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ROBUST EXPONENTIAL STABILITY OF A CLASS OF NONLINEAR SYSTEMS[1]

VOJTECH VESELÝ AND DANICA ROSINOVÁ

The paper addresses the problem of design of a robust controller for a class of nonlinear uncertain systems to guarantee the prescribed decay rate of exponential stability. The bounded deterministic uncertainties are considered both in a studied system and its input part. The proposed approach does not employ matching conditions.

1. INTRODUCTION

Dynamic systems with bounded uncertainties have been widely used to model physical systems. During the last two decades numerous papers dealing with a design of robust control schemes to stabilize such systems have been published, Brogliato and Neto [2], Corles [4], Leitman [14], Zhihua Qu and Dorsey [19], Zhihua Qu [20], Zhiming Gong et al [18], Niculescu et al [15]. The other approaches to the design of robust controller can be found in Jury [9], Poolla et al [16], Kozak [11], Konstantopoulos and Antsaklis [10], Prokop and Corriou [17] and others. In the case of nonlinear systems, significant results on analysis and design of robust control have been obtained. Various approaches have been studied for nonlinear systems, the Lyapunov function method being of central importance.

In this paper we consider the issue of robust exponential stability of a class of nonlinear and linear continuous-time systems and linear discrete-time uncertain systems using the direct Lyapunov method. These systems are described by a generalized dynamical model, where the upper bounds on both input and system uncertainties are supposed to be known. The robust exponential stability of nonlinear and linear systems plays an important role to guarantee robustness and dynamic performance quality of a system, [5, 15, 18]. In this paper the analytical method for a design of nonlinear control systems that guarantees the robust exponential stability is proposed. It is based on the semilinear representation, [1], of a nonlinear system, definition of exponential stability of generalized dynamical systems, [12], and the use

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of the polynomial formalism to shape the "eigenvalues" of nonlinear systems, [7, 8]. For a linear case the Lyapunov function method provides a useful tool for the design of static output feedback controller that guarantees the robust exponential stability of linear continuous-time systems. For the discrete-time systems the static state feedback controller is studied. In this paper the so called Chua's circuit, [3], which has become very popular recently as a benchmark example of a simple third order nonlinear system is used as an example for a design of controller that guarantees the exponential stability of the controlled uncertain system.

The paper is organized as follows. In Section 2 the problem formulation and some preliminary results are brought. The main results for nonlinear system are given in Section 3, for linear continuous-time systems in Section 4 and for linear discrete-time systems in Section 5. In Section 6 the obtained theoretical results are applied to Chua's circuit and the corresponding simulation results are provided.

2. PROBLEM FORMULATION AND PRELIMINARIES

Consider the following generalized uncertain dynamic system

$$\dot{x} = f(x) + \delta f(x) + (b(x) + \delta b(x))\, u \tag{1}$$

where $x \in R^n$ is the state, $u \in R^m$ is the control, $\delta f(x), f(x) : R^n \to R^n$, $\delta b(x)$, $b(x) : R^n \to R^{n\times m}$ are continuous and uniformly bounded functions of a class C^k, $k > 0$ is sufficiently large, differentiable on the set $R_\rho \times R^m$ with respect to system variables x, u, $R_\rho = \{x \in R^n : \|x\| \le \rho\}$, $\rho > 0$ and $f(0) = \delta f(0) = 0$, $\delta b(0) + b(0) \neq 0$, $f(x)$ and $b(x)$ are supposed to be known. In the next developments we employ Lemma 1, [1].

Lemma 1. The following equality is true for a nonlinear function $f(x)$

$$f(x) = A(x)\, x \tag{2}$$

where $A(x) \in R^{n\times n}$ is a nonlinear matrix with the entries

$$\begin{aligned} a_{ij}(x) &= \int_0^1 f_{ij}(x_1, x_2, \dots, x_{j-1}, \theta x_j, 0, 0, \dots, 0)\, \mathrm{d}\theta \\ f_{ij} &= \frac{\partial f_i}{\partial x_j}, \quad i, j \in N = \{1, 2, \dots, n\} \end{aligned}$$

and $x \in R_\rho$.

Owing to Lemma 1 the generalized uncertain dynamic system (1) can be rewritten in the form

$$\dot{x} = (A(x) + \delta A(x))\, x + (b(x) + \delta b(x))\, u. \tag{3}$$

The corresponding system without uncertainty, referred to as nominal model, is

$$\dot{x} = A(x)\, x + b(x)\, u \tag{4}$$

where $A(x)$ and $b(x)$ are supposed to be known. The unknown matrices $\delta A(x)$ and $\delta b(x)$ represent system and input uncertainties respectively. Assume that the upper bounds on these uncertainties are known. In other words, there exist two known symmetric positive definite constant matrices $Q \in R^{n\times n}$, $R \in R^{m\times m}$ and constants $\gamma_a \geq 0$ and $\gamma_b \geq 0$ such that

$$\begin{aligned} \delta A^T(x)\,\delta A(x) &\leq \gamma_a Q \\ \delta b^T(x)\,\delta b(x) &\leq \gamma_b R, \quad x \in R_\rho. \end{aligned} \tag{5}$$

Assumption (5) provides the upper bounds on uncertainties. Let us note that the uncertainties can be nonlinear and fast time-varying and no statistical information about them is assumed. Let the control algorithm be

$$u = -k^T(x)\,x \tag{6}$$

where $k(x) \in R^{n\times m}$. Then the closed loop uncertain system is

$$\dot{x} = (A(x) - b(x)\,k^T(x))\,x + (\delta A(x) - \delta b(x)\,k^T(x))\,x. \tag{7}$$

The corresponding nominal closed-loop system is

$$\dot{x} = (A(x) - b(x)\,k^T(x))\,x = A_c(x). \tag{8}$$

Before the exact statement of the studied problem let us introduce the following preliminaries.

Definition 1. The uncertain closed-loop system (7) is said to be robustly exponentially stable with a decay rate $\alpha > 0$ if there exists a Lyapunov function $V(x) : R^n \to R^+$, $V(x) \in C^1$, for the nominal closed-loop system (8) so that for all initial conditions $x(t_0) = x_0 \in R_\rho$ and for all admissible uncertainties given by (5) the following inequality holds along the solution of (7)

$$\frac{\mathrm{d}V(x)}{\mathrm{d}t} \leq -\alpha V(x) \tag{9}$$

with

$$\begin{aligned} c_1\|x\|^2 \leq V(x) &\leq c_2\|x\|^2 \qquad (10) \\ \dot{V}(x) &\leq -c_3\|x\|^2 \\ \|\mathrm{grad}\,V(x)\| &\leq c_4\|x\| \qquad (11) \end{aligned}$$

where

$$(\mathrm{grad}\,V(x))^T = \left[\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}\right]$$

and $\|\cdot\|$ denotes the standard Euclidean norm and $c_i > 0$, $i = 1, 2, 3, 4$.

Remark. A solution of (9) is

$$V(x,t) \le V(x_0,t_0)\, e^{-\alpha t}.$$

Using the inequality (11) for both $V(x,t)$ and $V(x_0,t_0)$

$$||x(t)|| \le \sqrt{\frac{c_2}{c_1}}\, ||x_0|| e^{-\alpha \frac{t}{2}}$$

for all $x \in R_\rho$. The last inequality implies that if the closed-loop system (7) is exponentially stable with a decay rate $\alpha > 0$, then $||x(t)||$ exponentially converge with a decay rate $\frac{\alpha}{2}$ for all $x \in R_\rho$. Let us finally state the problem studied in this paper.

Problem. Find the control algorithm (6) so that the closed-loop system (7) with the bounded uncertainties (5) is robustly exponentially stable with a prescribed decay rate α.

3. ROBUST CONTROL SYSTEM DESIGN

In this section the analytical method with polynomial formalism is used for a design of the state feedback nonlinear controller (6) that guarantees the exponential stability of the uncertain closed-loop system (7) with the bounded uncertainties (5). The nonlinear controller design procedure is divided into two steps. Firstly the matrix $k(x)$ is determined for the nominal model and decay rate α_1. The sufficient stability conditions for the closed-loop system (7) with uncertainties (5) guaranteeing the robust exponential stability with decay rate α are investigated in the second step. In the next developments for this section we assume $m = 1$.

Consider the nominal closed-loop system (8). The eigenvalues λ_i, $i = 1,2,\dots,n$ of the matrix $A_c(x)$ are given by the equation

$$A_c(\lambda, x) = \det(\lambda I - A_c(x)) = 0. \tag{12}$$

Let the characteristic polynomial $A_c(\lambda, x)$ is equal to the prescribed polynomial $A_d(\lambda, x)$

$$A_c(\lambda, x) = A_d(\lambda, x) \tag{13}$$

where

$$A_d(\lambda, x) = \Pi_{i=1}^{n}(\lambda - \lambda_{di}(x)) \tag{14}$$

and $\lambda_{di}(x)$, $i = 1,2,\dots,n$ are complex functions with $-\sigma \le \mathrm{Re}\,(\lambda_{di}(x)) \le -\varepsilon < 0$ for all $x \in R_\rho$, and $0 < \sigma < \infty$, $\varepsilon > 0$. The sufficient stability conditions of (12) with (13) are given in the following theorem, [7].

Theorem 1. For the system (8) with $x \in R_\rho$ and initial state $x(t_0) = x_0 \in R_\rho$ the inequality

$$||x(x_0, t)|| \le c(x_0)\, e^{-\varepsilon(t-t_0)} \tag{15}$$

holds if the following sufficient conditions are satisfied:

- There exist the partial derivatives

$$\frac{\partial a_{cij}(x)}{\partial x_k}, \quad i,j,k \in N, \quad A_c = \{a_{cij}\}_{n \times n}.$$

- There exists a vector $b_l(x) \in R^n$ with the derivatives $\frac{\partial b_{li}(x)}{\partial x_j}$, $i,\, j \in N$ and the following condition holds for $b_l(x)$.

$$Q(x) = \det[b_l A_c b_l \ldots A_c^{n-1} b_l] \neq 0.$$

- The polynomial equation (13) holds with $\lambda_{di} \neq \lambda_{dj}$, $i,\, j \in N$, $i \neq j$, $|\lambda_{di}(x)| < \infty$.

- The following condition is satisfied

$$\sup_{x \in R_\rho} \frac{n \cdot \mathrm{tr}(P_1(x))}{\det(P_1(x))} \leq \kappa < \infty$$

  where $\mathrm{tr}(\cdot)$ denotes the trace of the corresponding matrix, $P_1(x) = P(x)\, P(x)^*$, $P(x) = Q(x)\, N(x)$,

$$N(x) = \begin{bmatrix} \eta_1(x) & \eta_2(x) & \ldots & \eta_{n-1}(x) & 1 \\ \eta_2(x) & \eta_3(x) & \ldots & 1 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix}$$

$$A_c(\lambda, x) = \det(\lambda I - A_c) = \sum_{i=1}^{n} \eta_i(x)\, \lambda^i. \tag{16}$$

Proof of Theorem 1 is given in Gaiduk [7]. □

In the following it is assumed that the conditions of Theorem 1 are satisfied. From (15) it can be seen that by choosing functions $\lambda_{di}(x)$, $i \in N$ it is possible to achieve stability and desired quality of the nonlinear closed-loop system (8). In the next paragraphs our main results on exponential stability of the nominal as well as uncertain system are provided.

Theorem 2. The dynamic system (8) is exponentially stable with a decay rate α_1 for some control gain vector if the following sufficient conditions hold:

$$\det[b(x)A(x)b(x)\ldots A^{n-1}(x)b(x)] \neq 0, \quad x \in R_\rho. \tag{17}$$

- There exist both complex functions $\gamma_{di}(x)$, $i \in N$ with $-\sigma \leq \mathrm{Re}(\gamma_{di}(x)) \leq -\varepsilon < 0$ for all $x \in R_\rho$ and a matrix $H_d(x) \in R^{n \times n}$ with characteristic polynomial

$$H_d(\lambda, x) = \Pi_{i=1}^{n}(\lambda - \gamma_{di}(x)) \tag{18}$$

  such that the matrix

$$H_d(x)^T + H_d(x) \tag{19}$$

  has all eigenvalues in left half plane.

- There exists a candidate Lyapunov function $V(x) : R^n \rightarrow R_+$ for nominal closed-loop system (8) satisfying the conditions (10) and (11).
- The conditions of Theorem 1 are satisfied.

P r o o f . To prove Theorem 2 it is sufficient to provide the existence of a Lyapunov function conforming with Definition 1. Consider a candidate Lyapunov function of a nominal system (8), $V(x) : R^n \rightarrow R_+$, $V(x) \in C^1$. Owing to Lemma 1 the candidate Lyapunov function $V(x)$ can be written in the form

$$V(x) = x^T W(x)\, x \tag{20}$$

where $W(x) \in R^{n\times n}$ is a symmetric positive definite matrix $\forall\, x \in R_\rho$. To guarantee the exponential stability of the closed-loop system (8) with a decay rate $\alpha_1 > 0$ let us require that the following inequality holds:

$$\frac{\mathrm{d}V(x)}{\mathrm{d}t} \leq -\alpha_1 V(x). \tag{21}$$

For the time derivative of (20) along the solution of (8)

$$\frac{\mathrm{d}V}{\mathrm{d}t} = (\mathrm{grad}\, V)^T (A(x) - b(x)\, k^T(x))\, x = x^T (F(x) - B(x)\, k^T(x))\, x \tag{22}$$

where

$$\begin{aligned} \mathrm{grad}\, V &= D(x)\, x \\ F(x) &= D^T(x) A(x) \in R^{n\times n} \\ B(x) &= D^T(x) b(x) \in R^n. \end{aligned}$$

Substituting (22) into left hand side of (21) after simple manipulation

$$x^T [H(x) + H(x)^T]\, x \leq 0 \tag{23}$$

where

$$H(x) = F(x) + \alpha_1 W(x) - B(x)\, k^T(x).$$

Now, let us consider the characteristic polynomial of $H(x)$, denoted as $H(\lambda, x)$, to be equal to the prescribed one

$$H(\lambda, x) = H_d(\lambda, x). \tag{24}$$

When functions $\gamma_{di}(x)$, $i \in N$ are chosen under the second condition of Theorem 2, the polynomial equation (24) has a solution for entries of the vector $k(x)$ if and only if the first condition of Theorem 2 is satisfied. This completes the proof. □

Let us now provide the sufficient conditions to guarantee the exponential stability of uncertain closed-loop system (7) with uncertainty bounds (5).

Theorem 3. The uncertain dynamic system (7) with bounded uncertainties (5) and $x \in R_\rho$, is exponentially stable with a decay rate α if the following conditions are satisfied:

- Conditions of Theorem 1 and Theorem 2.
- There exist positive constants $\alpha_1, \epsilon_1, \epsilon_2$ such that the symmetric matrix $M(x)$ is negative semidefinite/definite,

$$M(x) = -2(\alpha_1 - \alpha)\, W(x) + \epsilon_1 Q + D(x)^T D(x)\,(\gamma_a \epsilon_1^{-1} + \gamma_b \epsilon_2^{-1}) + \epsilon_2 k(x)\, Rk(x)^T \tag{25}$$

  where $V(x) = x^T W(x)\, x$ is a candidate Lyapunov function of the nominal system (8) and $\operatorname{grad} V(x) = D(x)\, x$.

Proof. Similarly to Theorem 2 it is sufficient to prove the existence of a Lyapunov function $V(x)$ such that

$$\frac{\mathrm{d}V(x)}{\mathrm{d}t} \leq -\alpha V(x).$$

Consider a Lyapunov function candidate in the form (20). Assume that $k(x)$ is the vector guaranteeing that (21) holds (the existence of $k(x)$ is given by first condition). For the time derivative of $V(x)$ on the solution of uncertain system (7) then

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} \; \leq \; & -\alpha_1 V + \frac{1}{2} x^T (D(x)^T \delta A(x) + \delta A(x)^T D(x) \\ & -D(x)^T \delta b(x)\, k(x)^T - k(x)\, \delta b(x)^T D(x))\, x \end{aligned} \tag{26}$$

employing the equality

$$X^T Y + Y^T X = \left(\frac{Y}{\sqrt{\sigma_\gamma}} + X\sqrt{\sigma_\gamma}\right)^T \left(\frac{Y}{\sqrt{\sigma_\gamma}} + X\sqrt{\sigma_\gamma}\right) - \frac{Y^T Y}{\sigma_\gamma} - X^T X \sigma_\gamma$$

for

$$D^T \delta A + \delta A^T D \quad \text{and} \quad -(D^T \delta b k^T + k \delta b^T D)$$

and inequality

$$(B + C)^T (B + C) \leq (\epsilon + 1)\, B^T B + (\epsilon^{-1} + 1)\, C^T C, \quad \epsilon > 0, \tag{27}$$

after some manipulation we obtain

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} \; \leq \; & -\alpha V - (\alpha_1 - \alpha)\, V + \frac{1}{2} x^T \left\{ (\epsilon_1 + 1) \frac{\delta A^T \delta A}{\gamma_a} + (\epsilon_1^{-1} + 1)\, D^T D \gamma_a \right. \\ & - \frac{\delta A^T \delta A}{\gamma_a} - D^T D \gamma_a + (\epsilon_2 + 1) \frac{k \delta b^T \delta b k^T}{\gamma_b} + (\epsilon_2^{-1} + 1)\, D^T D \gamma_b \\ & \left. - \frac{k \delta b^T \delta b k^T}{\gamma_b} - D^T D \gamma_b \right\} x \end{aligned}$$

or

$$\frac{\mathrm{d}V(x)}{\mathrm{d}t} \leq -\alpha V(x) + \frac{1}{2} x^T M(x)\, x \tag{28}$$

where $M(x)$ is given by (25). This completes the proof. □

4. ROBUST CONTROL SYSTEM: LINEAR CONTINUOUS–TIME CASE

Consider the linear continuous-time system

$$\dot{x} = (A + \delta A)\,x + (B + \delta B)\,u$$

$$y = Cx \tag{29}$$

where $y \in R^l$ is the output vector of the system, A, B, C are real constant matrices of appropriate dimensions representing a nominal system. The unknown matrices δA, δB, of the corresponding dimensions, are piecewise continuous and bounded at every time and represent system and input uncertainties. They can be nonlinear and fast time-varying, no statistical information is assumed to be known. The following assumptions for the system (29) are considered.

Assumption 1. The uncertainties δA, δB satisfy the inequalities

$$\begin{array}{rcl} \delta A^T \delta A & \leq & \gamma_a Q_0 \\ \delta B^T \delta B & \leq & \gamma_b R \end{array} \tag{30}$$

where $Q_0 \in R^{n\times n}$, $R \in R^{m\times m}$ are symmetric positive definite matrices, γ_a, γ_b are positive constants.

Assumption 2. The system (29) is output feedback stabilizable. The necessary and sufficient conditions for output feedback stabilizability are given in Kučera and DeSouza [13].

The problem studied in this section can be stated as follows. Under Assumptions 1, 2 find the static output feedback matrix K

$$u = KCx \tag{31}$$

such that the resulting closed-loop system

$$\dot{x} = [A + \delta A + (B + \delta B)\,KC]\,x \tag{32}$$

is exponentially stable with a prescribed decay rate $\alpha > 0$. Sufficient conditions for exponential stabilizability of the uncertain system (32) are provided in Theorem 4.

Theorem 4. The uncertain closed-loop system (32) is output feedback robustly exponentially stabilizable with a prescribed decay rate $\alpha > 0$ if the following sufficient conditions hold:

- Assumption 1, 2.
- There exist positive constants $\alpha_1 \geq \alpha$, ϵ_1, ϵ_2 such that the symmetric matrix M is negative semidefinite/definite,

$$M = -(\alpha_1 - \alpha)\,P - Q_1 - (1-\epsilon_1)\,Q_0 + PP(\gamma_a \epsilon_1^{-1} + \gamma_b \epsilon_2^{-1}) - (1-\epsilon_2)\,C^T K^T RKC \tag{33}$$

where $Q_1 = Q_1^T > 0$, $Q = Q_0 + Q_1$, matrix P is calculated from the Lyapunov matrix equation

$$\left(A + \frac{\alpha_1}{2} I + BKC\right)^T P + P\left(A + \frac{\alpha_1}{2} I + BKC\right) + Q + C^T K^T RKC = 0. \tag{34}$$

The proof is omitted for brevity.

5. ROBUST CONTROL SYSTEM: LINEAR DISCRETE-TIME CASE

Consider the linear uncertain discrete-time system

$$x(t+1) = (F_d + \delta F_d)\, x(t) + (G_d + \delta G_d)\, u(t) \tag{35}$$

where $x(t) \in R^n$, $u(t) \in R^m$ are the state and control vectors respectively, $F_d \in R^{n\times n}$, $G_d \in R^{n\times m}$ are constant matrices. The unknown matrices δF_d, δG_d of appropriate dimensions represent system deterministic uncertainties which are upper-bounded and can be time-varying.

Assumption 3. The uncertainties δF_d, δG_d satisfy the inequalities

$$\begin{aligned} \delta F_d^T \delta F_d &\leq \gamma_{ad} Q_{ad} \\ \delta G_d^T \delta G_d &\leq \gamma_{bd} Q_{bd} \end{aligned} \tag{36}$$

where Q_{ad}, Q_{bd} are symmetric positive definite matrices, γ_{ad}, $\gamma_{bd} \geq 0$ are constants.

Assumption 4. The pair (F_d, G_d) is controllable.

Definition 1 for exponential stability of continuous-time systems can be modified to discrete-time system in an obvious way.

Definition 2. The uncertain system (35) is robustly exponentially stable with a decay rate $\alpha > 0$ if there exists a positive definite Lyapunov function $v(x) : R^n \to R_+$ and constants c_1, c_2, $c_3 > 0$ such that for all admissible uncertainties given by (36) the following inequalities hold along the solution of (35)

$$\Delta v(t) = v[x(t+1)] - v[x(t)] \leq -\alpha v[x(t)] \tag{37}$$

and

$$\begin{aligned} c_1 \|x(t)\|^2 &\leq v[x(t)] \leq c_2 \|x(t)\|^2 \\ \Delta v[x(t)] &\leq -c_3 \|x(t)\|^2. \end{aligned} \tag{38}$$

Consider now the constant state feedback control

$$u(t) = Kx(t), \quad K \in R^{m\times n}. \tag{39}$$

The corresponding uncertain closed-loop system is

$$x(t+1) = (F_d + G_d K + \delta F_d + \delta G_d K)\, x(t). \tag{40}$$

The aim of robust control design is to achieve the robust exponential stability of the closed-loop system (40) with a prescribed decay rate α. Sufficient conditions for robust stability of the system (40) are given in Theorem 5.

Theorem 5. The closed-loop system (40) is robustly exponentially stable with a decay rate $\alpha > 0$ if the following conditions are satisfied:

- Assumptions 3, 4.
- There exist a constant $\varphi_1 > 0$ and a symmetric positive definite matrix Q_d such that

$$M_d = -Q_d(1-\alpha) + (1+\varphi_1^{-1})\left[\varphi_2(1+\varphi_2)\,\gamma_{ad} Q_{ad} + (1+\varphi_2)\,\gamma_{bd} K^T Q_{bd} K\right] \tag{41}$$

  is negative semidefinite/definite matrix, where $\varphi_2 = \lambda_M(P_d)$, $\lambda_M(X)$ denotes the maximal eigenvalue of a symmetric matrix X, P_d is a solution to Lyapunov equation

$$\frac{1+\varphi_1}{1-\alpha}\,(F_d + G_d K)^T P_d (F_d + G_d K) - P_d = -Q_d. \tag{42}$$

Proof. The aim is to prove the existence of Lyapunov function $v(x)$ for the system (40) such that (37) holds. Consider

$$v[x(t)] = v(t) = x^T(t)\, P_d x(t)$$

where P_d is a positive definite matrix to be determined. Then

$$\Delta v(t) \leq x(t)^T \left\{[(F_d + G_d K) + (\delta F_d + \delta G_d K)]^{T'} \right.$$
$$\left. P_d[(F_d + G_d K) + (\delta F_d + \delta G_d K)] - P_d\right\} x(t)$$

or

$$\Delta v(t) \leq x(t)^T \left\{(1+\varphi_1)(F_d + G_d K)^T P_d (F_d + G_d K) - (1-\alpha)\, P_d - \alpha P_d \right.$$
$$\left. +(1+\varphi_1^{-1})(\delta F_d + \delta G_d K)^T P_d (\delta F_d + \delta G_d K)\right\} x(t) \tag{43}$$

where the inequality (27) was used in (43). Employing (27) on the last term in (43) we obtain

$$(\delta F_d + \delta G_d K)^T P_d (\delta F_d + \delta G_d K)$$
$$\leq (\varphi_2 + 1)\,\delta F_d^T P_d \delta F_d (\varphi_2^{-1} + 1)\, K^T \delta G_d^T P_d \delta G_d K$$
$$\leq \varphi_2(\varphi_2 + 1)\,\delta F_d^T \delta F_d + (\varphi_2 + 1)\, K^T \delta G_D^T \delta G_d K \tag{44}$$

for $\varphi_2 = \lambda_M(P_d)$. Substituting inequalities (36) and (44) into (43) gives

$$\Delta v(t) \le x(t)^T\{(\varphi_1+1)(F_d+G_dK)^T P_d(F_d+G_dK) - (1-\alpha)P_d$$

$$+(\varphi_1^{-1}+1)[\varphi_2(\varphi_2+1)\gamma_{ad}Q_{ad}+(\varphi_2+1)\gamma_{bd}K^TQ_{bd}K]\}x(t)-\alpha v(t) \quad (45)$$

or

$$\Delta v(t) \le -\alpha v(t) + x(t)^T M_d x(t) \quad (46)$$

where matrices M_d, P_d are determined by (41),(42) respectively. This completes the proof. □

To make M_d be a negative semidefinite/definite matrix, the robust control design strategy aims at determining feedback matrix K such that $\lambda_M(P_d)$ and $||K||$ are minimized, and the parameter φ_1 is to be optimized.

6. EXAMPLE: CONTROL OF CHUA'S CIRCUIT

The so called Chua's circuit has become popular recently as a benchmark example of simple third order nonlinear system exhibiting various forms of chaotic behaviour [6]. Its model in normalized form is given in [3].

$$\begin{aligned} \dot{x}_1 &= a_1(x_2 - x_1 - g(x)) + b_1u \\ \dot{x}_2 &= x_1 - x_2 + x_3 + b_2u \\ \dot{x}_3 &= -a_2x_2 - a_3x_3 + b_3u \end{aligned} \quad (47)$$

where

$$g(x) = M_0x_1 + 0.5(M_1 - M_0)\,[|x_1+1| - |x_1-1|]\,.$$

Owing to Lemma 1 the system (47) can be rewritten

$$\dot{x} = A(x)\,x + bu \quad (48)$$

where

$$\begin{aligned} A(x) &= \begin{bmatrix} -a_1(1+M_0+g_1(x)) & a_1 & 0 \\ 1 & -1 & 1 \\ 0 & -a_2 & -a_3 \end{bmatrix} \\ b^T &= [b_1 \quad b_2 \quad b_3]; \quad x^T = [x_1 \quad x_2 \quad x_3] \\ g_1(x) &= \frac{0.5(M_1-M_0)}{x_1}\,[|x_1+1| - |x_1-1|]\,. \end{aligned}$$

Choose the candidate Lyapunov function $V = 0.5x^Tx$ or $W(x) = 0.5I$ and $D(x) = I$. The characteristic polynomial of matrix $H(x)$, [8] is in the form

$$H(\lambda, x) = F_1(\lambda, x) + \sum\nolimits_{i=1}^{n} k_i(x)\nu_i(\lambda, x) \quad (49)$$

where

$$\begin{aligned}
F_1(\lambda, x) &= \det[\lambda I - (F(x) + \alpha_{11} W)], \quad \alpha_{11} = .5\alpha_1 \\
\nu_i(\lambda, x) &= c_i \mathrm{adj}\,(\lambda I - F_1(x))\, B(x) \;=\; \sum\nolimits_{j=0}^{n-1} r_{ij}\lambda^j \\
F_1(x) &= F(x) + \alpha_{11} W = A(x) + \alpha_{11} I \\
B(x) &= D^T(x) b, \quad F(x) \;=\; D^T(x) A(x) \;=\; A(x) \\
c_1 &= [1\ 0\ \ldots\ 0\ 0], \quad c_2 \;=\; [0\ 1\ 0\ \cdots\ 0], \ldots, \quad c_n \;=\; [0\ 0\ \cdots\ 0\ 1].
\end{aligned}$$

Let the polynomial (49) be equal to the prescibed polynomial $H_d(\lambda, x)$ (18), then the following algebraic equation is obtained for unknown vector $k(x)$

$$rk(x) = e \tag{50}$$

where $r = \{r_{ij}\}_{n \times n}$, $e^T = [e_0\ e_1\ \ldots\ e_{n-1}]$. The entries of e are the coefficients of the polynomial

$$H_d(\lambda, x) - H(\lambda, x) = e_0 + e_1\lambda + \ldots + e_{n-1}\lambda^{n-1}.$$

To demonstrate the ability of the proposed controller which guarantees the exponential stability of the Chua's circuit we carried out computer simulation for the following parameters of the nominal model

$$a_1 = 7, \quad a_2 = 14.286, \quad a_3 = 0, \quad M_0 = \frac{2}{7}, \quad M_1 = -\frac{1}{7}, \quad b_1 = 1, \quad b_2 = b_3 = 0$$

with initial state

$$x_{10} = -13, \qquad x_{20} = 20, \qquad x_{30} = -30$$

and

$$\gamma_{d1} = -1, \qquad \gamma_{d2} = -2, \qquad \gamma_{d3} = -3.$$

Simulation results are shown in Figures 1−4. Figure 1 shows the uncontrolled state trajectory, the transient process with the controller and $\alpha_1 = 10s^{-1}$ is given in Figure 2 and with $\alpha_1 = 20s^{-1}$ in Figure 3. $V(x)$ denotes the Lyapunov function. It is obvious that choosing different values of α_1 different behaviour of the controlled nominal system is achieved. Consider now the following uncertainties corresponding to (5), $\delta A(x) = \mathrm{diag}\{1\}_{3\times 3}, \delta b(x) = 0, \gamma_a = 1, \gamma_b = 0, Q = I, R = 0$. Then the matrix M from (25) is

$$M = -0.5\,(\alpha_1 - \alpha) I + \epsilon_1 I + \epsilon_1^{-1} I \leq 0.$$

For $\epsilon_1 = 1, \alpha_1 = 10, \alpha = 6$ is $M = 0$ and the investigated uncertain system is exponentially stable with a decay rate $\alpha = 6$. The dynamic behaviour of the controlled system with uncertainties is in Figure 4.

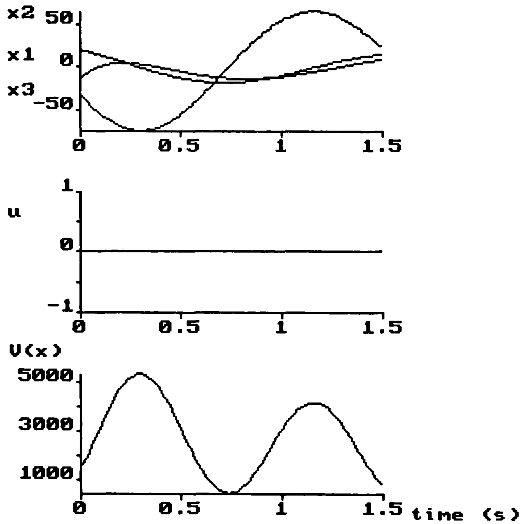

Fig. 1. Nominal system without control.

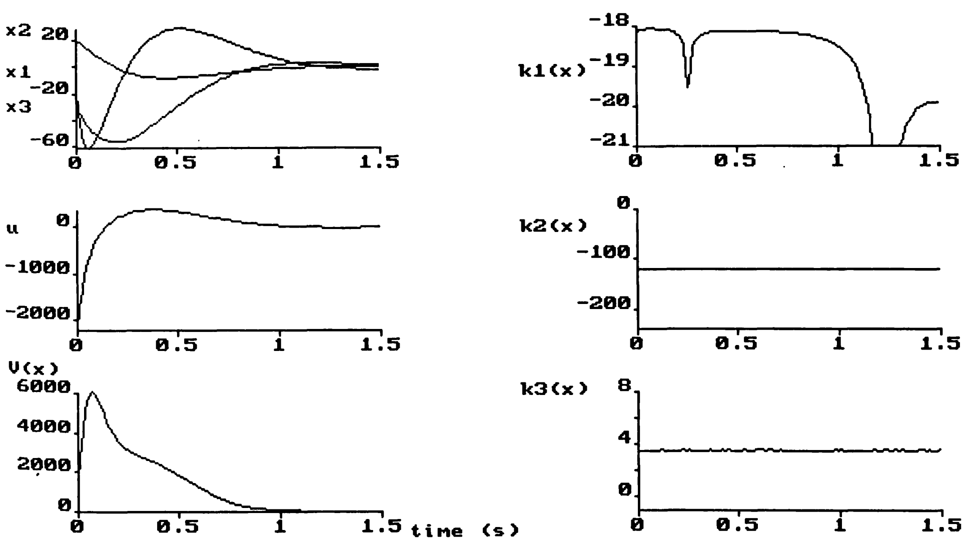

Fig. 2. Controlled nominal model with $\alpha_1 = 10s^{-1}$.

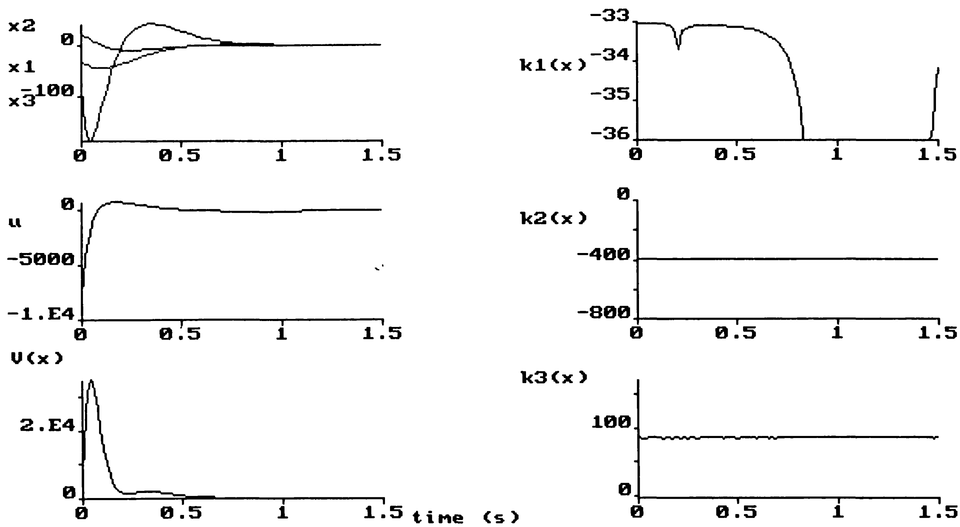

Fig. 3. Controlled nominal system with $\alpha_1 = 20s^{-1}$.

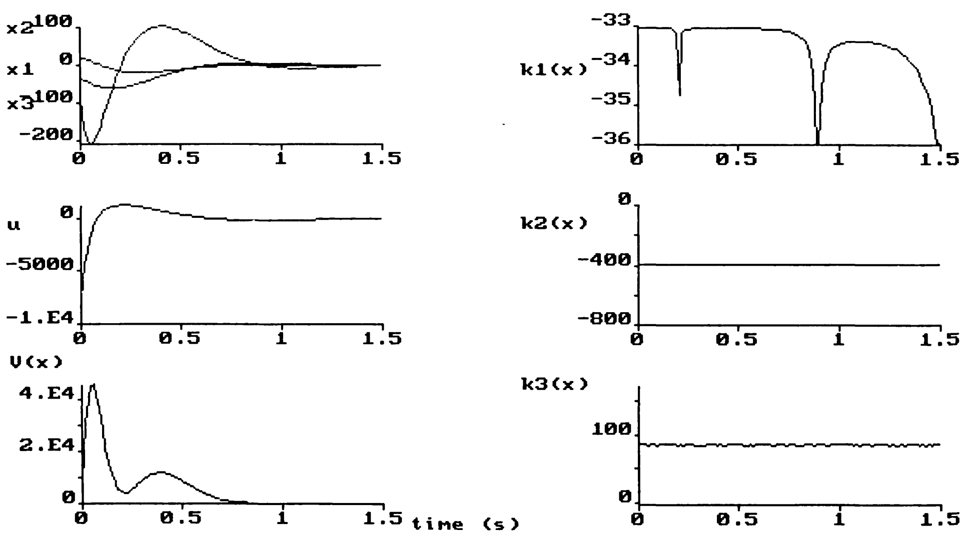

Fig. 4. Uncertain controlled system with $\alpha = 6s^{-1}$.

7. CONCLUSION

Robust exponential stability and robust control design for a class of nonlinear uncertain systems have been addressed. Sufficient conditions of exponential stability with a prescribed decay rate for nonlinear systems with bounded additive uncertainties are provided as the main result. The obtained stability conditions yield a constructive procedure of robust control design based on "shaping of eigenvalues". Linear continuous and discrete-time systems are studied as special cases. The obtained results for nonlinear robust control design are illustrated on example (so called Chua's circuit). Simulation demonstrates a promising behaviour of the designed robust controller.

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Prof. Ing. Vojtech Veselý, DrSc. and Ing. Danica Rosinová, CSc., Department of Automatic Control Systems, Faculty of Electrical Engineering and Information Technology, Slovak University of Technology, 812 19 Bratislava. Slovak Republic.
e-mails: vesely@kasr.elf.stuba.sk, rosinova@kasr.elf.stuba.sk