KYBERNETIKA — VOLUME 34 (1998), NUMBER 1, PAGES 3–15

TOTALLY COHERENT SET-VALUED PROBABILITY ASSESSMENTS

ANGELO GILIO AND SALVATORE INGRASSIA

We introduce the concept of total coherence of a set-valued probability assessment on a family of conditional events. In particular we give sufficient and necessary conditions of total coherence in the case of interval-valued probability assessments. Some relevant cases in which the set-valued probability assessment is represented by the unitary hypercube are also considered.

1. INTRODUCTION

A well established approach in managing uncertainty in Artificial Intelligence is probabilistic methodology (in this context, a discussion of symbolic and numerical approaches is given in [11]). In many applications, the assignment of a complete distribution can entail some difficulties because we are in a situation of partial knowledge, moreover very often we are interested in making inferences on a small number of conditional events or random quantities. In these cases de Finetti's approach, based on the well known coherence principle, allows us to introduce in a flexible and gradual way consistent probability assessments on arbitrary families of conditional events. This approach is also suitable in assessing qualitative or imprecise probabilistic judgements (see e.g. [3, 5, 6, 14, 19, 21, 22, 23, 24]).

In these cases a crucial problem is the checking of the coherence of (precise or imprecise) probability assessments on the given set of conditional events. For this aim, many algorithms have been proposed (see e.g. [4, 8, 9, 15, 18, 25]).

In this paper we analyze set-valued probability assessments on a family $\mathcal{F}$ of n conditional events. Imprecise probabilities (see [26] for a general approach) are useful to describe uncertainty in many real cases. For example, in a situation of partial knowledge, it may happen that some experts are in agreement in assigning a set-valued probability assessment $\mathcal{P} \in \mathcal{S}$, where $\mathcal{S}$ is a subset of the unitary hypercube of $\mathbb{R}^n$ (specified by the experts). An interesting case is obtained when uncertainty is managed by imprecise assessments such that $\mathcal{S}$ is some subset of the set $\mathcal{M}$ of all the precise (coherent) probability assessments on $\mathcal{F}$. In particular we are interested in convex subsets $\mathcal{S}$ of $[0,1]^n$, more specifically when $\mathcal{S}$ is an *interval* of $[0,1]^n$, say $[a_1, b_1] \times \cdots \times [a_n, b_n]$.

Two reasons which justify the interest in suitable subsets of $\mathcal{M}$ are: the determination of the set $\mathcal{M}$ is in general difficult and moreover, as observed in [12], in general $\mathcal{M}$ is not convex, as the following counterexample shows. Let $\mathcal{P} = (\alpha, \beta, \gamma)$ be a probability assessment on $\mathcal{F} = \{A|H,\ B|AH,\ AB|H\}$. As well-known, the coherence of $\mathcal{P}$ requires that the condition $\gamma = \alpha\beta$ be satisfied. Therefore the set $\mathcal{M}$ of all coherent probability assessments on $\mathcal{F}$ is the subset of points (α, β, γ) of the *unitary cube* in $I\!R^3$ such that $\gamma = \alpha\beta$. Given two points $\mathcal{P}_1 = (\alpha_1, \beta_1, \gamma_1)$, $\mathcal{P}_2 = (\alpha_2, \beta_2, \gamma_2)$ in $\mathcal{M}$ and $t \in (0,1)$ it is easy to verify that the point $\mathcal{P} = t\mathcal{P}_1 + (1-t)\mathcal{P}_2$ in general does not belong to $\mathcal{M}$, that is $\mathcal{P}$ is not coherent, so that $\mathcal{M}$ is not convex.

The paper is organized as follows. In the next section we give notations and some preliminaries, in particular, given a probability assessment $\mathcal{P}$ on a family $\mathcal{F}$ of conditional events, we recall the concept of generalized atoms associated to the pair $(\mathcal{F}, \mathcal{P})$ and give an algorithm for checking coherence of $\mathcal{P}$. In Section 3 we introduce the concept of total coherence of an imprecise assessment $\mathcal{P}_{\mathcal{S}}$ and give some related results. Finally in Section 4 we give some illustrative examples.

2. PRELIMINARIES AND NOTATIONS

Let $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\}$ be a set of conditional events, $\mathcal{P}$ be a probability assessment on $\mathcal{F}$. We denote by Π the partition of the certain event Ω obtained by developing the expression:

$$(E_1H_1 \vee E_1^cH_1 \vee H_1^c) \wedge (E_2H_2 \vee E_2^cH_2 \vee H_2^c) \wedge \ldots \wedge (E_nH_n \vee E_n^cH_n \vee H_n^c)\,. \quad (1)$$

We denote by $C_1, \ldots, C_m$ the atoms of Π contained in $H_0 = H_1 \vee H_2 \vee \ldots \vee H_n$ and, if $H_0 \subset \Omega$, we denote by C_0 the atom $H_0^c = H_1^cH_2^c \cdots H_n^c$. We say that $C_0, C_1, \ldots, C_m$ are the constituents (or atoms) corresponding to (or generated by) the family $\mathcal{F}$.

In [13] the set $\mathcal{Q} = \{Q_1, \ldots, Q_m\}$ of the generalized atoms associated to the atoms $C_1, \ldots, C_m$ has been introduced, where $Q_r = (q_{r1}, q_{r2}, \ldots, q_{rn}) \in [0,1]^n$ is given by

$$q_{ri} = \begin{cases} 1 & \text{if } C_r \subseteq E_iH_i \\ 0 & \text{if } C_r \subseteq E_i^cH_i \\ p_i & \text{if } C_r \subseteq H_i^c. \end{cases} \quad (2)$$

We say that the set $\mathcal{Q}$ is generated by (or relative to) the pair $(\mathcal{F}, \mathcal{P})$.

Let us consider the following system:

$$\begin{cases} p_i = \sum_{r=1}^m \lambda_r q_{ri} & i = 1, \ldots, n \\ \sum_{r=1}^m \lambda_r = 1 & \\ \lambda_r \geq 0 & r = 1, \ldots, m \end{cases} \quad (3)$$

in the m non negative unknowns $\lambda_1, \ldots, \lambda_m$. From a geometrical point of view, the compatibility of the system (3) means that $\mathcal{P}$ is in the convex hull $\mathcal{I}$ of $Q_1, \ldots, Q_m$.

As remarked in [13], $\mathcal{P} \in \mathcal{I}$ is a necessary but not sufficient condition for the coherence of $\mathcal{P}$.

We say that n events $E_1, \ldots, E_n$ are *logically independent* if their corresponding constituents are 2^n; this concept can be naturally generalized to conditional events and we state that $E_1|H_1, \ldots, E_n|H_n$ are *logically independent* if the number of their corresponding constituents is 3^n. The concept of logical dependence among conditional events has been deepened in [8] and [10].

Given the set $J_0 = \{1, 2, \ldots, n\}$, for any $J = (j_1, \ldots, j_k) \subset J_0$ let us define

$$\mathcal{F}_J = \{E_{j_1}|H_{j_1}, \ldots, E_{j_k}|H_{j_k}\}$$

and

$$\mathcal{P}_J = (p_{j_1}, \ldots, p_{j_k}).$$

Moreover let $\mathcal{I}_J$ be the convex hull of the generalized atoms relative to $(\mathcal{F}_J, \mathcal{P}_J)$ and Λ be the set of solutions of (3). For each $j \in J_0$, let us introduce the quantity:

$$M_j = \max_{(\lambda_1, \ldots, \lambda_m) \in \Lambda} \sum_{t : C_t \subseteq H_j} \lambda_t \tag{4}$$

and afterwards the set

$$I_0 = \{j \in J_0 \, : \, M_j = 0\}. \tag{5}$$

We point out that $M_j \leq 1$ for each $j = 1, \ldots, n$ and $I_0 \subset J_0$.

In [16] the following recursive procedure is given for checking coherence of a probability assessment $\mathcal{P}$ on a family $\mathcal{F}$ of conditional events.

Algorithm 1. Let the pair $(\mathcal{F}, \mathcal{P})$ be given.

1. Compute the generalized atoms relative to $(\mathcal{F}, \mathcal{P})$ and check the compatibility of the system (3);
2. If the system (3) is not compatible then $\mathcal{P}$ is not coherent and the procedure stops, otherwise compute the set I_0 given by (5);
3. If $I_0 = \emptyset$ then $\mathcal{P}$ is coherent and the procedure stops, otherwise set $(\mathcal{F}, \mathcal{P}) = (\mathcal{F}_{I_0}, \mathcal{P}_{I_0})$ and repeat steps 1–3.

Some geometrical aspects of this algorithm have been further considered in [18].

3. IMPRECISE PROBABILITIES

In this section we consider some aspects of coherence when uncertainty is managed by imprecise probabilities. We preliminarly observe that some definitions of (local type) coherence have been studied in [4, 15] and [25].

Let a family $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\}$ and a vector $\mathcal{A} = (\alpha_1, \ldots, \alpha_n)$ of lower bounds $P(E_i|H_i) \geq \alpha_i$, for $i = 1, \ldots, n$, be given. In [15] the vector of lower bounds $\mathcal{A}$ on $\mathcal{F}$ is defined as coherent if and only if there exists a (precise) coherent assessment $\mathcal{P} = (p_1, \ldots, p_n)$ on $\mathcal{F}$, with $p_i = P(E_i|H_i)$, such that $p_i \geq \alpha_i$ for each i.

In [4] the concept of *numerical generalized probabilistic* assessment is introduced as a *multivalued compact* and *convex* function ψ defined on a family of conditional events $\mathcal{E}$ with values in the set of parts of $I\!R$, $P(I\!R)$, with range $\Psi = \{\psi(E|H) = [p^*, p^{**}], p^* \leq p^{**}, E|H \in \mathcal{E}\}$. In that paper, the concept of coherence was introduced by means of a suitable definition, whose interpretation is based on the betting criterion.

In [25] some related results based on the approach proposed in [26] are given.

Here we consider a concept of *total (or global) coherence* of imprecise probability assessments. Given a family of n conditional events $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\}$ and a set $\mathcal{S} \subseteq [0,1]^n$, let us consider the set-valued probability assessment $\mathcal{P} \in \mathcal{S}$ on $\mathcal{F}$, denoted by $\mathcal{P}_{\mathcal{S}}$, where $\mathcal{P} = (p_1, \ldots, p_n)$, with $p_i = P(E_i|H_i)$ for $i = 1, \ldots, n$.

Definition 2. The set-valued probability assessment $\mathcal{P}_{\mathcal{S}}$ on $\mathcal{F}$ is defined *totally coherent* if the precise assessment $\mathcal{P}$ on $\mathcal{F}$ is coherent for every $\mathcal{P} \in \mathcal{S}$.

The next proposition provides a first case of total coherence. Preliminarly we recall that the relation of inclusion $\subseteq$ can be extended to conditional events, by defining (see [20]):

$$B|A \subseteq D|C \iff AB \subseteq CD \text{ and } D^cC \subseteq B^cA.$$

Obviously the relation $\subseteq$ is transitive.

Proposition 3. Given a family $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\}$ of n conditional events, let $\mathcal{S} = \{\mathcal{P}\}$ be a set of precise probability assessments $\mathcal{P} = (p_1, \ldots, p_n)$ on $\mathcal{F}$. If $E_1|H_1 \subseteq E_2|H_2 \subseteq \cdots \subseteq E_n|H_n$, then $\mathcal{P}_{\mathcal{S}}$ is totally coherent if and only if $\mathcal{S}$ is a subset of the (convex) set $\mathcal{M}$ of the points $(p_1, \ldots, p_n)$ of the unitary hypercube of $[0,1]^n$ such that $p_1 \leq p_2 \leq \cdots \leq p_n$.

Proof. Given a probability assessment $\mathcal{P} = (p_1, p_2)$ on the family $\{B|A,\ D|C\}$, with $B|A \subseteq D|C$, $\mathcal{P}$ is coherent if and only if $p_1 \leq p_2$ (see e.g. Proposition 7 in [6]). As the relation $\subseteq$ is transitive, then the set of all coherent probability assessments $\mathcal{P}$ on $\mathcal{F}$ coincides with the set $\mathcal{M} = \{(p_1, \ldots, p_n) \in I\!R^n : 0 \leq p_1 \leq p_2 \leq \cdots \leq p_n \leq 1\}$. Therefore the set $\mathcal{P}_{\mathcal{S}}$ is totally coherent if and only if $\mathcal{S} \subseteq \mathcal{M}$. □

Now let us consider *interval-valued probability assessments* on a set of conditional events $\mathcal{F}$.

Given $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in [0,1]^n$, with $\alpha_i \leq \beta_i$ for $i = 1, \ldots, n$, let $\mathcal{P}_{\alpha,\beta}$ be the interval-valued probability assessment on $\mathcal{F}$ such that

$$\alpha_i \leq p_i \leq \beta_i \quad i = 1, \ldots, n, \tag{6}$$

where $p_i = P(E_i|H_i)$. We write $\mathcal{P}_{\alpha,\beta} = \{[\alpha_1, \beta_1], \cdots, [\alpha_n, \beta_n]\}$.

In particular we denote by $\mathcal{P}_{0,1} = \{[0,1], \cdots, [0,1]\}$, $\mathcal{P}_{0,\beta} = \{[0,\beta_1], \cdots, [0,\beta_n]\}$ and $\mathcal{P}_{\alpha,1} = \{[\alpha_1, 1], \cdots, [\alpha_n, 1]\}$ the interval-valued probability assessments on $\mathcal{F}$ respectively defined by $0 \leq p_i \leq 1$, $0 \leq p_i \leq \beta_i$ and $\alpha_i \leq p_i \leq 1$ for $i = 1, \ldots, n$. Finally if $\alpha_i = \beta_i = p_i$ for some i, we write $\mathcal{P}_{\alpha,\beta} = \{[\alpha_1, \beta_1], \cdots, p_i, \cdots, [\alpha_n, \beta_n]\}$.

We point out that $\mathcal{P}_{\alpha,\beta}$ is associated with an interval contained in the unitary hypercube $[0,1]^n$. We denote by $\mathcal{P}_1,\ldots,\mathcal{P}_{2^n}$ the 2^n probability assessments relative to the vertices of this interval, that is $\mathcal{P}_i = (p_1,\ldots,p_n) \in \{\alpha_1,\beta_1\} \times \cdots \times \{\alpha_n,\beta_n\}$ for $i = 1,\ldots,2^n$; these probability assessments will be referred to as the *vertices* of the interval-valued probability assessment $\mathcal{P}_{\alpha,\beta}$.

The Definition 2 in the case of interval-valued probability assessments can be specialized as follows.

Definition 4. We say that an interval-valued probability assessment $\mathcal{P}_{\alpha,\beta}$ is *totally coherent* if every precise conditional probability assessment $\mathcal{P} = (p_1,\ldots,p_n)$ on $\mathcal{F}$, with

$$\alpha_i \leq p_i \leq \beta_i, \quad i = 1,\ldots,n,$$

is coherent.

Now let us give some results about total coherence. The first theorem concerns a necessary condition for total coherence.

Theorem 5. Let $\mathcal{P}_1,\ldots,\mathcal{P}_{2^n}$ be the vertices of an interval-valued probability assessment $\mathcal{P}_{\alpha,\beta}$ on a family $\mathcal{F}$ of conditional events, and let $\mathcal{Q}_1 = \{Q_{11},\ldots,Q_{1m}\},\ldots,$ $\mathcal{Q}_{2^n} = \{Q_{2^n1},\ldots,Q_{2^nm}\}$ be the sets of the generalized atoms relative to $(\mathcal{P}_1,\mathcal{F}),\ldots,$ $(\mathcal{P}_{2^n},\mathcal{F})$. If $\mathcal{P}_{\alpha,\beta}$ is totally coherent then for every subscript j the point $\mathcal{P}_j$ belongs to the convex hull of $\mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_{2^n}$.

Proof. Assume that $\mathcal{P}_{\alpha,\beta}$ is totally coherent. Then, for every $j = 1,\ldots,2^n$, $\mathcal{P}_j$ is coherent and thus it belongs to the convex hull of $\mathcal{Q}_j$ and consequently it also belongs to the convex hull of $\mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_{2^n}$. □

In the following, we shall apply the above theorem as a criterion for checking non total coherence of $\mathcal{P}_{\alpha,\beta}$.

In order to prove the main result of this section, we need the following lemma.

Lemma 6. Let $\mathcal{P} = (p_1,\ldots,p_n)$ be a coherent probability assessment on a family $\mathcal{F} = \{E_1|H_1,\ldots,E_n|H_n\}$ and p^*, p^{**} be two probability evaluations for another conditional event $E_{n+1}|H_{n+1}$. If the assessments $(p_1,\ldots,p_n,p^*)$ and $(p_1,\ldots,p_n,p^{**})$ on $\mathcal{F} \cup \{E_{n+1}|H_{n+1}\}$ are coherent, then the assessment $(p_1,\ldots,p_n,p_{n+1})$ on $\mathcal{F} \cup \{E_{n+1}|H_{n+1}\}$ is coherent for every $p_{n+1} \in [p^*,p^{**}]$.

Proof. Given a conditional event $E_{n+1}|H_{n+1}$, it is well-known that the probability assessment $(p_1,\ldots,p_n,p_{n+1})$ is a coherent extension of the assessment $\mathcal{P}$ (defined on $\mathcal{F}$) to the family $\mathcal{F} \cup \{E_{n+1}|H_{n+1}\}$ if and only if p_{n+1} belongs to a suitable interval $[p',p''] \subseteq [0,1]$. Moreover, the hypothesis of coherence of the assessments $(p_1,\ldots,p_n,p^*)$ and $(p_1,\ldots,p_n,p^{**})$ implies that the values p^* and p^{**} belong to $[p',p'']$, and then $[p^*,p^{**}] \subseteq [p',p'']$. Thus from $p_{n+1} \in [p^*,p^{**}]$ it follows $p_{n+1} \in [p',p'']$ and then $(p_1,\ldots,p_n,p_{n+1})$ is a coherent probability assessment on the family $\{E_1|H_1,\ldots,E_n|H_n,E_{n+1}|H_{n+1}\}$. □

Theorem 7. An interval-valued assessment $\mathcal{P}_{\alpha,\beta}$ on $\mathcal{F}$ is totally coherent if and only if every precise assessment $\mathcal{P} = (x_1, \ldots, x_n)$ on $\mathcal{F}$ with

$$x_i \in \{\alpha_i, \beta_i\}, \quad i = 1, \ldots, n,$$

is coherent.

Proof. The necessary condition follows by the definition of total coherence. The sufficient condition is obtained by a recursive application of Lemma 6. Let us assume the coherence of the assessment $\mathcal{P} = (x_1, \ldots, x_n)$ for every $(x_1, \ldots, x_n) \in \{\alpha_1, \beta_1\} \times \cdots \times \{\alpha_n, \beta_n\}$. Then from coherence of $(\alpha_1, x_2, \ldots, x_n)$ and $(\beta_1, x_2, \ldots, x_n)$, by Lemma 6, the coherence of $(p_1, x_2, \ldots, x_n)$ follows for every $p_1 \in [\alpha_1, \beta_1]$. Analogously, from coherence of $(p_1, \alpha_2, x_3, \ldots, x_n)$ and $(p_1, \beta_2, x_3, \ldots, x_n)$ the coherence of $(p_1, p_2, x_3, \ldots, x_n)$ follows for every $(p_1, p_2) \in [\alpha_1, \beta_1] \times [\alpha_2, \beta_2]$, and so on. In this way we obtain that the assessment $\mathcal{P} = (p_1, \ldots, p_n)$ is coherent for every $(p_1, \ldots, p_n) \in [\alpha_1, \beta_1] \times \cdots \times [\alpha_n, \beta_n]$, that is $\mathcal{P}_{\alpha,\beta}$ is totally coherent. □

In conclusion, the total coherence of the interval-valued probability assessment $\mathcal{P}_{\alpha,\beta} = \{[\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n]\}$ amounts to the coherence of the 2^n probability assessments

$$\mathcal{P} = (x_1, \ldots, x_n) \in \{\alpha_1, \beta_1\} \times \cdots \times \{\alpha_n, \beta_n\}.$$

Theorem 7 is the basis of some results which will be given below.

Proposition 8. Let $\mathcal{F} = \{E_1, \ldots, E_n\}$ be a family of n events. If $E_1, \ldots, E_n$ are logically independent, then the interval-valued assessment $\mathcal{P}_{0,1} = \{[0,1], \ldots, [0,1]\}$ on $\mathcal{F}$ is totally coherent.

Proof. In this case $\mathcal{F}$ is a family of unconditional events. Then the coherence of a given assessment $\mathcal{P}$ on $\mathcal{F}$ amounts to the condition $\mathcal{P} \in \mathcal{I}$, where $\mathcal{I}$ is the convex hull of the generalized atoms $Q_1, \ldots, Q_m$. Since $E_1, \ldots, E_n$ are logically independent there are 2^n atoms and, as $H_0 = \Omega$ (yielding $C_0 = \emptyset$), the corresponding generalized atoms are the 2^n vertices $Q_1, \ldots, Q_{2^n}$ of the unitary hypercube of $\mathbb{R}^n$. For every assessment $\mathcal{P} = (x_1, \ldots, x_n) \in \{0,1\}^n$, it is $\mathcal{P} = Q_h$ for a certain subscript h and then $\mathcal{P} \in \mathcal{I}$. Hence the interval-valued assessment $\mathcal{P}_{0,1} = \{[0,1], \ldots, [0,1]\}$ is totally coherent. □

A different proof of Proposition 8 is given in [2].

Remark 9. With analogous arguments, Proposition 8 can be extended to the case of a family $\mathcal{F} = \{E_1|H, \ldots, E_n|H\}$, with $E_1, \ldots, E_n, H$ logically independent.

Lemma 10. Let $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\}$ be a family of n logically independent conditional events and let $\mathcal{P} = (x_1, \ldots, x_n)$ be a probability assessment on $\mathcal{F}$ such that $x_i \in \{0,1\}$ for each $i = 1, \ldots, n$. Then there are 2^n distinct generalized atoms generated by $(\mathcal{F}, \mathcal{P})$ and hence $\mathcal{P} \in \mathcal{Q}$, that is there exists at least a subscript k such that $\mathcal{P} = Q_k$.

Proof. As $x_i \in \{0,1\}$ for $i = 1, \dots, n$, then by (2) it follows that $q_{ri} \in \{0,1\}$ for $r = 1, \dots, m$ and $i = 1, \dots, n$ so that the Q_r's are vertices of the unitary hypercube of $I\!R^n$. Now let us prove that the set of distinct generalized atoms coincides with the set of vertices of the hypercube. Let $(x_1, \dots, x_n)$ be any vertex of the hypercube. Developing the expression

$$(A_1 \vee H_1^c) \wedge \cdots \wedge (A_n \vee H_n^c)$$

where it is $A_i = E_i H_i$ or $A_i = E_i^c H_i$ according to whether $x_i = 1$ or $x_i = 0$, $i = 1, \dots, n$, we obtain 2^n atoms. If we consider the $2^n - 1$ atoms different from $C_0 = H_1^c \cdots H_n^c$, then we can easily verify that all the corresponding generalized atoms coincide with the vertex $(x_1, \dots, x_n)$. □

We observe that for every $j = 1, \dots, n$, the atom $A_1 \cdots A_n$ is contained in H_j and in the following it will be denoted by C_1.

Proposition 11. Let $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n\}$ be a family of n logically independent conditional events. Then the interval-valued assessment $\mathcal{P}_{0,1} = \{[0,1], \dots, [0,1]\}$ on $\mathcal{F}$ is totally coherent.

Proof. Since $E_1|H_1, \dots, E_n|H_n$ are logically independent there are 3^n atoms and $3^n - 1$ generalized atoms for any probability assessment $\mathcal{P}$ on $\mathcal{F}$. Given an assessment $\mathcal{P} = (x_1, \dots, x_n) \in \{0,1\}^n$, Lemma 10 implies that the generalized atom Q_1 associated with the atom $C_1 = A_1 \cdots A_n$ coincides with $\mathcal{P}$. Then, applying the Algorithm 1, the condition $\mathcal{P} \in \mathcal{I}$ corresponding to the compatibility of the system

$$\mathcal{P} = \sum_{h=1}^{m} \lambda_h Q_h, \quad \sum_{h=1}^{m} \lambda_h = 1, \quad \lambda_h \geq 0$$

is satisfied in particular when $\lambda_1 = 1$ and $\lambda_h = 0$ for $h \neq 1$. Then, for every H_j it is

$$\sum_{h: C_h \subseteq H_j} \lambda_h = \lambda_1 = 1,$$

so that it results $M_j = 1$ for every j and hence $I_0 = \emptyset$. Thus, for every $(x_1, \dots, x_n) \in \{0,1\}^n$ the assessment $\mathcal{P} = (x_1, \dots, x_n)$ is coherent and therefore the interval-valued assessment $\mathcal{P}_{0,1} = \{[0,1], \dots, [0,1]\}$ on $\mathcal{F}$ is totally coherent. □

Proposition 12. Let $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n\}$ be a family of n conditional events. If $H_1, \dots, H_n$ are pairwise incompatible then the interval-valued assessment $\mathcal{P}_{0,1} = \{[0,1], \dots, [0,1]\}$ on $\mathcal{F}$ is totally coherent.

Proof. Since $H_i H_j = \emptyset$ for $i \neq j$, the atoms $C_1, \dots, C_m$ are the following ones:

$$\begin{array}{lllllll}
C_1 & = & E_1 H_1 H_2^c \cdots H_n^c & \quad & C_2 & = & E_1^c H_1 H_2^c \cdots H_n^c \\
C_3 & = & H_1^c E_2 H_2 H_3^c \cdots H_n^c & & C_4 & = & H_1^c E_2^c H_2 H_3^c \cdots H_n^c \\
\cdots & & \cdots & & \cdots & & \cdots \\
C_{2n-1} & = & H_1^c \cdots H_{n-1}^c E_n H_n & & C_{2n} & = & H_1^c \cdots H_{n-1}^c E_n^c H_n \; .
\end{array}$$

Given an assessment $\mathcal{P} = (p_1, \ldots, p_n)$, the generalized atoms are

$$\begin{array}{rclcrcl} Q_1 & = & (1, p_2, \ldots, p_n) & & Q_2 & = & (0, p_2, \ldots, p_n) \\ Q_3 & = & (p_1, 1, p_3, \ldots, p_n) & & Q_4 & = & (p_1, 0, p_3, \ldots, p_n) \\ \ldots & & \ldots & & \ldots & & \ldots \\ Q_{2n-1} & = & (p_1, \ldots, p_{n-1}, 1) & & Q_{2n} & = & (p_1, \ldots, p_{n-1}, 0) \ . \end{array}$$

We observe that if $\mathcal{P} \in \{0,1\}^n$ then $Q_h \in \{0,1\}^n$ for every subscript h. Moreover, for each $k = 1, \ldots, n$ it is $\mathcal{P} = Q_{2k-1}$ or $\mathcal{P} = Q_{2k}$ according to whether $p_k = 1$ or $p_k = 0$. In other words, n of the $2n$ generalized atoms, say $Q_{i_1}, \ldots, Q_{i_n}$, coincide with $\mathcal{P}$, and $i_k = 2k-1$ or $i_k = 2k$, for $k = 1, \ldots, n$.

Then, for every $\mathcal{P} \in \{0,1\}^n$, the condition $\mathcal{P} \in \mathcal{I}$ is satisfied, that is the system in the unknowns $\lambda_1, \ldots, \lambda_{2n}$

$$\mathcal{P} = \sum_{h=1}^{2n} \lambda_h Q_h, \quad \sum_{h=1}^{2n} \lambda_h = 1, \quad \lambda_h \geq 0$$

is compatible. In particular the system has the following solutions: $\boldsymbol{\lambda}_1 : \lambda_{i_1} = 1$ and $\lambda_h = 0$ for $h \neq i_1; \ldots; \boldsymbol{\lambda}_n : \lambda_{i_n} = 1$ and $\lambda_h = 0$ for $h \neq i_n$. Then every linear convex combination with positive coefficients of $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_n$ is a solution $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{2n})$ of the system such that

$$\lambda_{2j-1} + \lambda_{2j} > 0, \quad j = 1, \ldots, n.$$

Moreover, for each j, it is

$$\sum_{h: C_h \subseteq H_j} \lambda_h = \lambda_{2j-1} + \lambda_{2j} > 0,$$

so by Algorithm 1 we get $I_0 = \emptyset$. Therefore $\mathcal{P}$ is coherent and $\mathcal{P}_{0,1} = \{[0,1], \ldots, [0,1]\}$ is totally coherent. □

Remark 13. We observe that in Bayesian uncertainty modeling, if we regard a given event E as evidence and n events $H_1, \ldots, H_n$ as hypotheses constituting a partition of the certain event Ω, then the probabilities $P(E|H_j)$, $j = 1, \ldots, n$, play the role of the likelihood and by Proposition 12 they are coherent. The more realistic case in which $H_1, \ldots, H_n$ do not constitute a partition of Ω, has been considered in [7], where the checking of coherence of the assessments $\{P(H_j), P(E|H_j), j = 1, \ldots, n\}$ has been studied in the context of automatic medical diagnosis.

Proposition 14. Let $(p_1, \ldots, p_n)$ be a coherent probability assessment on a family $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\}$ and $E_{n+1}|H_{n+1}$ be a further conditional event, with $H_{n+1} \wedge (H_1 \vee \cdots \vee H_n) = \emptyset$. Then the interval-valued assessment $\{p_1, \ldots, p_n, [0,1]\}$ on $\mathcal{F} \cup \{E_{n+1}|H_{n+1}\}$ is totally coherent.

Proof. Let $C_0, C_1, \ldots, C_m$ be the atoms corresponding to the family $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\}$. Then, as $H_{n+1} \wedge (H_1 \vee \cdots \vee H_n) = \emptyset$, for the family $\mathcal{F}' = \mathcal{F} \cup \{E_{n+1}|H_{n+1}\}$ the atoms are the following ones

$$C_1' = C_1 H_{n+1}^c, \qquad C_2' = C_2 H_{n+1}^c, \qquad \cdots \qquad C_m' = C_m H_{n+1}^c$$
$$C_{m+1}' = C_0 E_{n+1} H_{n+1}, \quad C_{m+2}' = C_0 E_{n+1}^c H_{n+1}, \quad \cdots \quad C_0' = C_0 H_{n+1}^c.$$

Then there are $m+2$ generalized atoms $Q_1', \dots, Q_{m+1}', Q_{m+2}'$ with $Q_{m+1}' = (p_1, \dots, p_n, 1)$ and $Q_{m+2}' = (p_1, \dots, p_n, 0)$.

We observe that the total coherence of $\{p_1, \dots, p_n, [0,1]\}$ amounts to coherence of the two assessments $(p_1, \dots, p_n, 0)$ and $(p_1, \dots, p_n, 1)$. As concerns the assessment $\mathcal{P}' = (p_1, \dots, p_n, 1)$ on $\mathcal{F}'$, since $\mathcal{P}' = Q_{m+1}'$ the system

$$\mathcal{P}' = \sum_{h=1}^{m+2} \lambda_h Q_h', \quad \sum_{h=1}^{m+2} \lambda_h = 1, \quad \lambda_h \geq 0 \tag{7}$$

is satisfied by the solution $\lambda_h = 0$, for $h \neq m+1$, $\lambda_{m+1} = 1$. Thus we get

$$\sum_{h: C_h \subseteq H_j} \lambda_h = \begin{cases} 0 & \text{for } j = 1, \dots, n \\ 1 & \text{for } j = n+1 \end{cases}$$

so that $M_{n+1} = 1$ and, by coherence of $\mathcal{P}$, it follows (see Algorithm 5.2 in [17]) that the assessment $\mathcal{P}' = (\mathcal{P}, 1) = (p_1, \dots, p_n, 1)$ on $\mathcal{F}' = \mathcal{F} \cup \{E_{n+1}|H_{n+1}\}$ is coherent.

With analogous arguments, considering $\mathcal{P}' = (\mathcal{P}, 0) = (p_1, \dots, p_n, 0)$, as $\mathcal{P}' = Q_{m+2}'$, the system (7) is compatible with $M_{n+1} = 1$ so that $(p_1, \dots, p_n, 0)$ is coherent too and we conclude that the interval-valued assessment $\{p_1, \dots, p_n, [0,1]\}$ is totally coherent. □

Remark 15. We observe that, from the point of view of the *fundamental theorem of probability* of de Finetti, the previous result amounts to stating that the interval $[p', p'']$ of the coherent extensions of the assessment $\mathcal{P}$ on $\mathcal{F}$ to $E_{n+1}|H_{n+1}$ coincides with the interval $[0,1]$.

4. EXAMPLES

In this section we give some applications of the previous results: in the first one the total coherence is attained; in the second one the total coherence is not verified; in the third one we consider a case in which the intervals are narrower than $[0,1]$.

Example 16. Consider the interval-valued probability assessment $\mathcal{P}_{0,1}$ on $\mathcal{F} = \{E|H,\ H^c|(E^cH \vee H^c)\}$. In the following we prove that $\mathcal{P}_{0,1}$ is totally coherent. As

$$\begin{aligned} H^c \wedge (E^cH \vee H^c) &= H^c \\ H \wedge (E^cH \vee H^c) &= E^cH \\ (E^cH \vee H^c)^c &= (E^cH)^c \wedge H = (E \vee H^c) \wedge H = EH, \end{aligned}$$

according to (1), we have

$$(EH \vee E^cH \vee H^c) \wedge (H^c \vee E^cH \vee EH) = EH \vee E^cH \vee H^c.$$

Then the constituents generated by $\mathcal{F}$ are: $C_1 = EH$, $C_2 = E^cH$, $C_3 = H^c$ and, given a (precise) assessment $\mathcal{P} = (p_1, p_2)$, the generalized atoms generated by $(\mathcal{F}, \mathcal{P})$ are: $Q_1 = (1, p_2)$, $Q_2 = (0,0)$ and $Q_3 = (p_1; 1)$. In this case, coherence of $\mathcal{P}$ amounts to

$$0 \leq p_1, p_2 \leq 1 \quad \text{and} \quad \mathcal{P} \in \mathcal{I}.$$

For any $\mathcal{P} = (p_1, p_2) \in \{0,1\}^2$, there exists a subscript h such that $\mathcal{P} = Q_h$. Therefore $\mathcal{P}_{0,1} = \{[0,1],[0,1]\}$ is totally coherent.

Example 17. Consider the interval-valued probability assessment $\mathcal{P}_{0,1}$ on $\mathcal{F} = \{E|H,\, EH|(EH \vee H^c),\, E^cH|(E^cH \vee H^c)\}$. Here we prove that $\mathcal{P}_{0,1}$ is not totally coherent. In fact, as

$$\begin{aligned} EH \wedge (EH \vee H^c) &= EH \\ (EH)^c \wedge (EH \vee H^c) &= H^c \\ (EH \vee H^c)^c &= (EH)^c \wedge H = (E^c \vee H^c) \wedge H = E^cH, \end{aligned}$$

and

$$\begin{aligned} E^cH \wedge (E^cH \vee H^c) &= E^cH \\ (E^cH)^c \wedge (E^cH \vee H^c) &= H^c \\ (E^cH \vee H^c)^c &= (E^cH)^c \wedge H = (E \vee H^c) \wedge H = EH, \end{aligned}$$

according to (1) the constituents generated by $\mathcal{F}$ are: $C_1 = EH$, $C_2 = E^cH$ and $C_3 = H^c$. Hence, given a (precise) assessment $\mathcal{P} = (p_1, p_2, p_3)$, the generalized atoms generated by $(\mathcal{F}, \mathcal{P})$ are: $Q_1 = (1,1,p_3)$, $Q_2 = (0, p_2, 1)$ and $Q_3 = (p_1, 0, 0)$.

One can immediately see that $\mathcal{P}_{0,1} = \{[0,1],[0,1],[0,1]\}$ is not totally coherent. In fact the set of the generalized atoms generated by the vertices of $\mathcal{P}_{0,1}$ and $\mathcal{F}$ is $\{(1,1,0), (1,1,1), (0,0,1), (0,1,1), (0,0,0), (1,0,0)\}$, which does not contain the assessments $\mathcal{P} = (1,0,1)$ and $\mathcal{P} = (0,1,0)$; then Theorem 5 implies that $\mathcal{P}_{0,1}$ is not totally coherent.

Example 18. Let us consider the interval-valued probability assessment $\mathcal{P}_{\alpha,\beta} = \{[\frac{1}{4}, \frac{1}{2}], [\frac{1}{3}, \frac{2}{3}], [\frac{2}{5}, \frac{5}{11}]\}$ on $\mathcal{F} = \{C|A,\, C|B,\, C|(A \vee B)\}$, where A, B and C are logically independent. We prove that $\mathcal{P}_{\alpha,\beta}$ is totally coherent. For this aim, from Theorem 7 we need to check coherence of the eight precise assessments corresponding to the vertices of the interval $[\frac{1}{4}, \frac{1}{2}] \times [\frac{1}{3}, \frac{2}{3}] \times [\frac{2}{5}, \frac{5}{11}]$. In [16] it is proved that the coherence of an assessment $\mathcal{P} = (\alpha, \beta, \gamma)$ on the given family $\mathcal{F}$ reduces to the condition "$\mathcal{P} \in \mathcal{I}$", where $\mathcal{I}$ is the convex hull of the generalized atoms which in our case are: $Q_1 = (1,1,1)$, $Q_2 = (0,0,0)$, $Q_3 = (1, \beta, 1)$, $Q_4 = (0, \beta, 0)$, $Q_5 = (\alpha, 1, 1)$ and $Q_6 = (\alpha, 0, 0)$. This condition is represented by the following system:

$$\begin{cases} \lambda_1 + \lambda_3 + \alpha(\lambda_5 + \lambda_6) = \alpha \\ \lambda_1 + \lambda_5 + \beta(\lambda_3 + \lambda_4) = \beta \\ \lambda_1 + \lambda_3 + \lambda_5 = \gamma \\ \sum_{i=1}^{6} \lambda_i = 1\,, \lambda_i \geq 0. \end{cases}$$

Some calculations show that the above system is compatible for all the eight vertices. In particular the assessment $\left(\frac{1}{4}, \frac{1}{3}, \frac{5}{11}\right)$ belongs to the face of the convex hull delimited by the generalized atoms $(0,0,0)$, $\left(1, \frac{1}{3}, 1\right)$ and $\left(\frac{1}{4}, 1, 1\right)$; the assessment $\left(\frac{1}{2}, \frac{2}{3}, \frac{2}{5}\right)$ belongs to the face delimited by the generalized atoms $\left(\frac{1}{2}, 0, 0\right)$, $\left(0, \frac{2}{3}, 0\right)$ and $(1,1,1)$. The points representing the other assessments are internal to the corresponding convex hulls.

We point out that the family $\mathcal{F} = \{C|A, C|B, C|(A \vee B)\}$ is related to the *disjunction rule* of Adams, see [1]. It has been also investigated in [16] where it is proved that, for each $\alpha, \beta \in [0,1]$, the assessment (α, β, γ) is coherent if and only if $\gamma \in [\gamma', \gamma'']$, with

$$\gamma' = \frac{\alpha\beta}{\alpha + \beta - \alpha\beta}, \quad \gamma'' = \frac{\alpha + \beta - 2\alpha\beta}{1 - \alpha\beta}. \tag{8}$$

On the basis of the above results, we can give an alternative proof of the total coherence of the interval-valued probability assessment $\mathcal{P}_{\alpha,\beta} = \left\{\left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{1}{3}, \frac{2}{3}\right], \left[\frac{2}{5}, \frac{5}{11}\right]\right\}$ on $\mathcal{F}$. Consider the pair $(\alpha, \beta) = \left(\frac{1}{4}, \frac{1}{3}\right)$: by (8) we obtain that the assessment $\left(\frac{1}{4}, \frac{1}{3}, \gamma\right)$ is coherent if and only if $\gamma \in \left[\frac{1}{6}, \frac{5}{11}\right]$. Thus the assessments $\left(\frac{1}{4}, \frac{1}{3}, \frac{2}{5}\right)$ and $\left(\frac{1}{4}, \frac{1}{3}, \frac{5}{11}\right)$ are coherent. In the same way, one can prove that the assessments associated with the other vertices of $\mathcal{P}_{\alpha,\beta}$ are coherent and therefore $\mathcal{P}_{\alpha,\beta}$ is totally coherent.

(Received November 7, 1997.)

REFERENCES

[1] E. W. Adams: The Logic of Conditionals. D. Reidel, Dordrecht 1975.

[2] A. Capotorti and B. Vantaggi: The consistency problem in belief and probability assessments. In: Proceedings of the Sixth International Conference on "Information Processing and Management of Uncertainty in Knowledge-Based Systems" (IPMU '96), Granada 1996, pp. 55–59.

[3] G. Coletti: Numerical and qualitative judgements in probabilistic expert systems. In: Proceedings of the International Workshop on "Probabilistic Methods in Expert Systems" (R. Scozzafava, ed.), SIS, Roma 1993, pp. 37–55.

[4] G. Coletti: Coherent numerical and ordinal probabilistic assessments. IEEE Trans. Systems Man Cybernet. *24* (1994), 12, 1747–1754.

[5] G. Coletti, A. Gilio and R. Scozzafava: Conditional events with vague information in expert systems. In: Uncertainty in Knowledge Bases (Lecture Notes in Computer Science 521; B. Bouchon-Meunier, R. R. Yager, L. A. Zadeh, eds.), Springer-Verlag, Berlin - Heidelberg 1991, pp. 106–114.

[6] G. Coletti, A. Gilio and R. Scozzafava: Comparative probability for conditional events: a new look through coherence. Theory and Decision *35* (1993), 237–258.

[7] G. Coletti and R. Scozzafava: Learning from data by coherent probabilistic reasoning. In: Proceedings of ISUMA-NAFIPS '95, College Park 1995, pp. 535–540.

[8] G. Coletti and R. Scozzafava: Characterization of coherent conditional probabilities as a tool for their assessment and extension. J. Uncertainty, Fuzziness and Knowledge-Based Systems *4* (1996), 2, 103–127.

[9] G. Di Biase and A. Maturo: Checking the coherence of conditional probabilities in expert systems: remarks and algorithms. In: Mathematical Models for Handling Partial

Knowledge in Artificial Intelligence (G. Coletti, D. Dubois and R. Scozzafava, eds.), Plenum Press, New York 1995, pp. 191–200.

[10] S. Doria and A. Maturo: A hyperstructure of conditional events for Artificial Intelligence. In: Mathematical Models for Handling Partial Knowledge in Artificial Intelligence (G. Coletti, D. Dubois and R. Scozzafava, eds.), Plenum Press, New York 1995, pp. 201–208.

[11] D. Dubois and H. Prade: Probability in automated reasoning: from numerical to symbolic approaches. In: Probabilistic Methods in Expert Systems, Proc. of the International Workshop (R. Scozzafava, ed.), SIS, Roma 1993, pp. 79–104.

[12] S. Holzer: On coherence and conditional prevision. Boll. Un. Mat. Ital. *4* (1985), 4–B, 441–460.

[13] A. Gilio: Criterio di penalizzazione e condizioni di coerenza nella valutazione soggettiva della probabilità. Boll. Un. Mat. Ital. *7* (1990), 4–B, 645–660.

[14] A. Gilio: Conditional events and subjective probability in management of uncertainty. In: Uncertainty in Intelligent Systems (B. Bouchon–Meunier, L. Valverde and R. R. Yager, eds.), Elsevier Science Publishing B. V., North–Holland, 1993, pp. 109–120.

[15] A. Gilio: Probabilistic consistency of conditional probability bounds. In: Advances in Intelligent Computing – IPMU'94 (Lecture Notes in Computer Science 945; B. Bouchon–Meunier, R. R. Yager and L. A. Zadeh, eds.), Springer–Verlag, Berlin – Heidelberg 1995, pp. 200–209.

[16] A. Gilio: Algorithms for precise and imprecise conditional probability assessments. In: Mathematical Models for Handling Partial Knowledge in Artificial Intelligence (G. Coletti, D. Dubois and R. Scozzafava, eds.), Plenum Press, New York 1995, pp. 231–254.

[17] A. Gilio: Algorithms for conditional probability assessments. In: Bayesian Analysis in Statistics and Econometrics (D. A. Berry, K. M. Chaloner and J. K. Geweke, eds.), J. Wiley, New York 1996, pp. 29–39.

[18] A. Gilio and S. Ingrassia: Geometrical aspects in checking coherence of probability assessments. In: Proceedings of the Sixth International Conference on "Information Processing and Management of Uncertainty in Knowledge–Based Systems" (IPMU'96), Granada, 1996, pp. 55–59.

[19] A. Gilio and R. Scozzafava: Le probabilità condizionate coerenti nei sistemi esperti In: Ricerca Operativa e Intelligenza Artificiale, Atti Giornate di Lavoro A.I.R.O., IBM, Pisa 1988, pp. 317–330.

[20] I. R. Goodman and H. T. Nguyen: Conditional objects and the modeling of uncertainties. In: Fuzzy Computing Thoery, Hardware and Applications (M. M. Gupta and T. Yamakawa, eds.), North–Holland, New York 1988, pp. 119–138.

[21] F. Lad: Coherent prevision as a linear functional without an underlying measure space: the purely arithmetic structure of logical relations among conditional quantities. In: Mathematical Models for Handling Partial Knowledge in Artificial Intelligence (G. Coletti, D. Dubois and R. Scozzafava, eds.), Plenum Press, New York 1995, pp. 101–111.

[22] G. Regoli: Comparative probability assessments and stochastic independence. In: Proceedings of the Sixth International Conference on "Information Processing and Management of Uncertainty in Knowledge–Based Systems" (IPMU'96), Granada 1996, pp. 49–53.

[23] R. Scozzafava: How to solve some critical examples by a proper use of coherent probability. In: Uncertainty in Intelligent Systems (B. Bouchon–Meunier, L. Valverde and R. R. Yager, eds.), Elsevier Science Publishing B.V., North–Holland, Amsterdam 1993, pp. 121–132.

[24] R. Scozzafava: Subjective probability versus belief functions in artificial intelligence. Internat. J. Gen. Systems *22* (1994), 197–206.

[25] P. Vicig: An algorithm for imprecise conditional probability assessments in expert systems. In: Proceedings of the Sixth International Conference on "Information Processing and Management of Uncertainty in Knowledge–Based Systems" (IPMU'96), Granada 1996, pp. 61–66.

[26] P. Walley: Statistical Reasoning with Imprecise Probabilities. Chapman and Hall, London 1991.

Prof. Dr. Angelo Gilio, Dipartimento di Matematica, Viale Andrea Doria 6, Università di Catania, 95125 Catania. Italy.
e-mail: gilio@dipmat.unict.it

Dr. Salvatore Ingrassia, Istituto di Statistica, Facoltà di Economia, Università di Catania, Corso Italia 55, 95129 Catania. Italy.
e-mail: ingrax@dipmat.unict.it