KYBERNETIKA – VOLUME 34 (1998), NUMBER 5, PAGES 535 – 543

MODIFIED MINIMAX QUADRATIC ESTIMATION OF VARIANCE COMPONENTS

VIKTOR WITKOVSKÝ

The paper deals with modified minimax quadratic estimation of variance and covariance components under full ellipsoidal restrictions. Based on the, so called, linear approach to estimation variance components, i. e. considering useful local transformation of the original model, we can directly adopt the results from the linear theory. Under normality assumption we can can derive the explicit form of the estimator which is formally find to be the Kuks–Olman type estimator.

1. INTRODUCTION

We consider a general linear model with variance and covariance components

$$\left(y, X\beta, V(\vartheta) = \sum_{i=1}^{p} \vartheta_i V_i\right), \tag{1}$$

where y denotes the n-dimensional vector of observations; X is a given $(n \times k)$-dimensional design matrix; $\beta \in \mathrm{R}^k$ is the vector of unknown first order parameters; $V(\vartheta)$ is an $(n \times n)$ variance-covariance matrix which is linear combination of unknown variance-covariance components $\vartheta = (\vartheta_1, \dots, \vartheta_p)'$, $\vartheta \in \Theta \subseteq \mathrm{R}^p$, and known symmetric matrices V_i, $i = 1, \dots, p$. In general, we do not assume normal distribution of the vector y. However, we assume the existence of the matrices of the third and fourth moments, i. e.,

$$E(\varepsilon \otimes \varepsilon\varepsilon') = \Phi, \quad E(\varepsilon\varepsilon' \otimes \varepsilon\varepsilon') = \Psi, \tag{2}$$

where $\varepsilon = y - X\beta$, and $\otimes$ stands for the Kronecker product of matrices.

We are interested in quadratic (plus constant) estimation of the linear function of the variance and covariance components $g'\vartheta$, g being a p-dimensional vector of constants.

The natural parameter space for variance-covariance components is $\Theta = \{\vartheta : V(\vartheta) \geq 0\}$. We assume that Θ includes an open set in R^p. Here, we consider the situation when the additional information is available in the form of restricted parameter space $\Theta_E \subseteq \Theta$ defined by

$$\Theta_E = \{\vartheta : (\vartheta - \bar{\vartheta})' H(\vartheta - \bar{\vartheta}) \leq 1\}, \tag{3}$$

where $\bar{\vartheta} \in \Theta$ is the given center point of the ellipsoid, and H is a given symmetric positive definite matrix. This is the case of full ellipsoidal restrictions on variance and covariance components.

If we denote by R a $(p \times p)$-matrix such that $H = R'R$, then the restricted parameter space Θ_E can be written also as $\Theta_E = \{\vartheta : R\vartheta \in \Omega\}$, Ω being a unit ball in R^p centered at $R\bar{\vartheta}$. Note that if $\Omega = R\bar{\vartheta}$, i.e. Ω contains only one point, then $\Theta_E = \{\vartheta : R(\vartheta - \bar{\vartheta}) = 0\}$, what is the case of linear restrictions. This special case is of no great interest because the obvious reparametrization leads directly to the solution by classical estimation methods. Anyway, it is possible to give the explicit quadratic estimators of the variance components function with linear restrictions.

The information given by ellipsoidal restrictions might be useful when we want to introduce prior information about variance and covariance components. This is frequently the case if we are working with repeated experiments, or experiments designed in several subsequent stages, etaps or epochs. For more details see e.g. Kubáček et al [4].

Maximum likelihood estimation and Bayes estimation are suitable approaches to find estimates which respect the restrictions. But unfortunately, in general, it is impossible to express such estimators in an algebraic form and thus it is also impossible to characterize their statistical properties.

In the present paper we consider modified minimax quadratic estimation of the linear function of variance and covariance components $g'\vartheta$ with ellipsoidal restrictions. The estimators are by definition quadratic (plus constant), but on the other hand, they do not necessarily fullfil given ellipsoidal restrictions. The good property is that such estimators improve the (modified) risk, if the additional information is given, if compared with the risk of the traditional unbiased quadratic estimators of variance components function, e.g. Rao's $MINQE(U, I)$.

Methodologically, we will adopt recent results of the linear theory by using the so called linear approach to estimation variance and covariance components. The results on linear minimax estimation (the linear theory) comes basically from the paper by J. Pilz [5], but see also J. Kozák [3], N. Gaffke and B. Heiligers [1], and B. Heiligers [2].

2. LINEAR APPROACH TO ESTIMATION VARIANCE COMPONENTS

In the present section we give a short survey of the linear approach to estimation of variance and covariance components. For more details see J. Volaufová and V. Witkovský [14], J. Volaufová [13], and also F. Pukelsheim [6], and S.R. Searle et al [12].

The linearized model is a linear model based on the second tensor power of suitably chosen maximal invariant z, i.e. on the hypervector $(z \otimes z) = vec(zz')$. Here $vec(A)$ denotes the vector which is composed from the columns of the matrix A arranged one below the other.

The following notation will be used: Let $\vartheta_0 \in \Theta$ and $\Psi_0 \in \{\Psi : \Psi(\vartheta_0)\}$ be fixed. Then $V_0 = V(\vartheta_0) = \sum_{i=1}^{p} \vartheta_{0i} V_i$; $T_0 = V_0 + XX'$; U_0 is defined by the equations $T_0^+ = U_0' U_0$ and $U_0 T_0 U_0' = I$; $M_0 = I - U_0 X (X' T_0^+ X)^+ X' U_0'$; and F_0 is defined by the equations $M_0 = F_0F_0'$ and $F_0'F_0 = I$. A^+ denotes the Moore–Penrose g-inverse of the matrix A.

Then the vector $z = F_0'U_0y$ stands for the maximal invariant with respect to the group of translations $y \mapsto y + X\beta$, for all $\beta \in \mathrm{R}^k$. In the next we will consider the N-dimensional hypervector $(z \otimes z) = vec(zz')$ for which the following holds true:

$$E(vec(zz')) \equiv Q_0\vartheta = \left[vec(F_0'U_0V_1U_0'F_0) : \ldots : vec(F_0'U_0V_pU_0'F_0)\right]\vartheta, \quad (4)$$

$$\begin{aligned} Var(vec(zz')) \equiv \Sigma(\vartheta_0,\vartheta,\Psi) &= \left(F_0'U_0 \otimes F_0'U_0\right)\Psi\left(F_0'U_0 \otimes F_0'U_0\right)' \quad (5) \\ &\quad -\left(vec(F_0'U_0V(\vartheta)U_0'F_0)\right)\left(vec(F_0'U_0V(\vartheta)U_0'F_0)\right)'. \end{aligned}$$

Moreover, under normality assumptions

$$Var(vec(zz')) \equiv \Sigma(\vartheta_0,\vartheta) = \left(F_0'U_0 \otimes F_0'U_0\right)(I+T_n)(V(\vartheta)\otimes V(\vartheta))\left(F_0'U_0 \otimes F_0'U_0\right)', \quad (6)$$

where T_n is such a matrix that $T_n\, vec(A) = vec(A')$ for arbitrary $(n \times n)$-matrix A. For more details see C. R. Rao and J. Kleffe [10], pp. 52–53.

For fixed ϑ_0 and Ψ_0 we will denote $\Sigma_0 = \Sigma(\vartheta_0,\vartheta_0,\Psi_0)$. Under normality assumptions we get $\Sigma_0 = \Sigma(\vartheta_0,\vartheta_0)$. Moreover, for the sake of simplicity, we will assume that $\mathcal{R}(Q_0) \subseteq \mathcal{R}(\Sigma_0)$, where $\mathcal{R}(A)$ denotes the linear space generated by the columns of A.

The linear model

$$\left(vec(zz'), Q_0\vartheta, \Sigma_0\right) \quad (7)$$

is our working approximation to the true model

$$\left(vec(zz'), Q_0\vartheta, \Sigma\right), \quad (8)$$

locally at the point (ϑ_0, Ψ_0). The model (7) will be denoted as a linearized model.

Finally, we mention the important property of the model (7). J. Volaufová and V. Witkovský [14] proved that under the assumption of normality distribution of the original vector of observations y the following identity holds true:

$$\Sigma_0 Q_0 = 2Q_0, \quad (9)$$

what means that $\mathcal{R}(\Sigma_0Q_0) \subseteq \mathcal{R}(Q_0)$ holds true as well. This is, however, sufficient condition for OLSE (the ordinary least squares estimator) to be identical with *BLUE* (the best linear unbiased estimator) in the linear model (7). Because the model (7) coincides with the true model only locally at the point (ϑ_0, Ψ_0), the *BLUE*, in general, is in fact only the *WLSE* (the weighted least squares estimator). See G. Zyskind [15], C. R. Rao and S. K. Mitra [11], and C. R. Rao [9].

Consider now the linear model (7). The linear function $g'\vartheta$ is linearly and unbiasedly estimable if and only if $g \in \mathcal{R}(Q_0'Q_0)$. The *OLSE* and the *BLUE* of the unbiasedly estimable function $g'\vartheta$ in the model (7) are:

$$\widehat{g'\vartheta}_{OLSE} = g'(Q_0'Q_0)^- Q_0'\, vec(zz'), \quad (10)$$

$$\widehat{g'\vartheta}_{BLUE} = g'(Q_0'\Sigma_0^- Q_0)^- Q_0'\Sigma_0^-\, vec(zz'). \quad (11)$$

Considering the linear restrictions $R(\vartheta - \bar{\vartheta}) = 0$, $g'\vartheta$ is linearly and unbiasedly estimable if and only if $g \in \mathcal{R}(Q_0'Q_0 + R'R)$. If we assume $\mathcal{R}(R') \subseteq \mathcal{R}(Q_0')$, so $g \in \mathcal{R}(Q_0'Q_0)$, then we get

$$\widehat{\widehat{g'\vartheta}}_{OLSE} = g'\left(M_{R'}Q_0'Q_0M_{R'}\right)^{+}\left(Q_0'\mathrm{vec}(zz') - Q_0'Q_0\bar{\vartheta}\right) + g'\bar{\vartheta}, \tag{12}$$

$$\widehat{\widehat{g'\vartheta}}_{BLUE} = g'\left(M_{R'}Q_0'\Sigma_0^{-}Q_0M_{R'}\right)^{+}\left(Q_0'\Sigma_0^{-}\mathrm{vec}(zz') - Q_0'\Sigma_0^{-}Q_0\bar{\vartheta}\right) + g'\bar{\vartheta}, \tag{13}$$

where $M_{R'} = I - R^{+}R = I - R'(RR')^{-1}R$.

Note that under normality assumptions, because of the identity $\Sigma_0 Q_0 = 2Q_0$, in the model (7), the *OLSE* is equivalent with the *BLUE* (*WLSE*).

On the other hand, the Rao's *MINQE*(U, I), the minimum norm quadratic estimator, unbiased and invariant, of $g'\vartheta$, $g \in \mathcal{R}(K_0)$, depending on a priori chosen point $\vartheta_0 \in \Theta$, is given as

$$\widehat{g'\vartheta}_{MINQE} = g'K_0^{-}q_0, \tag{14}$$

where the $(p \times p)$-matrix $K_0 = Q_0'Q_0$ and the p-dimensional vector of quadratics $q_0 = Q_0'\mathrm{vec}(zz')$. The elements of K_0 can be expressed as:

$$\{K_0\}_{i,j} = tr\left((MV_0M)^{+}V_i(MV_0M)^{+}V_j\right), \quad i,j = 1,\ldots,p, \tag{15}$$

where $M = I - XX^{+} = I - X(X'X)^{-}X$ and $V_0 = V(\vartheta_0)$. The elements of the vector of quadratics q_0 are:

$$q_{0i} = y'(MV_0M)^{+}V_i(MV_0M)^{+}y, \quad i = 1,\ldots,p. \tag{16}$$

The equations (10) and (14) shows that the *OLSE* of the function $g'\vartheta$ in the model (7) is identical with Rao's *MINQE*(U, I).

For more details see C. R. Rao [7], C. R. Rao [8], and C. R. Rao and J. Kleffe [10].

3. LINEAR MINIMAX ESTIMATION IN THE LINEARIZED MODEL

In this section we will derive the Θ_E-*MILE* – the minimax linear estimator of $g'\vartheta$, the linear function of variance and covariance components, subject to constraints given by Θ_E, in the the linearized model (7), i.e. the approximation to the true model which depends on prior choice of (ϑ_0, Ψ_0). Under given setup we can directly follow the results of the linear theory given by J. Pilz [5], and N. Gaffke and B. Heiligers [1].

We consider the linearized model (7) with the restrictions (3). Moreover, we will assume that Σ_0 and $Q_0'\Sigma_0^{-1}Q_0$ are full ranked matrices. In the model (7), we are interested in the linear plus constant estimation of the linear function $g'\vartheta$, i.e. $\widehat{g'\vartheta} \in LCE$, where LCE is the class of estimators given by

$$LCE = \left\{\widehat{g'\vartheta} : \widehat{g'\vartheta} = g'L\mathrm{vec}(zz') + g'l, L \in \mathrm{R}^{p\times N}, l \in \mathrm{R}^{p}\right\}. \tag{17}$$

The goodness of an estimator will be evaluated by the risk function based on the mean squared error function:

$$R(g'\vartheta, \widehat{g'\vartheta}) = E\left[(\widehat{g'\vartheta} - g'\vartheta)^2\right] = g'L\Sigma_0L'g + \left(g'(I - LQ_0)\,\vartheta - g'l\right)^2. \tag{18}$$

According to J. Pilz [5], Lemma 1, it is sufficient to consider the class of estimators LCE with $l = (I - LQ_0)\bar{\vartheta}$, so

$$R(g'\vartheta, \widehat{g'\vartheta}) = g'L\Sigma_0 L'g + \left(g'(I - LQ_0)(\vartheta - \bar{\vartheta})\right)^2. \tag{19}$$

The minimax linear estimator of $g'\vartheta$ subject to Θ_E, i.e. the Θ_E-*MILE* in the model (7), is defined as $\widehat{g'\vartheta}_{MILE}$, such that

$$\sup_{\vartheta\in\Theta_E} R\left(g'\vartheta, \widehat{g'\vartheta}_{MILE}\right) = \inf_{\widehat{g'\vartheta}\in LCE} \sup_{\vartheta\in\Theta_E} R(g'\vartheta, \widehat{g'\vartheta}). \tag{20}$$

J. Pilz [5] established the duality between linear minimax estimation and least favorable prior distribution for linear Bayes estimation: For arbitrary prior distribution P on Θ_E, which is characterized by the moment matrix $N_P = \int_{\vartheta\in\Theta_E}(\vartheta - \bar{\vartheta})(\vartheta - \bar{\vartheta})'P(d\vartheta)$, we can derive, according to Pilz, the Bayes linear estimator $\widehat{g'\vartheta}_{BLE}$ of $g'\vartheta$ as

$$\widehat{g'\vartheta}_{BLE} = g'N_P Q_0'(Q_0 N_P Q_0' + \Sigma_0)^{-1}(vec(zz') - Q_0'\bar{\vartheta}) + g'\bar{\vartheta}. \tag{21}$$

The estimator minimizes the Bayes risk function

$$r(g'\vartheta, \widehat{g'\vartheta}) = E\left[R(g'\vartheta, \widehat{g'\vartheta})\right] = g'L\Sigma_0 L'g + g'(I - LQ_0)N_P(I - LQ_0)'g. \tag{22}$$

So, the $\widehat{g'\vartheta}_{BLE}$ is defined by $L = N_P Q_0'(Q_0 N_P Q_0' + \Sigma_0)^{-1}$, and directly we get the Bayes risk

$$r(g'\vartheta, \widehat{g'\vartheta}_{BLE}) = g'N_P g - g'N_P Q_0'(Q_0 N_P Q_0' + \Sigma_0)^{-1} Q_0 N_P g. \tag{23}$$

The Θ_E-*MILE* is of the form (21) with the moment matrix N_* of the least favorable distribution on Θ_E. The problem is to find the explicit form of N_*. This is not easy in general – for arbitrary compact and symmetric set of restrictions. However, in the case of ellipsiod restrictions Θ_E it was found by Pilz that the least favorable distribution exists and is concentrated at a single point $\vartheta_* \in \Theta_E$.

Such a point is precisely

$$\vartheta_* = \bar{\vartheta} \pm \lambda \left(H + Q_0'\Sigma_0^{-1}Q_0\right)^{-1} g, \tag{24}$$

with $\lambda^2 = g'\left(H + Q_0'\Sigma_0^{-1}Q_0\right)^{-1} H \left(H + Q_0'\Sigma_0^{-1}Q_0\right)^{-1} g$. For more details see J. Pilz [5], Lemma 7, and N. Gaffke and B. Heiligers [1], Lemma 2 and Example 1.

This implies that the explicit form of $N_* = (\vartheta_* - \bar{\vartheta})(\vartheta_* - \bar{\vartheta})'$ is given by

$$N_* = \frac{\left(H + Q_0'\Sigma_0^{-1}Q_0\right)^{-1} gg' \left(H + Q_0'\Sigma_0^{-1}Q_0\right)^{-1}}{g'\left(H + Q_0'\Sigma_0^{-1}Q_0\right)^{-1} H \left(H + Q_0'\Sigma_0^{-1}Q_0\right)^{-1} g}. \tag{25}$$

Now, we have the explicit form of the Θ_E-*MILE* of $g'\vartheta$:

$$\widehat{g'\vartheta}_{MILE} = g'N_* Q_0'(Q_0 N_* Q_0' + \Sigma_0)^{-1}(vec(zz') - Q_0'\bar{\vartheta}) + g'\bar{\vartheta}. \tag{26}$$

On the other hand, by using the identity

$$\begin{aligned}(Q_0 N_* Q_0' + \Sigma_0)^{-1} &= (Q_0 t_* t_*' Q_0' + \Sigma_0)^{-1} \\ &= \Sigma_0^{-1} - (1 + t_*' Q_0' \Sigma_0^{-1} Q_0 t_*)^{-1} \Sigma_0^{-1} Q_0 t_* t_*' Q_0' \Sigma_0^{-1}, \quad (27)\end{aligned}$$

where $t_* = \vartheta_* - \bar{\vartheta}$, we get

$$\widehat{g'\vartheta}_{MILE} = \frac{g' t_*}{1 + t_*' Q_0' \Sigma_0^{-1} Q_0 t_*} t_*' Q_0' \Sigma_0^{-1} (\mathrm{vec}(zz') - Q_0' \bar{\vartheta}) + g' \bar{\vartheta}. \quad (28)$$

Observing that $(g' t_*)^2 / (1 + t_*' Q_0' \Sigma_0 Q_0 t_*)$ is the Bayes risk of the least favorable distribution, see (23) and (27), which are equal to

$$r(g'\vartheta, \widehat{g'\vartheta}_{MILE}) = g' \left(H + Q_0' \Sigma_0^{-1} Q_0\right)^{-1} g, \quad (29)$$

finally, we have

$$\begin{aligned}\widehat{g'\vartheta}_{MILE} &= g' \left(H + Q_0' \Sigma_0^{-1} Q_0\right)^{-1} Q_0' \Sigma_0^{-1} (vec(zz') - Q_0' \bar{\vartheta}) + g' \bar{\vartheta} \\ &= g' \left(H + Q_0' \Sigma_0^{-1} Q_0\right)^{-1} (Q_0' \Sigma_0^{-1} vec(zz') + H \bar{\vartheta}) \\ &= g' \left(H + Q_0' \Sigma_0^{-1} Q_0\right)^{-1} \left((Q_0' \Sigma_0^{-1} Q_0) \hat{\vartheta} + H \bar{\vartheta}\right) \quad (30)\end{aligned}$$

where $\hat{\vartheta} = (Q_0' \Sigma_0^{-1} Q_0)^{-1} Q_0' \Sigma_0^{-1} vec(zz')$. This is the Kuks–Olman type of estimator of the linear function $g'\vartheta$.

Note, that if $\Sigma_0 Q_0 = 2 Q_0$ holds true, see (9), we have

$$\widehat{g'\vartheta}_{MILE} = g' \left(H + \frac{1}{2} K_0\right)^{-1} \left(H \bar{\vartheta} + \frac{1}{2} K_0 \hat{\vartheta}\right), \quad (31)$$

where $K_0 = 2 Q_0' \Sigma_0^{-1} Q_0 = Q_0' Q_0$ is the critical matrix for $MINQE(U, I)$ estimation given by (15). If K_0 has full rank, then $\hat{\vartheta}$ is unique and represents the $MINQE(U, I)$ of ϑ. The risk of $\widehat{g'\vartheta}_{MILE}$ is then given by

$$R(g'\vartheta, \widehat{g'\vartheta}_{MILE}) = r(g'\vartheta, \widehat{g'\vartheta}_{MILE}) = g' \left(H + \frac{1}{2} K_0\right)^{-1} g. \quad (32)$$

4. MODIFIED MINIMAX QUADRATIC ESTIMATION AND ITS APPLICATIONS

The previous results can be directly used also in the context of the original model (1), however, the new interpretation should be established.

The linearized model (7) is only approximation to the true model (8). So, the risk function $R(g'\vartheta, \widehat{g'\vartheta})$ given by (18) and (19) is only approximation to the true risk function. In the setup of the true model $(vec(zz'), Q_0\vartheta, \Sigma)$ the risk function $R(g'\vartheta, \widehat{g'\vartheta})$ given by (19) is considered as a modified risk function. Moreover, note that the model (8), and also the associated modified risk function, depends on the prior choice of the point (ϑ_0, Ψ_0). To emphasize this fact we will denote the modified risk function based on the model (8) as $R(g'\vartheta, \widehat{g'\vartheta}|\vartheta_0, \Psi_0)$.

After all, in the context of the original linear model with variance and covariance components (1), we define the minimax linear estimator $\widehat{g'\vartheta}_{MILE}$ of the estimable function $g'\vartheta$, subject to restrictions given by Θ_E, which is based on the linearized model (7), as the modified $(\vartheta_0, \Psi_0, \Theta_E)$-*MIQE* – the modified minimax quadratic (plus constant) estimator.

The results on modified minimax estimation are summarized in the following theorem:

Theorem 1. Consider the linear model with variance and covariance components (1). Let $\vartheta_0 \in \Theta$ and $\Psi_0 \in \{\Psi : \Psi(\vartheta_0)\}$ be fixed.

Then the modified minimax quadratic estimator, $(\vartheta_0, \Psi_0, \Theta_E)$-*MIQE*, of the estimable linear function $g'\vartheta$, with respect to the ellipsoidal restrictions

$$\Theta_E = \{\vartheta : (\vartheta - \bar{\vartheta})'H(\vartheta - \bar{\vartheta}) \leq 1\}, \tag{33}$$

is given as

$$\widehat{g'\vartheta}_{MIQE} = g' \left(H + Q_0'\Sigma_0^{-1}Q_0\right)^{-1} \left((Q_0'\Sigma_0^{-1}Q_0)\hat{\vartheta} + H\bar{\vartheta}\right), \tag{34}$$

where $\hat{\vartheta} = (Q_0'\Sigma_0^{-1}Q_0)^{-1}Q_0'\Sigma_0^{-1}vec(zz')$, and z is suitable maximal invariant, the matrix Q_0 is defined by (4) and Σ_0 is defined by (5).

The modified risk is

$$R(g'\vartheta, \widehat{g'\vartheta}_{MIQE}|\vartheta_0, \Psi_0) = g' \left(H + Q_0'\Sigma_0^{-1}Q_0\right)^{-1} g. \tag{35}$$

Moreover, if we assume that the vector of observations y is normally distributed, the (ϑ_0, Θ_E)-*MIQE* is given as

$$\widehat{g'\vartheta}_{MIQE} = g' \left(H + \frac{1}{2}K_0\right)^{-1} \left(H\bar{\vartheta} + \frac{1}{2}K_0\hat{\vartheta}\right), \tag{36}$$

where $\hat{\vartheta}$ is the MINQE(U,I) based on a prior value ϑ_0. The modified risk is then

$$R(g'\vartheta, \widehat{g'\vartheta}_{MIQE}|\vartheta_0) = g' \left(H + \frac{1}{2}K_0\right)^{-1} g, \tag{37}$$

where K_0 is the critical matrix for *MINQE*(U, I) given by (15).

At the end we mention two possible applications of the modified minimax quadratic estimation:

1. The computation of $MINQE(U, I)$ which is based on the prior value ϑ_0 requires computation of the matrix

$$(MV_0M)^+ = V_0^{-1} - V_0^{-1}X(X'V_0^{-1}X)^-X'V_0^{-1}, \tag{38}$$

see (15) and (16). The computation tends to be very difficult if the model has complicated design and large number of observations. On the other hand, in many situations exists such $\vartheta_0^* \in \Theta$ that $V_0 = V(\vartheta_0^*) = I$. The $MINQE(U, I)$ based on such ϑ_0^* is denoted as $0 - MINQE(U, I)$. Numerical evaluation of $0 - MINQE(U, I)$ is considerably simplified.

If the additional information Θ_E is given, we suggest to use the modified minimax quadratic estimator based on ϑ_0^*, which respects the information, and has simple numerical solution. The reason is that for any fixed $\vartheta_0 \in \Theta$ the modified minimax quadratic estimator improves the modified risk $R(g'\vartheta, \widehat{g'\vartheta}_{MIQE}|\vartheta_0) = g'\left(H + \frac{1}{2}K_0\right)^{-1}g$, see (37), according to the risk of the $MINQE(U, I)$ estimator, $R(g'\vartheta, \widehat{g'\vartheta}_{MINQE}|\vartheta_0) = 2g'K_0^{-1}g$.

2. The second application is based on the dual problem. Suppose now that we want to compute $MINQE(U, I)$, (because of its good properties as e.g. unbiasedness). The problem is how to choose the prior ϑ_0 if we have the additional information Θ_E. The possible solution is to take such ϑ_0^* that it minimizes the modified minimax risk $R(g'\vartheta, \widehat{g'\vartheta}_{MIQE}|\vartheta_0) = g'\left(H + \frac{1}{2}K_0\right)^{-1}g$ through all $\vartheta_0 \in \Theta_E$, i.e. such ϑ_0^* that

$$g'\left(H + \frac{1}{2}K_0^*\right)^{-1} g \leq g'\left(H + \frac{1}{2}K_0\right)^{-1} g, \quad \text{for all } \vartheta_0 \in \Theta_E, \tag{39}$$

where the critical matrix K_0 is defined by (15).

ACKNOWLEDGEMENT

The research has been supported by the grant No. 1/4196/97 from the Scientific Grant Agency of the Slovak Republic.

(Received February 14, 1997.)

REFERENCES

[1] N. Gaffke and B. Heiligers: Bayes, admissible, and minimax linear estimators in linear models with restricted parameter space. Statistics *20* (1989), 4, 487–508.

[2] B. Heiligers: Linear Bayes and minimax estimation in linear models with partially restricted parameter space. J. Statist. Plann. Inference *36* (1993), 175–184.

[3] J. Kozák: Modified minimax estimation of regression coefficients. Statistics *16* (1985), 363–371.

[4] L. Kubáček, L. Kubáčková and J. Volaufová: Statistical Models with Linear Structures. Publishing House of the Slovak Academy of Sciences, Bratislava 1995.

[5] J. Pilz: Minimax linear regression estimation with symmetric parameter restrictions. J. Statist. Plann. Inference *13* (1986), 297–318.

[6] F. Pukelsheim: Estimating variance components in linear models. J. Multivariate Anal. *6* (1976), 626–629.

[7] C. R. Rao: Estimation of variance and covariance components – MINQUE theory. J. Multivariate Anal. *1* (1971), 257–275.

[8] C. R. Rao: Minimum variance quadratic unbiased estimation of variance components. J. Multivariate Anal. *1* (1971), 445–456.

[9] C. R. Rao: Unified theory of linear estimation. Sankhyā Ser. B *33* (1971), 371–394.

[10] C. R. Rao and J. Kleffe: Estimation of Variance Components and Applications. Statistics and Probability, Volume 3. North-Holland, Amsterdam – New York – Oxford – Tokyo 1988.

[11] C. R. Rao and K. Mitra: Generalized Inverse of Matrices and Its Applications. Wiley, New York – London – Sydney – Toronto 1971.

[12] S. R. Searle, G. Casella, and Ch. E. McCulloch: Variance Components. (Wiley Series in Probability and Mathematical Statistics.) Wiley, New York – Chichester – Brisbane – Toronto – Singapore 1992.

[13] J. Volaufová: A brief survey on the linear methods in variance-covariance components model. In: Model-Oriented Data Analysis (W. G. Müller, H. P. Wynn, and A. A. Zhigljavsky, eds.), Physica-Verlag, Heidelberg 1993, pp. 185–196.

[14] J. Volaufová and V. Witkovský: Estimation of variance components in mixed linear model. Appl. Math. *37* (1992), 139–148.

[15] G. Zyskind: On canonical forms, nonnegative covariance matrices and best and simple least square estimator in linear models. Ann. Math. Statist. *38* (1967), 1092–1110.

RNDr. Viktor Witkovský, CSc., Institute of Measurement Science, Slovak Academy of Sciences, Dúbravská cesta 9, 842 19 Bratislava. Slovak Republic.
e-mail: umerwitk@savba.sk