $\alpha_j \in \mathbb{R}$, the matrix appearing in condition (6) rewritten for $\widehat{\mathcal{S}}_j$ becomes

$$\left[\begin{array}{cc} A-\lambda E & B\,H_{j,1} \\ 0 & (\alpha_j-\lambda)\,I \\ \hline C & D\,H_{j,1} \end{array}\right], \tag{32}$$

which, through a block-row inter-change, can be recast as follows:

$$\begin{bmatrix} A-\lambda E & B\,H_{j,1} \\ C & D\,H_{j,1} \\ 0 & (\alpha_j-\lambda)\,I \end{bmatrix}. \tag{33}$$

Since the eigenvalues $\widehat{\lambda}_i$ of $\widehat{\mathcal{S}}_j$ coincide with the eigenvalues λ_i of $\mathcal{S}$ and with α_j, it is evident from (33) that matrix (32) has full column-rank for all $\lambda = \widehat{\lambda}_i$: for $\widehat{\lambda}_i \neq \alpha_j$, this is yielded by the γ-detectability of $\mathcal{S}$, while for $\widehat{\lambda}_i = \alpha_j$ this is yielded by (21a). This proves the γ-detectability of $\widehat{\mathcal{S}}_j$ in the case $\alpha_j \in \mathbb{R}$. A similar proof holds for the case $\alpha_j \notin \mathbb{R}$.

As for the impulse-free property of $\widehat{\mathcal{S}}_j$, taking into account that $\widehat{\mathcal{S}}_j$ is the series connection of $\mathcal{K}_{b,j}$ and $\mathcal{S}$, the matrix appearing in the left hand side of condition (6) rewritten for $\widehat{\mathcal{S}}_j$ becomes

$$\left[\begin{array}{cc|cc} E & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline A & B\,H_{j,1} & E & 0 \\ 0 & \alpha_j\,I & 0 & I \end{array}\right], \tag{34}$$

which, through block-row and block-column inter-changes, can be recast as follows:

$$\begin{bmatrix} E & 0 & 0 & 0 \\ A & E & B H_{j,1} & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & \alpha_j I & I \end{bmatrix}. \tag{35}$$

Taking into account that the assumed impulse-free property of $\mathcal{S}$ yields condition (4), it is evident from (35) that the rank of matrix (34) is equal to the rank of

$$\begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix}$$

plus the rank of

$$\begin{bmatrix} A & B H_{j,1} \\ 0 & \alpha_j I \end{bmatrix},$$

which implies the impulse-free property of $\widehat{\mathcal{S}}_j$. Thus, the proof that Step 5 of Procedure 1 can be actually done has been completed, and the demonstration of item (i) is ended.

As for item (ii), notice, by Step 5, that the compensator $\mathcal{K}$ obtained through Procedure 1 is designed so that the over-all control system $\widetilde{\mathcal{S}}$ has all its eigenvalues $\widetilde{\lambda}_i$ with $\mathrm{re}(\widetilde{\lambda}_i) < \gamma$, and is impulse-free. This proves that such a $\mathcal{K}$ guarantees requirements (a) and (b) to be satisfied for $\widetilde{\mathcal{S}}$. The following lemma will be useful for checking that the same $\mathcal{K}$ guarantees also requirement (c) to be satisfied for the over-all control system $\widetilde{\mathcal{S}}$ thus obtained.

Lemma 4. Consider the compensator $\mathcal{K}$ designed by applying Procedure 1 and the over-all control system $\widetilde{\mathcal{S}}$ thus obtained, and assume that requirements (a) and (b) are satisfied by $\widetilde{\mathcal{S}}$. For each $j = 1, 2, \ldots, \mu$, call $\widehat{\mathcal{S}}_j$ the series connection of $\mathcal{K}_{b,i}$ and $\mathcal{S}$, having $\widehat{u}_j(t)$ and $d_i(t)$, $i = 1, 2, \ldots, \mu$, as inputs, and call $\widehat{x}^j(t) \in \mathbb{R}^{\widehat{n}_j}$ its pseudo-state; denote by $\widehat{E}_j$, $\widehat{A}_j$, $\widehat{B}_j$, $\widehat{C}_j$, $\widehat{D}_j$, $\widehat{M}_{j,i}$, $\widehat{N}_{j,i}$, $i = 1, 2, \ldots, \mu$, the matrices characterizing the description of $\widehat{\mathcal{S}}_j$ in a form similar to (24). If the following relations hold:

$$\mathrm{Im}\begin{bmatrix} \widehat{M}_{j,j} \\ \widehat{N}_{j,j} \end{bmatrix} \subseteq \mathrm{Im}\begin{bmatrix} \widehat{A}_j - \alpha_j \widehat{E}_j \\ \widehat{C}_j \end{bmatrix}, \quad j = 1, 2, \ldots, \mu, \tag{36a}$$

$$\mathrm{Im}\begin{bmatrix} 0 \\ V \end{bmatrix} \subseteq \mathrm{Im}\begin{bmatrix} \widehat{A}_j - \alpha_j \widehat{E}_j \\ \widehat{C}_j \end{bmatrix}, \quad j = 1, 2, \ldots, \mu_0, \tag{36b}$$

then requirement (c) is satisfied for $\widetilde{\mathcal{S}}$.

Proof. Call $w_a(t)$ the state of $\mathcal{K}_a$. For any $j \in \{1, 2, \ldots, \mu\}$, consider $\widetilde{\mathcal{S}}$ under a null reference signal $r(t)$, under null disturbance functions $d_i(t)$ for $i = 1, 2, \ldots, \mu$, $i \neq j$, and $d_j(t) \in \mathcal{D}_j$. Assume α_j (the case $\alpha_j \notin \mathbb{R}$ admits a similar proof). Since $\widetilde{\mathcal{S}}$

is exponentially stable, then there exists for $\widetilde{\mathcal{S}}$ a unique steady-state response (of the same exponential type as $d_j(t)$) in all the variables of $\widetilde{\mathcal{S}}$. Hence, if a set of signals of the same exponential type as $d_j(t)$ is found so that the equations describing $\widehat{\mathcal{S}}_j$, $\mathcal{K}_{b,i}$, $i = 1, 2, \ldots, \mu$, $i \neq j$, and $\mathcal{K}_a$ are satisfied by such signals, then these signals are the (unique) steady-state response of $\widetilde{\mathcal{S}}$ to the inputs under consideration. In addition, (36a) implies that for any $d_j(0) \in \mathbb{R}^{m_j}$ there exists a $\widehat{x}_0^j \in \mathbb{R}^{\widehat{n}_j}$ such that

$$0 = (\widehat{A}_j - \alpha_j \widehat{E}_j)\, \widehat{x}_0^j \, e^{\alpha_j t} + \widehat{M}_{j,j}\, d_j(0)\, e^{\alpha_j t}, \quad \forall t \in \mathbb{R},\ t \geq 0, \tag{37a}$$

$$0 = \widehat{C}_j\, \widehat{x}_0^j \, e^{\alpha_j t} + \widehat{N}_{j,j}\, d_j(0)\, e^{\alpha_j t}, \quad \forall t \in \mathbb{R},\ t \geq 0. \tag{37b}$$

Therefore, the steady-state response in all the variables of $\widetilde{\mathcal{S}}$ in necessarily characterized by:

$$\widehat{x}^j(t) = \widehat{x}_0^j \, e^{\alpha_j t}, \tag{38a}$$

$$\widehat{u}_j(t) = 0, \tag{38b}$$

$$w_{b,i}(t) = 0,\ \widehat{u}_i(t) = 0, \quad i = 1, 2, \ldots, \mu,\ i \neq j, \tag{38c}$$

$$w_a(t) = 0, \tag{38d}$$

$$y(t) = 0, \tag{38c}$$

so that the steady-state response in the $y(t)$ variable is identically zero, thus implying the exponential convergence to zero of the error $e(t)$, for all the initial conditions of $\widetilde{\mathcal{S}}$. Then, the arbitrariness of $j \in \{1, 2, \ldots, \mu\}$ and the linearity of $\widetilde{\mathcal{S}}$, together with a similar reasoning for the case $r(t) \in \mathcal{R}_j$, $j \in \{1, 2, \ldots, \mu_0\}$, and $d_i(t) = 0$, $i = 1, 2, \ldots, \mu$, complete the proof. □

On the basis of Lemma 4, the proof that the compensator $\mathcal{K}$ obtained by Procedure 1 guarantees also requirement (c), will be completed by showing that (36) hold. Now, as for (36a), for each $i \in \{1, 2, \ldots, \mu\}$, the following condition will be proved to hold for $\widehat{\mathcal{S}}_i$:

$$\operatorname{Im} \begin{bmatrix} \widehat{M}_{i,i} \\ \widehat{N}_{i,i} \end{bmatrix} \subseteq \begin{bmatrix} \widehat{A}_i - \alpha_i \widehat{E}_i \\ \widehat{C}_i \end{bmatrix}; \tag{39}$$

as for (36b), a similar proof can show that, for each $i \in \{1, 2, \ldots, \mu_0\}$, the following condition holds for $\widehat{\mathcal{S}}_i$:

$$\operatorname{Im} \begin{bmatrix} 0 \\ V \end{bmatrix} \subseteq \begin{bmatrix} \widehat{A}_i - \alpha_i \widehat{E}_i \\ \widehat{C}_i \end{bmatrix}. \tag{40}$$

Then, in order to prove (39), taking into account that $\widehat{\mathcal{S}}_i$ is the series connection of $\mathcal{K}_{b,i}$ and $\mathcal{S}$, substitute the corresponding expressions of $\widehat{E}_i$, $\widehat{A}_i$, $\widehat{C}_i$, $\widehat{M}_{i,i}$, $\widehat{N}_{i,i}$

into condition (39); both in the case $\alpha_i \in I\!R$ and $\alpha_i \notin I\!R$, by suitable elementary operations, condition (39) can be seen to hold if and only if

$$\operatorname{Im}\begin{bmatrix} M_i \\ N_i \\ 0 \end{bmatrix} \subseteq \operatorname{Im}\begin{bmatrix} A-\alpha_i E & B H_{i,1} \\ C & D H_{i,1} \\ 0 & 0 \end{bmatrix}. \tag{41}$$

By the choice of matrix $H_{i,1}$ satisfying (21a), one has

$$\operatorname{Im}\begin{bmatrix} A-\alpha_i E & B \\ C & D \end{bmatrix} = \operatorname{Im}\begin{bmatrix} A-\alpha_i E & B H_{i,1} \\ C & D H_{i,1} \end{bmatrix}. \tag{42}$$

Therefore, relations (18a) and (42) imply (41), and hence (39). This completes the proof of (36), thus showing that $\mathcal{K}$ guarantees also requirement (b) to be satisfied for $\widetilde{\mathcal{S}}$, and completing the proof of item (ii).

It is stressed that Lemma 4 gives an algebraic form of the *internal model principle* for generalized systems, for the classes of disturbance functions and of reference signals under consideration, when they enter the plant $\mathcal{S}$ only partially. Namely, the meaning of relations (36) is that, for each $j = 1, 2, \ldots, \mu$, system $\widehat{\mathcal{S}}_j$ is able to generate free output responses equal to all the signals $V\,r(t)$, $r(t) \in \mathcal{R}$, if $j \le \mu_0$, and to the output responses of $\mathcal{S}$ to all the disturbance functions $d_j(t) \in \mathcal{D}_j$.

5. CONCLUSIONS

The tracking and regulation problem has been considered for a class of generalized systems, in case of exponential reference signals and of disturbance functions. First, the notions of steady-state response and of blocking zero, which are classical for linear time-invariant systems, have been given for generalized systems. Then, the tracking and regulation problem has been stated and solved for the class of generalized systems under consideration, giving a general design procedure. As a corollary of the effectiveness proof of the design procedure, an algebraic version of the internal model principle has been stated for generalized systems.

Future work will regard the possibility of taking into account the presence of structured perturbations (affecting the plant under consideration) in the design of the compensator.

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