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NULL EVENTS AND STOCHASTICAL INDEPENDENCE

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In this paper we point out the lack of the classical definitions of stochastical independence (particularly with respect to events of 0 and 1 probability) and then we propose a definition that agrees with all the classical ones when the probabilities of the relevant events are both different from 0 and 1, but that is able to focus the actual stochastical independence also in these extreme cases. Therefore this definition avoids inconsistencies such as the possibility that an event A can be at the same time stochastically independent and logically dependent on an event B. In a forthcoming paper we will deepen (in this context) the concept of *conditional independence* (which is just sketched in the last section of the present paper) and we will deal also with the extension of these results to the general case of any (finite) number of events.

1. INTRODUCTION

Stochastical independence plays a central role both in probability theory and in its application to the treatment of uncertainty in expert systems. On the other hand, the classical definition may give rise to the counterintuitive situation that an event A can be stochastically independent of an event B while being at the same time logically dependent on it. We recall that two events A, B are logically independent (to avoid cumbersome notation, we drop the conjunction operator, denoting the intersection of two events A, B simply by AB) when none of the four atoms AB^c,AB, A^cB, A^cB^c is impossible: in other words, each one of the two events A, B remains possible even when the outcome of the other is known. For example, two *incompatible* events A, B, with $P(A) = 0$, are clearly logically dependent, nevertheless they satisfy the classical definition of independence (the product rule), since

$$P(AB) = 0 = P(A)\,P(B).$$

It has been extensively discussed elsewhere (see, for example, [1] and [2], which are more recent papers with a large bibliography) that the possibility of dealing with zero probabilities is a very crucial feature, even in the case of a *finite* family of events: in fact, ignoring the possible existence of null events (which amounts to a stronger requirement of *coherence*) drastically restricts the class of admissible probability assessments and the possibility of extending in any case a coherent conditional probability.

Consider the following experiment: toss once a coin and, putting

$$S = \text{ the coin stands (e.g., leaning against the wall),}$$

consider the following outcomes

$$A_1 = S^c \text{ and head,}$$

$$A_2 = S^c \text{ and tail,}$$

$$A_3 = \text{ S and it gives head in a second toss,}$$

$$A_4 = \text{ S and it gives tail in a second toss,}$$

and the assessments

$$P(A_1) = 1/2,\ P(A_2) = 1/2,$$

$$P(A_3|A_3 \vee A_4) = P(A_4|A_3 \vee A_4) = 1/2,$$

which are clearly coherent. Recall (see [1]) that in our approach conditional probability is *directly* introduced as a function whose domain is an *arbitrary* set of conditional events and that must satisfy only the requirement of *coherence*: so there is *no need of assuming positive probability* for the conditioning event.

If we required positivity of the probability of conditioning events in the above example, putting $x_i = P(A_i) = P_o(A_i)$ (by P_o we denote, in general, the extension of P to the atoms: in this case P_o coincides with P), we should demand compatibility of the following system

$$(S_1) \begin{cases} x_1 = (1/2)(x_1 + x_2 + x_3 + x_4) \\ x_2 = (1/2)(x_1 + x_2 + x_3 + x_4) \\ x_3 = (1/2)(x_3 + x_4) \\ x_4 = (1/2)(x_3 + x_4) \\ x_3 + x_4 > 0 \\ x_1 + x_2 + x_3 + x_4 = 1 \\ x_i \geq 0 \end{cases}$$

which has no solution. Instead, dropping the condition $x_3+x_4 > 0$, we get a solution of the system (S_1) which allows (trivially) a "Kolmorogovian" representation of the first two assessments.

Then, going on with the algorithm expounded in [1] and [2] (which requires the consideration of a sequence of linear systems, each one referring to the set $\mathcal{A}_1$ of atoms that in the previous system were giving null probability P_o to some conditioning event), we get a second system (S_2), i.e.

$$(S_2) \begin{cases} y_3 = (1/2)(y_3 + y_4) \\ y_4 = (1/2)(y_3 + y_4) \\ y_3 + y_4 = 1 \\ y_i \geq 0 \end{cases}$$

which allows to suitably represent, by another probability $P_1(A_i) = y_i$ defined on $\mathcal{A}_1$, also the two remaining assessments.

Another interesting (and real) situation in which zero probability events come naturally to the fore is that of the so-called *first digit problem* [4].
It has been observed that empirical results concerning the distribution of the first significant digit of a large body of statistical data (in a wide sense: physical and chemical constants, partial and general census and election results, etc) show a peculiarity that has been considered paradoxical, i.e. there are more "constants" with low order first significant digits than high. In fact, the observed frequency of the digit k $(1 \leq k \leq 9)$ is *not* $1/9$, but is given by

$$P(E_k) = \log_{10}\left(1 + \frac{1}{k}\right),$$

where E_k is the event "the first significant digit of the observed constant is k", i.e.

$$E_k = \bigcup_{n=0}^{\infty} I_{kn},$$

with

$$I_{kn} = [k \cdot 10^n, (k+1) \cdot 10^n).$$

These intervals, in spite of their increasing (with n) cardinality, might obviously (to guarantee the summability of the relevant series) have probability converging to zero. Moreover, since any kind of "regularity" in a statistical table should be apparent also in every table obtained from it by any change of units, it follows that the sought probability P should be "scale-invariant", i.e.

$$P(I) = P(\lambda I)$$

for any interval I and real λ. By choosing as λ a power of 10, it follows that, for any integer k between 1 and 9, and for any natural number n,

$$P(I_{kn}) = 0.$$

In a finitely additive setting, these equalities are compatible with the above value of $P(E_k)$. On the other hand, for any given natural number n,

$$P(E_k I_{kn}) = P(I_{kn}) = 0 = P(E_k)\, P(I_{kn}),$$

while E_k and I_{kn} are clearly *not* independent (neither logically nor stochastically).

2. DEFINITIONS

Let us consider a family of conditional events $\mathcal{C} = \{A_i|B_i, A_i, B_i\}$, with $B_i \neq \emptyset$, and a coherent probability P on $\mathcal{C}$. We denote by $\emptyset$ and by Ω the impossible and the certain event, respectively. Note that an *unconditional* event A is the same as $A|\Omega$.

Definition 1 (de Finetti [3]). — An event $A \in \mathcal{C}$ is *infinitely less probable* than an event $B \in \mathcal{C}$ (in symbols $A \prec B$) if $P(A|A \vee B) = 0$ (and so $P(B|A \vee B) = 1$).

This means (if the probabilities of the two events A and B have been coherently assessed equal to zero) that once we have reached, by the aforementioned algorithm, the first linear system giving positive probability to $A \vee B$, it still gives zero probability to A.

Definition 2. Two events $A, B \in \mathcal{C}$ are of the *same degree of probability* (in symbols $A \simeq B$) if neither $A \prec B$ nor $B \prec A$.
Then $A \simeq B$ if and only if $P(A|A \vee B)\, P(B|A \vee B) > 0$. In particular, they are called *equivalent* (in probability) when $P(A|A \vee B) = P(B|A \vee B) = 1/2$.

Definition 3. An event $A \in \mathcal{C}$ is *stochastically independent* of an event $B \in \mathcal{C}$ ($B \neq \emptyset$ and $B \neq \Omega$) with respect to a probability P (in symbols $A\$B$: we choose the dollar symbol since it looks like intertwining S and I, the initials of *stochastically independent*) when one of the following conditions holds:
(i) $0 < P(A|B) = P(A|B^c) < 1$;
(ii) $P(A|B) = P(A|B^c) = 0$ and $AB \simeq AB^c$
(iii) $P(A|B) = P(A|B^c) = 1$ and $A^cB \simeq A^cB^c$.

Notice that this definition implies that we must have also $A \neq \emptyset$ and $A \neq \Omega$. It follows that independence can be considered only for *possible* events, avoiding the counterintuitive situation of the classical definition that implies that any event A is stochastically independent of both $\emptyset$ and Ω.

Example 1 (see [4]). We know that a natural number n has been obtained by one of two given methods A or A^c: the method is secretly chosen by a friend of ours with probabilities $P(A) = 1$ and $P(A^c) = 0$ (for example, according to whether the outcome of the experiment in the first example of the Introduction is, respectively, $A_1 \vee A_2$ or S). In method A, n is the number of times our friend tosses a coin until the first outcome of "head"; in method A^c, our friend ask a mathematician to choose at his will and tell him a natural number n (for example, the mathematician could choose the factorial of the maximum integer less than e^{27}): notice that if we judge that choice as "uniform", the probability distribution expressing all possible choices should be (in a finitely additive setting) $P(n) = 0$ for any n.
Then our friend tell us *only* the number n (call E this event): on the basis of this data, we must "guess" (i.e., reassess the relevant probabilities) whether n has been obtained by method A or A^c. Clearly

$$P(E|A) = \frac{1}{2^n} \neq P(E|A^c) = 0,$$

so that E and A are *not* independent. Nevertheless, since

$$P(E) = P(A)\,P(E|A) + P(A^c)\,P(E|A^c) = \frac{1}{2^n},$$

we have

$$P(AE) = P(A)\,P(E|A) = \frac{1}{2^n} = P(A)\,P(E),$$

i.e. the two events verify the classical definition of independence.

Example 2. Recall that also the two events E_k and I_{kn} of the last example of the Introduction should be regarded as independent according to the classical definition, while they are *not*. In fact, for any given natural number n we have

$$P(E_k|I_{kn}) = 1,$$

which is different from

$$P(E_k|I_{kn}^c) = \frac{P(E_k I_{kn}^c)}{P(I_{kn}^c)} = P(E_k I_{kn}^c) = P(E_k)\,P(I_{kn}^c|E_k) = P(E_k)\cdot 1 = P(E_k).$$

In the following example, notice that both conditional probabilities corresponding to those introduced in Definition 3 are equal to zero.

Example 3. The two (incompatible) events A_1 and A_3 considered in the first example of the Introduction are *not* independent, since $P(A_3|A_1) = P(A_3|A_1^c) = 0$, but $A_1A_3 \not\simeq A_1^cA_3$: in fact $P(\emptyset|A_3)\,P(A_3|A_3) = 0$, i.e. $\emptyset \not\simeq A_3$. (According to the classical definition, these two events should instead be regarded as independent).

Proposition 1. For any P and for any possible event A, even if $P(A) = 0$ or $P(A) = 1$, one has (not $A\$A$), i.e. the relation $\$$ is *irreflexive*.

Proposition 2. Let A, B be two possible events and P a coherent probability. If $A\$B$ with respect to P, then $A^c\$B$ and $A\$B^c$.

3. MAIN RESULTS

Theorem 1. $A\$B$ implies $P(A|B) = P(A)$. Conversely, assuming $P(B) < 1$ and $0 < P(A) < 1$, then

$$P(A|B) = P(A) \text{ implies } A\$B.$$

Proof. Assume $A\$B$: clearly, if $P(B) < 1$,

$$P(A|B) = P(A|B^c) = \frac{P(AB^c)}{P(B^c)} = \frac{P(A)[1 - P(B|A)]}{1 - P(B)}$$

so that

$$P(A|B) - P(AB) = P(A) - P(AB)$$

and finally $P(A|B) = P(A)$. Notice that the latter relation holds trivially, for any A, when $P(B) = 1$.
Conversely, if $P(B) < 1$, and assuming $0 < P(A) < 1$ and $P(A|B) = P(A)$, we have

$$P(A|B^c) = \frac{P(AB^c)}{P(B^c)} = \frac{P(A)\,P(B^c|A)}{P(B^c)} = \frac{P(A) - P(B)\,P(A)}{1 - P(B)} = P(A)$$

so that $P(A|B^c) = P(A|B)$. The conclusion follows from (i) of Definition 3. □

Remark 1. When $P(B) = 1$, so that $P(A|B) = P(A)$, the relation $A\$B$ may not hold, since the probability $P(A|B^c)$ can take any value of the interval [0,1]: in fact, for any assessment of this probability, putting (we denote by the same letter P all probabilities)

$$x_1 = P(AB^c)\,,\; x_2 = P(AB)\,,\; x_3 = P(A^cB)\,,\; x_4 = P(A^cB^c)\,,$$

the following systems are compatible: the first one

$$\begin{cases} x_2 = P(A)(x_2 + x_3) \\ x_1 = P(A|B^c)(x_1 + x_4) \\ x_2 + x_3 = 1 \\ x_1 + x_4 = 0 \\ x_1 + x_2 + x_3 + x_4 = 1 \\ x_i \geq 0 \end{cases}$$

which has the solution $x_1 = x_4 = 0$, $x_2 = P(A)$, $x_3 = 1 - P(A)$, and the next

$$\begin{cases} y_1 = P(A|B^c)(y_1 + y_4) \\ y_1 + y_4 = 1 \\ y_i \geq 0 \end{cases}$$

which is satisfied for any $y_1 = P(A|B^c) \geq 0$. Consider now the case $P(A) = 0$: the equality $P(A|B) = P(A)$ does not imply $A\$B$, even if it implies (for $P(B) \neq 1$) that $P(A|B) = P(A|B^c) = 0$, since it does not follow necessarily that $P(B|A)\,P(B^c|A) > 0$. In fact the assessments $P(B|A) = 1$ and $P(B^c|A) = 0$ are coherent, because the following systems are compatible:

$$\begin{cases} x_2 = x_1 + x_2 \\ x_2 + x_3 = P(B) \\ x_1 + x_2 = 0 \\ x_1 + x_2 + x_3 + x_4 = 1 \\ x_i \geq 0 \end{cases}$$

which has the solution $x_1 = x_2 = 0$, $x_3 = P(B)$, $x_4 = 1 - P(B)$, and the next

$$\begin{cases} y_2 = y_1 + y_2 \\ y_1 + y_2 = 1 \\ y_i \geq 0 \end{cases}$$

which has the solution $y_1 = 0$, $y_2 = 1$. If $P(A) = 1$, the equality $P(A|B) = P(A)$ does not guarantee that $A\$B$, but (if $P(B) \neq 1$) only that $P(A|B) = P(A|B^c)$, while $P(B|A^c)\,P(B^c|A^c) > 0$ may not hold. In fact, the following systems are compatible:

$$\begin{cases} x_2 = x_2 + x_3 \\ x_1 = x_1 + x_4 \\ x_1 + x_2 = 1(x_1 + x_2 + x_3 + x_4) \\ x_3 = 0(x_3 + x_4) \\ x_4 = x_3 + x_4 \\ x_1 + x_2 + x_3 + x_4 = 1 \\ x_i \geq 0 \end{cases}$$

which has the solution $x_3 = x_4 = 0$, $x_1 = 1 - x_2$, and the next

$$\begin{cases} y_3 = 0 \\ y_4 = y_3 + y_4 \\ y_3 + y_4 = 1 \\ y_i \geq 0 \end{cases}$$

which has the solution $y_3 = 0$, $y_4 = 1$ (this ends Remark 1).

Theorem 2. Let A, B be two possible events and P a coherent probability. The following conditions hold:

(a) if $0 < P(A) < 1$, then $A\$B$ implies $B\$A$;

(b) if $P(A) = P(B) = 0$, or $P(A) = P(B) = 1$, or $P(A) = 0$ and $P(B) = 1$, or $P(A) = 1$ and $P(B) = 0$, then

$$A\$B \text{ implies (not } B\$A)$$

Proof. (a) Since

$$P(B|A) = \frac{P(B)\,P(A|B)}{P(A)},$$

using Theorem 1 we get $P(B|A) = P(B)$, and so

$$P(B|A) = \frac{1 - P(A)}{1 - P(A)} P(B) = \frac{1 - P(A|B)}{P(A^c)} P(B) =$$

$$= \frac{P(A^c|B)}{P(A^c)} P(B) = \frac{P(A^c B)}{P(A^c)} = P(B|A^c).$$

(b) Let $P(A) = P(B) = 0$. This gives $P(B|A^c) = 0$ and so, if it were $B\$A$, then $P(B|A) = 0 = P(AB|A)$. On the other hand, from $A\$B$ it follows $P(A|B) = P(A|B^c) = 0$ and $P(AB|A)\,P(AB^c|A) > 0$ (contradiction).
Let $P(A) = P(B) = 1$. Now, $A\$B$ implies $P(A|B) = P(A|B^c) = 1$ and

$$P(A^c B|A^c)\,P(A^c B^c|A^c) > 0;$$

in particular, $P(B^c|A^c) > 0$. On the other hand, $P(B|A) = 1$, and so $B\$A$ would imply $P(B|A^c) = 1$, i.e. $P(B^c|A^c) = 0$ (contradiction).
When $P(A) = 1$ and $P(B) = 0$, obviously $P(B|A) = 0$. So, since $A\$B$ implies $P(A|B) = P(A|B^c) = 1$ and $P(B|A^c)\,P(B^c|A^c) > 0$, assuming $B\$A$ we would get the contradiction $0 = P(B|A) = P(B|A^c)$.
In the case $P(A) = 0$, $P(B) = 1$ the proof is similar. □

Theorem 3. $A\$B$ implies $P(AB) = P(A)\,P(B)$. Conversely, if $0 < P(A) < 1$ and $0 < P(B) < 1$, then

$$P(AB) = P(A)\,P(B) \text{ implies } A\$B.$$

Proof. The initial statement follows easily from the first implication under Theorem 1.
Conversely, since the product rule implies $P(A|B) = P(A)$ and $P(B|A) = P(B)$, one has

$$P(A|B^c) = \frac{P(A)\,P(B^c|A)}{P(B^c)} = \frac{P(A)(1 - P(B|A)}{1 - P(B)},$$

and so $A\$B$. □

Remark 2. When $P(B) = 0$, the equality $P(AB) = P(A)\,P(B)$ holds for any $P(A)$, while the equality $P(A|B) = P(A)$ may not hold, and so, by Theorem 1, neither $A\$B$. If $P(B) = 1$, both equalities hold for any A, but (as it has been already noticed in Remark 1) this does not imply $A\$B$. If $P(A) = 0$, the product rule is satisfied for any B, and we have also $P(A|B) = P(A|B^c) = 0$, but it does not follow that $P(B|A)\,P(B^c|A) > 0$ (this also has been shown in Remark 1). Finally, if $P(A) = 1$, both equalities hold (the second one with 1 in place of 0), but not necessarily the corresponding inequality having A^c in place of A holds.

The last theorem shows that (with our definition) stochastical independence is stronger than logical independence.

Theorem 4. If $A\$B$, then A and B are logically independent.

Proof. If A and B were logically dependent, this would correspond to one (at least) of the following three situations: $AB = \emptyset$; $A \subseteq B$ or $B \subseteq A$; $A \vee B = \Omega$. So we give the proof in three steps:

(i) $AB \neq \emptyset$;

(ii) (not $A \subseteq B$) and (not $B \subseteq A$);

(iii) $A \vee B \neq \Omega$.

(i) Suppose $AB = \emptyset$: we prove that (not $A\$B$). The proof is trivial when $P(A)\,P(B) > 0$ and $P(B) \neq 1$, or $P(A) > 0$ and $P(B) = 0$. In the remaining cases, it is enough to verify that the condition $P(B|A)\,P(B^c|A) > 0$ does not hold, since $P(B|A) = P(\emptyset|A) = 0$.

(ii) If it were $A \subseteq B$, and so $P(A|B^c) = 0$, we should have also $P(A|B) = 0$ and $P(B|A)\,P(B^c|A) > 0$ (since $A\$B$). But the latter condition cannot be true, since $P(B|A) = 1$ and so $P(B^c|A) = 0$. On the other hand, if it were $B \subseteq A$, then $P(A^c|B) = 0$, i.e. $P(A|B) = 1$; from $A\$B$ it follows $P(A|B^c) = 1$ and $P(B|A^c)\,P(B^c|A^c) > 0$, but the latter inequality cannot hold, since $P(B|A^c) = 0$.

(iii) Let $A \vee B = \Omega$: then $B^c \subseteq A$, so that $P(A|B^c) = 1$. From $A\$B$ it follows $P(A|B) = 1$ and $P(B|A^c)\,P(B^c|A^c) > 0$, while $P(B^c|A^c) = 0$, since $A^cB^c = (A \vee B)^c = \emptyset$. □

4. CONDITIONAL INDEPENDENCE

Definition 4. An event A is *stochastically independent* of an event B *conditionally* on E ($BE \neq \emptyset \neq B^cE$) with respect to a probability P (in symbols $A\$B|E$) when one of the following conditions holds:

(i) $0 < P(A|BE) = P(A|B^cE) < 1$;

(ii) $P(A|BE) = P(A|B^cE) = 0$ and $ABE \simeq AB^cE$

(iii) $P(A|BE) = P(A|B^cE) = 1$ and $A^cBE \simeq A^cB^cE$.

Almost all properties established in the previous sections can be suitably extended to conditional independence.

We show now that, as in the classical case, if $A\$B$ we may have situations in which, for every event E, the condition $A\$B|E$ may *not* hold.

Let us consider, assuming $P(A) = P(B) = 0$, the case $P(A|B) = P(A|B^c) = 0$ with $AB \simeq AB^c$ (i.e. $0 < p = P(B|A) < 1$). It is enough proving that the assignments $P(A|BE) = 1$ and $P(A|B^cE) = 0$ are coherent. We must consider the conditional events $A|\Omega, B|\Omega, A|B, A|B^c, B|A, A|BE, A|B^cE$, so starting with the following system

$$\begin{cases} x_1 + x_2 + x_5 + x_6 = 0(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8) \\ x_2 + x_3 + x_6 + x_7 = 0(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8) \\ x_2 + x_6 = 0(x_2 + x_3 + x_6 + x_7) \\ x_1 + x_5 = 0(x_1 + x_4 + x_5 + x_8) \\ x_2 + x_6 = p(x_1 + x_2 + x_5 + x_6) \\ x_2 = 1(x_2 + x_3) \\ x_1 = 0(x_1 + x_4) \\ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 1 \\ x_i \geq 0 \end{cases}$$

whose unknowns are

$$x_1 = P(AB^cE)\,,\ x_2 = P(ABE)\,,\ x_3 = P(A^cBE)\,,\ x_4 = P(A^cB^cE)\,,$$
$$x_5 = P(AB^cE^c)\,,\ x_6 = P(ABE^c)\,,\ x_7 = P(A^cBE^c)\,,\ x_8 = P(A^cB^cE^c)\,,$$

and whose solutions are such that $x_1 = x_2 = x_3 = x_5 = x_6 = x_7 = 0$ and $x_4 + x_8 > 0$. It follows that the second system is

$$\begin{cases} y_2 + y_6 = 0(y_2 + y_3 + y_6 + y_7) \\ y_1 + y_5 = p(y_1 + y_2 + y_5 + y_6) \\ y_2 = 1(y_2 + y_3) \\ y_1 + y_2 + y_3 + y_5 + y_6 + y_7 = 1 \\ y_i \geq 0 \end{cases}$$

which has the solution $y_2 = y_3 = y_6 = 0$; finally, the third system is

$$\begin{cases} z_1 + z_5 = p(z_1 + z_2 + z_5 + z_6) \\ z_2 = 1(z_2 + z_3) \\ z_1 + z_2 + z_3 + z_5 + z_6 = 1 \\ z_i \geq 0 \end{cases}$$

which has the solution $z_3 = z_6 = 0$, $z_1 + z_5 = p$, $z_2 = 1 - p$.

5. CONCLUSIONS

We showed that our definition of stochastical independence is stronger than the classical one: this circumstance avoids some inconsistencies, but others remain (for example, part (b) of Theorem 2). So we are looking for an "improved" definition, which should be still stronger than the classical one, but different from the one given in this paper: to pursue this aim we will rely on the idea of taking into account the different "layers of zeros" (in a sense related to that of Definition 1) corresponding to the relevant null conditional probabilities.

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REFERENCES

[1] G. Coletti and R. Scozzafava: Characterization of coherent conditional probabilities as a tool for their assessment and extension. Internat. J. Uncertainty, Fuzziness and Knowledge-Based Systems *4* (1996), 103–127.
[2] G. Coletti and R. Scozzafava: Exploiting zero probabilities. In: Proceedings EUFIT '97, ELITE Foundation, Aachen, 1997, pp. 1499–1503.
[3] B. de Finetti: Les probabilité nulles. Bull. Sci. Math. *60* (1936), 275–288.
[4] R. Scozzafava: A survey of some common misunderstandings concerning the role and meaning of finitely additive probabilities in statistical inference. Statistica *44* (1984), 21–45.

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