FINITE-TO-ONE FUZZY MAPS AND FUZZY PERFECT MAPS

FRANCISCO GALLEGO LUPIÁÑEZ

In this paper we define, for fuzzy topology, notions corresponding to finite-to-one and k-to-one maps. We study the relationship between these new fuzzy maps and various kinds of fuzzy perfect maps. Also, we show the invariance and the inverse inveriance under the various kinds of fuzzy perfect maps (and the finite-to-one fuzzy maps), of different properties of fuzzy topological spaces.

1. INTRODUCTION AND DEFINITIONS

In General Topology, the open maps has been very studied. The efforts has been particularly concentrated on determining which classes of spaces are invariant or inverse invariant under various kinds of such maps: open compact maps, open finite-to-one maps, open k-to-one maps,... [1,2,3,6,9,10,11,17,18,19,21,26].

Definition 1. [19] Let X and Y be topological spaces and $f: X \to Y$ be a map. The map is said finite-to-one if $f^{-1}(y)$ is a finite subset of X for each $y \in Y$, and is k-to-one if $f^{-1}(y)$ consists of exactly k points in X for each $y \in Y$.

In this paper we define for fuzzy topology notions corresponding to these maps. We study the relationship between these new fuzzy maps and various kinds of fuzzy perfect maps. Also, we show the invariance and the inverse invariance under the various kinds of fuzzy perfect maps (and the finite-to-one fuzzy maps) of different properties of fuzzy topological spaces in the Chang's sense.

All the maps are assumed to be continuous and onto.

First, we summarize the definitions which we will use along this paper.

Definition 2. [13] Let (X, τ) be a topological space and $\omega(\tau)$ be the set of all semicontinuous functions from (X, τ) to the unit interval equipped with the usual topology, then $(X, \omega(\tau))$ is called the weakly induced fuzzy topological space by (X, τ) .

Definition 3. [13] A fuzzy extension of a topological property is said to be good, when its is possessed by $(X, \omega(\tau))$ if, and only if, the original property is possessed by (X, τ) .

Definition 4. [20] If μ is a fuzzy set in X, the set $\{x \in X | \mu(x) > 0\}$ is called the support of μ and is denoted by supp μ .

Definition 5. [20] Let μ_1, μ_2 be two fuzzy sets in X, μ_1 is said to be quasicoincident with μ_2 , denoted by $\mu_1 q \mu_2$, if there exists $x \in X$ such that $\mu_1(x) > \mu'_2(x)$ where μ'_2 is the fuzzy complement of μ_2 .

Definition 6. [20] A fuzzy set in X is called a fuzzy point if it takes the value 0 for all $y \in X$ except one, say $x \in X$. If this value at x is α ($0 < \alpha \le 1$) we denote this fuzzy point by x_{α} .

Definition 7. [4] Let f be a map from X to Y. Let μ be a fuzzy set in Y, then the inverse of μ , denoted as $f^{-1}(\mu)$, is defined by $f^{-1}(\mu)(x) = \mu(f(x))$ for all x in X. Conversely, if μ is a fuzzy set in X, the image of μ , written as $f(\mu)$ is a fuzzy set in Y given by

 $f(\mu)(y) = \begin{cases} \sup_{z \in f^{-1}(y)} \{\mu(z)\}, & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$

Definition 8. [4] A map f from a fuzzy topological space (X, τ) to a fuzzy topological space (Y, s) is F-continuous if the inverse of each open fuzzy set is open fuzzy in (X, τ) .

Definition 9. [27] A map f from a fuzzy topological space (X, τ) to a fuzzy topological space (Y, s) is fuzzy closed (resp. fuzzy open) if $f(\mu)$ is fuzzy closed (resp. fuzzy open) in (Y, s), for each fuzzy closed (resp. fuzzy open) set μ in X.

Definition 10. [4] A family of fuzzy sets $\{\mu_j | j \in J\}$ is a cover of a fuzzy set μ if $\mu \leq \bigvee \{\mu_j | j \in J\}$. A subcover of $\{\mu_j | j \in J\}$ is a subfamily of it which is also a cover. A fuzzy topological space is compact in the Chang's sense, if each cover has a finite subcover.

Definition 11. [12] A fuzzy set μ in (X, T) is fuzzy compact in the Lowen's sense, if for all family of fuzzy open sets $\{\mu_j | j \in J\}$ such that $\mu \leq \bigvee \{\mu_j | j \in J\}$ and for all $\varepsilon > 0$ there exists a finite subfamily $\{\mu_j | j \in J_0\}$ such that $\mu - \varepsilon \leq \bigvee \{\mu_j | j \in J_0\}$.

Definition 12. [8] A collection \mathcal{U} of fuzzy sets in (X, T) is called *q*-cover of a fuzzy sets μ if for each $x \in \operatorname{supp} \mu$, $x_{\mu(x)}q \bigvee_{U \in \mathcal{U}} U$. If each member of \mathcal{U} is fuzzy open, then \mathcal{U} is called an open *q*-cover. A fuzzy set μ in a fuzzy topological space (X, T) is called *q*-compact if for every open *q*-cover \mathcal{U} , of μ , there exists a finite subcollection \mathcal{U}_0 of \mathcal{U} such that $\sup\{1 - \bigvee_{U \in \mathcal{U}_0} U\}(z)|z \in \operatorname{supp} \mu\} < \mu(x)$ for every $x \in \operatorname{supp} \mu$.

Definition 13. [8] A map f from a fuzzy topological space (X, T) to a fuzzy topological space (Y, S) is called fuzzy perfect, in the Ghosh's sense, if f is onto, fuzzy closed, F-continuous and $f^{-1}(y_{\alpha})$ is q-compact, for each fuzzy point y_{α} in Y.

Definition 14. [23] A map f from a fuzzy topological space (X, T) to a fuzzy topological space (Y, S) is called fuzzy perfect in the Srivastava and Lal's sense, if f is onto, fuzzy closed, F-continuous and $f^{-1}(y_{\alpha})$ is fuzzy compact in the Lowen's sense, for each fuzzy point y_{α} in Y.

Definition 15. [5] A map f from a fuzzy topological space (X, T) to a fuzzy topological space (Y, S) is called fuzzy perfect in the Christoph's sense, if f is onto, fuzzy closed, F-continuous and $f^{-1}(y_{\alpha})$ is compact in the Chang's sense, for each fuzzy point y_{α} in Y.

Let (X, τ) and (Y, s) be two topological spaces and $f : (X, \tau) \to (Y, s)$ an onto map. We denote $\tilde{f} : (X, \omega(\tau)) \to (Y, \omega(s))$ the map given by $\tilde{f}(x_{\alpha})(y) = \sup_{t \in f^{-1}(y)} \{x_{\alpha}(t)\}$ for each fuzzy point x_{α} in X.

Definition 16. Let (X,T), (Y,S) be two fuzzy topological spaces and $\tilde{f}: (X,T) \to (Y,S)$ be a fuzzy map. The map will be called finite-to-one fuzzy if supp $\tilde{f}^{-1}(y_{\alpha})$ is a finite subset of X for each fuzzy point y_{α} in Y, and will be called k-to-one fuzzy if supp $\tilde{f}^{-1}(y_{\alpha})$ consists of exactly k points in X for each fuzzy point y_{α} in Y.

Lemma 1.1. If the map $\tilde{f}: (X, \omega(\tau)) \to (Y, \omega(s))$ is open fuzzy then the map $f: (X, \tau) \to (Y, s)$ is open.

Proof. If A is open in (X, τ) , we have $A = \chi_A^{-1}((a, 1])$ for all $a \in [0, 1), \chi_A \in \omega(\tau)$, and then $\tilde{f}(\chi_A) = \chi_{f(A)} \in \omega(s)$ and f(A) is open in (Y, s).

Proposition 1.2. If the map $\tilde{f}: (X, \omega(\tau)) \to (Y, \omega(s))$ is finite-to-one fuzzy (resp. k-to-one fuzzy), then the map $f: (X, \tau) \to (Y, s)$ is finite-to-one (resp. k-to-one).

Proof. If there exists $y \in Y$ such that $f^{-1}(y)$ is infinite (resp. card $f^{-1}(y) \neq k$) let

$$y_1(y') = \begin{cases} 1 & \text{if } y' = y \\ 0 & \text{if } y' \neq y \end{cases}$$

Then, for each $x \in X$,

$$\widetilde{f}^{-1}(y_1)(x) = y_1(f(x)) = \begin{cases} 1 & \text{if } f(x) = y \\ 0 & \text{if } f(x) \neq y \end{cases}$$

Thus, supp $\tilde{f}^{-1}(y_1)$ and $f^{-1}(y)$ have equal cardinals, and supp $\tilde{f}^{-1}(y_1)$ is infinite (resp. its cardinal is $\neq k$).

2. RELATIONSHIP BETWEEN THESE FUZZY MAPS AND FUZZY PERFECT MAPS

It is known that every open k-to-one continuous and onto map is a perfect map [1, Lemmas 1 and 2]. Then it is natural to study the relationship between k-to-one fuzzy maps and the various kinds of fuzzy perfect maps.

Remark 2.1. In general, \tilde{f} open-k-to-one fuzzy map $\neq \tilde{f}$ fuzzy perfect map in the Christoph's sense.

Indeed, let $1_x : (X,T) \to (X,T)$ be the identity fuzzy map. Clearly, 1_x is 1-to-one fuzzy map. But $1_x^{-1}(x_1)$ for x_1 crisp fuzzy point in X, is not fuzzy compact in the Chang's sense, because if $x_1 \leq \bigvee_{j \in (0,1)} \mu_j$, then $1 = \sup\{\mu_j(x)|j \in (0,1)\}$, and if μ_j is an open fuzzy set in X such that $\mu_j(x) = j$, for all $j \in (0,1)$, there is not a finite subfamily $J_0 \subset (0,1)$ such that $1 = \sup\{\mu_j(x)|j \in J_0\}$. Then, 1_x is not fuzzy perfect map in the Christoph's sense.

Proposition 2.2. If $\tilde{f}: (X,T) \to (Y,S)$ is a finite-to-one fuzzy map, then $\tilde{f}^{-1}(y_{\alpha})$ is fuzzy compact in the Lowen's sense for each fuzzy point y_{α} in Y.

Proof. Let supp $\tilde{f}^{-1}(y_{\alpha}) = \{x_r | r = 1, ..., k\}$ for some fuzzy pont y_{α} in Y, and $\{\mu_j | j \in J\}$ be a family of open fuzzy sets in X such that $\tilde{f}^{-1}(y_{\alpha}) \leq \bigvee_{j \in J} \mu_j$. Then $\alpha \leq \sup\{\mu_j(x_r) | j \in J\}$ for every r = 1, ..., k. Thus, for each $\varepsilon > 0$ and each r = 1, ..., k there exists $j_r \in J$ such that $\alpha - \varepsilon < \mu_{j_r}(x_r)$. We denote $J_0 = \{j_r | r = 1, ..., k\}$, and we have that $\alpha - \varepsilon < \sup\{\mu_j(x_r) | j \in J_0\}$ for every r = 1, ..., k, and $\tilde{f}^{-1}(y_{\alpha}) - \varepsilon \leq \bigvee_{j \in J_0} \mu_j$. Hence $\tilde{f}^{-1}(y_{\alpha})$ is fuzzy compact in the Lowen's sense. \Box

Corollary 2.3. If \tilde{f} is an onto, *F*-continuous, finite-to-one closed fuzzy map, then \tilde{f} is fuzzy perfect in the Srivastava and Lal's sense.

Proposition 2.4. If $\tilde{f}: (X,T) \to (Y,S)$ is a finite-to-one fuzzy map, then $\tilde{f}^{-1}(y_{\alpha})$ is *q*-compact for each fuzzy point y_{α} in Y.

Proof. Let supp $\tilde{f}^{-1}(y_{\alpha}) = \{x_r | r = 1, ..., k\}$ for some fuzzy point y_{α} in Y, and \mathcal{U} be an open q-cover of $\tilde{f}^{-1}(y_{\alpha})$. Then for each $x \in \text{supp } \tilde{f}^{-1}(y_{\alpha}), x_{\alpha}q \bigvee_{U \in \mathcal{U}} \mathcal{U}$. Thus, we have that $\alpha > 1 - \bigvee_{U \in \mathcal{U}} \mathcal{U}(x_r)$ for each r = 1, ..., k, and there exists an $U_r \in \mathcal{U}$ such that $U_r(x_r) > 1 - \alpha$ for each r = 1, ..., k. Let $\mathcal{U}_0 = \{U_r | r = 1, ..., k\}$. We have that \mathcal{U}_0 is a finite subcollection of \mathcal{U} and $(1 - \bigvee_{U \in \mathcal{U}_0} \mathcal{U})(x_r) = 1 - \sup\{\mathcal{U}(x_r)|\mathcal{U} \in \mathcal{U}_0\} < \alpha$. Thus, $\sup\{(1 - \bigvee_{U \in \mathcal{U}_0} \mathcal{U})(x_r)|r = 1, ..., k\} < \alpha = f^{-1}(y_{\alpha})(x)$ for every $x \in \operatorname{supp} \tilde{f}^{-1}(y_{\alpha})$. Hence $\tilde{f}^{-1}(y_{\alpha})$ is q-compact.

Corollary 2.5. If \tilde{f} is an onto, *F*-continuous, finite-to-one closed fuzzy map, then \tilde{f} is fuzzy perfect in the Ghosh's sense.

3. INVARIANCE THEOREMS

Now, we show the invariance's theorems for the various kinds of fuzzy perfect maps:

Proposition 3.1. If the map $\tilde{f}: (X, \omega(\tau)) \to (Y, \omega(s))$ verifies that $\tilde{f}^{-1}(y_{\alpha})$ is *q*-compact for every fuzzy point y_{α} in Y, then $f^{-1}(y)$ is compact for every $y \in Y$.

Proof. For every open cover \mathcal{U} of $f^{-1}(y)$, we have that $\mathcal{U}^* = \{\chi_u | U \in \mathcal{U}\}$ is open q-cover of $\tilde{f}^{-1}(y_a)$. Indeed: for each $x \in \text{supp } \tilde{f}^{-1}(y_a)$ is $\tilde{f}^{-1}(y_a)(x) = y_a(f(x)) = \alpha$, then f(x) = y, $x_{\tilde{f}^{-1}(y_a)(x)} + \chi_{\bigcup_{U \in \mathcal{U}} U}(x) = \alpha + 1 > 1$, thus $x_{\tilde{f}^{-1}(y_a)(x)} q \chi_{\bigcup_{U \in \mathcal{U}} U}$.

By the hypothesis, there exists a finite subcollection \mathcal{U}_0^* of \mathcal{U}^* such that $\sup\{(1 - \bigvee_{\chi_U \in \mathcal{U}_0^*} \chi_U)(x) | x \in \sup \tilde{f}^{-1}(y_\alpha)\} < \tilde{f}^{-1}(y_\alpha)(x)$, for every $x \in \sup \tilde{f}^{-1}(y_\alpha)$ (or, equivalently $x \in f^{-1}(y_\alpha)$). Then, $(\bigvee_{\chi_U \in \mathcal{U}_0^*} \chi_U)(x) + \tilde{f}^{-1}(y_\alpha)(x) > 1$ for every $x \in f^{-1}(y)$ [8, Corollary 2.1], $\chi_{\bigcup_{U \in \mathcal{U}_0} U}(x) + \alpha > 1$ for every $x \in f^{-1}(y)$ (where $\mathcal{U}_o \subset \mathcal{U}$ is a finite subfamily), $x \in \bigcup_{U \in \mathcal{U}_0} U$ for all $x \in f^{-1}(y)$ and, finally \mathcal{U}_0 covers $f^{-1}(y)$. Thus $f^{-1}(y)$ is compact.

Corollary 3.2. If the map $\tilde{f}: (X, \omega(\tau)) \to (Y, \omega(s))$ is fuzzy perfect in the Ghosh's sense, then the map $f: (X, \tau) \to (Y, s)$ is perfect.

Proof. It follows from the proposition and Lemmas 2.1 and 2.2 in [15]. \Box

Proposition 3.3. If the map $\tilde{f}: (x, \omega(\tau)) \to (Y, \omega(s))$ is fuzzy perfect in the Srivastava and Lal's sense (or in the Christoph's sense), then the map $f: (X, \tau) \to (Y, s)$ is perfect.

Proof. We have proved this in [15].

Theorem 3.4. Let $(X, \omega(\tau)), (Y, \omega(s))$ be two weakly induced fuzzy topological spaces and $\tilde{f}: (X, \omega(\tau)) \to (Y, \omega(s))$ be fuzzy perfect in the Ghosh's sense (resp. in the Srivastava and Lal's sense, or in the Christoph's sense), if:

(a) $(X, \omega(\tau))$ verifies the fuzzy version of a property \mathcal{P} of topological spaces,

- (b) the property \mathcal{P} is invariant under perfect maps,
- (c) the fuzzy version of \mathcal{P} is a good extension of \mathcal{P} .

Then $(Y, \omega(s))$ verifies the fuzzy version of \mathcal{P} .

Proof. The map $f: (X, \tau) \to (Y, s)$ is perfect by Corollary 3.2 and Proposition 3.3. Then the result is clear.

Corollaries 3.5.

- 1. Hausdorff fuzzy topological spaces [24] are invariant by fuzzy perfect maps in the various senses.
- 2. All the good extensions of paracompactness are invariant by fuzzy perfect maps in the various senses.

Theorem 3.6. Let $(X, \omega(\tau)), (Y, \omega(s))$ be two weakly induced fuzzy topological spaces and $\tilde{f}: (X, \omega(\tau)) \to (Y, \omega(s))$ be fuzzy perfect in the Ghosh's sense (resp. in the Srivastava and Lal's sense, or in the Christoph's sense), if:

- (a) $(Y, \omega(s))$ verifies the fuzzy version of a property \mathcal{P} of topological spaces.
- (b) the property \mathcal{P} is inverse invariant under perfect maps,
- (c) the fuzzy version of \mathcal{P} is a good extension of \mathcal{P} .

Then $(X, \omega(\tau))$ verifies the fuzzy version of \mathcal{P} .

Proof. The map $f: (X, \tau) \to (Y, s)$ is perfect by Corollary 3.2 and Proposition 3.3. Then the result is clear.

Corollaries 3.7.

- 1. Haussdorff fuzzy topological spaces are inverse invariant by fuzzy perfect maps in the various senses.
- 2. All the good extensions of paracompactness are inverse invariant by the various kinds of fuzzy perfect maps.

(Received December 30, 1996.)

REFERENCES

- A. V. Arhangel'skii: A criterion for the existence of a bicompact element in a continuous decomposition. A theorem on the invariance of weight under open-closed finiteto-one mappings. Soviet Math. Dokl. 7 (1966), 249-253.
- [2] A. V. Arhangel'skii: A theorem of the metrizability of the inverse image of a metric space under an open-closed finite-to-one mapping. Example and unsolved problems. Soviet Math. Dokl. 7 (1966), 1258-1262.
- [3] J. Chaber: Open finite-to-one images of metric spaces. Topology Appl. 14 (1982), 241-246.
- [4] C.L. Chang: Fuzzy topological spaces. J. Math. Anal. Appl. 24 (1968), 182-190.
- [5] F. T. Christoph: Quotient fuzzy topology and local compactness, J. Math. Anal. Appl. 57 (1977), 497-504.
- [6] M. M. Čoban: Open finite-to-one mappings. Soviet Math. Dokl. 8 (1967), 603-605.
- [7] M. E. A. El-Monsef, F. M. Zeyada, S. N. El-Deeb and I. M. Hanafy: Good extensions of paracompactness. Math. Japon. 37 (1992), 195-200.
- [8] B. Ghosh: Directed family of fuzzy sets and fuzzy perfect maps. Fuzzy Sets and Systems 75 (1995), 93-101.

- [9] R. F. Gittings: Finite-to-one open maps of generalized metric spaces. Pacific J. Math. 59 (1975), 33-41.
- [10] R. F. Gittings: Open mapping theory. In: Set-theoretic Topology (G. M. Reed, ed.), Academic Press, New York 1977, pp. 141-191.
- J. K. Kohli: Finite-to-one maps and open extensions of maps. Proc. Amer. Math. Soc. 48 (1975), 464-468.
- [12] R. Lowen: Fuzzy topological spaces and fuzzy compactness. J. Math. Anal. Appl. 56 (1976), 621-633.
- [13] R. Lowen: A comparison of different compactness notions in fuzzy topological spaces. J. Math. Anal. Appl. 64 (1978), 446-454.
- [14] M. K. Luo: Paracompactness in fuzzy topological spaces. J. Math. Anal. Appl. 130 (1988), 55-77.
- [15] F.G. Lupiáñez: Fuzzy perfect maps and fuzzy paracompactness. Fuzzy Sets and Systems, to appear.
- [16] H. W. Martin: Weakly induced fuzzy topological spaces. J. Math. Anal. Appl. 78 (1980), 634-639.
- [17] A. Okuyama: Note on inverse images under finite-to-one mappings. Proc. Japan Acad. 43 (1967), 953-956.
- [18] C. M. Pareek: Open finite-to-one mappings on p-spaces. Math. Japon. 28 (1983), 9-13.
- [19] V. V. Proizvolov: Finitely multiple open mappings. Soviet Math. Dokl. 7 (1966), 35– 38.
- [20] P.-M. Pu and Y.-M. Liu: Fuzzy topology I. Neighborhood structure of a fuzzy point and Moore-Smith convergence. J. Math. Anal. Appl. 76 (1980), 571-599.
- [21] R.S. Sadyrkhanov: A finite-to-one criterion for covering maps. Soviet Math. Dokl. 28 (1983), 587-590.
- [22] A. P. Shostak: Two decades of fuzzy topology: basic ideas, notions, and results. Russian Math. Surveys 44 (6) (1989), 125–186.
- [23] R. Srivastava and S. N. Lal: On fuzzy proper maps. Mat. Vesnik 38 (1986), 337-342.
- [24] R. Srivastava, S. N. Lal and A. K. Srivastava: Fuzzy Hausdorff topological spaces, J. Math. Anal. Appl. 81 (1981), 497-506.
- [25] R. Srivastava, S. N. Lal and A. K. Srivastava: Fuzzy T₁-topological spaces. J. Math. Anal. Appl. 102 (1984), 442-448.
- [26] Y. Tanaka: On open finite-to-one maps. Bull. Tokyo Gakugei Univ., Ser. IV, 25 (1973), 1-13..
- [27] C.K. Wong: Fuzzy topology: product and quotient theorems. J. Math. Anal. Appl. 45 (1974), 512-521.

Dr. Francisco Gallego Lupiáñez, Departamento de Geometría y Topologia, Facultad de Matemáticas, Universidad Complutense de Madrid, 28040-Madrid. Spain. e-mail: fg_lupianez@mat.ucm.es