KYBERNETIKA — VOLUME 33 (1997), NUMBER 4, PAGES 445-461

BLOCK BIALTERNATE SUM WITH APPLICATIONS TO COMPUTATION OF STABILITY BOUNDS

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The block bialternate sum for partitioned matrices is introduced in this paper and its basic properties are established. Using the block bialternate sum, exact values of the maximal stability range of the parameter in integral control systems and singularly perturbed systems as also the minimal range of the gain parameter in a high-gain feedback system are determined. The proposed method is claimed to be computationally superior to all other existing methods.

1. INTRODUCTION

Stability is an important property to be analyzed for all practical control systems. For cases of integral control, singularly perturbed systems and high-gain feedback systems, the bound of a scalar parameter has to be determined beyond which the system stability is lost. Fuller [2] showed that the Kronecker sum of the $n \times n$ system matrix with itself may be used for such stability analysis. The key property of the Kronecker sum which makes it suitable for this purpose is that its eigenvalues are the pairwise sums of the eigenvalues of the original matrix. However, any block structure in the original matrix is lost while computing the Kronecker sum. Hyland and Collins [3] defined a block Kronecker sum which has similar properties to the Kronecker sum while preserving the block matrices of the original system. In spite of this, the block Kronecker sum (as also the Kronecker sum) is not convenient for analytical purposes owing to its high dimension of n^2. Tesi and Vicino [11] gave a lower order $n(n+1)/2$ formula for finding the robust stability bound based on the Lyapunov sum matrix properties as given in Fuller [2]. Mustafa [8] used the concept of the block Kronecker sum to define a block Lyapunov sum of order $n(n+1)/2$. This reduced the dimension of the matrix being used for computation while still retaining the block structure of the original system matrix. Fuller [2] in the same work provided an alternative critical criteria for stability involving the system matrix of order n as well as the bialternate sum of the matrix with itself which is of dimension $n(n-1)/2$ only. In the present paper, a block bialternate sum of the system which combines the advantages of the low order bialternate sum with those of the block Kronecker sum is defined, and its basic properties are established.

The block bialternate sum thus defined is used with advantage to solve various two block-structured stability problems. Firstly, the radius of integral controllability, defined by Lunze [5] and Morari [7] as the maximal integral gain for closed-loop stability as the integral gain increases from zero, is determined using this approach. The second problem solved using this method is that of determining the stability bound ε_0 of a singularly perturbed system such that it is stable $\forall \varepsilon \in [0, \varepsilon_0)$ where ε is the singular perturbation parameter. This problem has earlier been solved by Sen and Datta [9] using the bialternate sum to obtain an $(n_1 + n_2)(n_1 + n_2 - 1)/2$ dimensional eigenvalue formula, n_1 and n_2 being the dimensions of the two blocks in the two block-structured system. Both the first and the second problem have been solved by Mustafa [8] using the block Lyapunov sum leading to a much lower $n_1 n_2$ dimensional formulae in both cases. For the singularly perturbed system, however, his approach requires two extra assumptions to be made for the solution. The method presented in the present paper requires only one of those assumptions, thus incurring considerably less loss of generality for the solution. Moreover, while still retaining the overall dimension of the eigenvalue formula at $n_1 n_2$ for both cases, the dimensions of two block matrices required to be inverted have been reduced from $n_i(n_i+1)/2$ to $n_i(n_i-1)/2$ for $i = 1, 2$ as compared to Mustafa's formulae, leading to a considerable saving in computation, particularly for higher order systems. Finally, another stability problem has also been addressed using this same approach. The lower bound g_0 of the high gain parameter g in high-gain feedback systems such that the systems are stable $\forall g \in (g_0, \infty)$ has been computed by Sen, Ghosh and Datta [10] using Fuller's [2] approach. The present approach yields a much lower dimensional formula for the solution of the minimal stability bound.

The paper has been organized as follows. In Section 2 of the paper, the critical stability criteria developed by Fuller [2] have been stated in terms of the Kronecker, Lyapunov and bialternate sum of the system matrix A with itself. Then, the Kronecker sum, the block Kronecker sum as well as the bialternate sum have been defined and their basic properties have been stated, particularly those required for the stability analysis. While the Kronecker sum and block Kronecker sum have been essentially compiled from Brewer [1] and Hyland and Collins [3] respectively, the properties of the bialternate sum have been taken in part from Fuller [2] while the rest have been developed analogous to the properties of the block Lyapunov matrix as in Mustafa [8]. In Section 3, the block bialternate sum has been defined and the structure of the matrix for systems with a 2-block structure has been derived. In Section 4, the properties of the block bialternate sum have been exploited to solve the stability problems as stated above. Finally, some lengthy proofs have been given in the Appendix.

2. DEFINITIONS AND PROPERTIES

2.1. Critical stability criteria (CSC)

Let us consider a nominally stable system A, which is subject to disturbances. When this system encounters instability, due to perturbations, one or more of its eigenvalues cross over into the right half of the s-plane (RHP) from the left half (LHP).

At the boundary of stability, either a real eigenvalue becomes 0 or a complex conjugate pair of eigenvalues pass onto the imaginary axis or both these cases occur simultaneously.

The Kronecker sum matrix $A \oplus A$ has eigenvalues $\lambda_i + \lambda_j \ \ \forall\, 1 \le i, j \le n$, where λ_i $(1 \le i \le n)$ are the eigenvalues of A. Thus, we have

$$\det(A \oplus A) = \prod_{i,j=1}^{n} (\lambda_i + \lambda_j),$$

which becomes 0 iff either or both of the above cases occur. So, a critical stability criteria (CSC) is that

$$(-1)^{n^2} \det(A \oplus A) > 0. \tag{1}$$

But, in this case, we note a redundancy in the sense that every eigenvalue of the form $(\lambda_i + \lambda_j)\ \forall\, i \neq j$ is repeated twice. This is removed, causing a considerable saving in dimension, by considering the Lyapunov sum matrix $A \overline{\oplus} A$ which also has eigenvalues $(\lambda_i + \lambda_j)$ but only for $1 \le i \le j \le n$. So, the CSC can alternatively be expressed as

$$(-1)^{(1/2)n(n+1)} \det(A \overline{\oplus} A) > 0. \tag{2}$$

This naturally leads us to consider the possibility of a further reduction in the dimension of the matrix used for determining the CSC. We note that any crossing over of the eigenvalues *through the origin* is detected by $\det(A)$. So, it is required only to check if a pair of complex conjugate roots cross over through the imaginary axis. This is done using the bialternate sum matrix $A \overline{\overline{\oplus}} A$ (Fuller [8]) which has eigenvalues $\lambda_i + \lambda_j, \forall\, 1 \le i < j \le n$. Thus, the third CSC is

$$\begin{array}{lrcl} \text{(i)} & (-1)^n \det(A) & > & 0 \\ \text{(ii)} & (-1)^{(1/2)n(n-1)} \det(A \overline{\overline{\oplus}} A) & > & 0 \end{array}. \tag{3}$$

Let us now consider the construction and some basic properties of these matrices.

2.2. Kronecker sum

If $A \in \Re^{n \times n}$ and $B \in \Re^{n \times n}$, then $(A \otimes B) \in \Re^{n^2 \times n^2}$ is the Kronecker product of matrices A and B (Brewer [1]) such that

$$A \otimes B \ := \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ & \ddots & \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix}. \tag{4}$$

For $A \in \Re^{n \times n}$ and $B \in \Re^{n \times n}$, $(A \oplus B) \in \Re^{n^2 \times n^2}$ is the Kronecker sum of matrices A and B such that

$$A \oplus B = A \otimes I_n + I_n \otimes B. \tag{5}$$

It is useful to know that there exists a permutation matrix (so called because it only interchanges certain rows and columns) $K_{mn} \in \Re^{mn \times mn}$ with the property

$$K_{mn} = K_{nm}^T = K_{nm}^{-1} \tag{6}$$

such that for $B \in \Re^{m\times m}$ and $A \in \Re^{n\times n}$,

$$B \otimes A = K_{mn} \cdot (A \otimes B) \cdot K_{nm} \tag{7}$$

which justifies the nomenclature as in (Magnus [6]). For the case when $m = n$, $K_{mn} = K_{nn} \in \Re^{n^2\times n^2}$ with the property

$$K_{nn} = K_{nn}^T = K_{nn}^{-1}. \tag{8}$$

2.3. Block Kronecker sum

Let us consider the case when the matrices A and B are partitioned matrices. Then the operations of $\otimes$ or $\oplus$ destroy the block structure. Hyland and Collins [3] overcame this drawback by defining block-structured versions of $\otimes$ and $\oplus$ which retain all important properties without destroying the partitioning. Let the matrix $A \in \Re^{n\times n}$ be partitioned as

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1r} \\ & \ddots & \\ A_{r1} & \cdots & A_{rr} \end{bmatrix} \tag{9}$$

where $A_{ij} \in \Re^{n_i\times n_j}$ and $\sum_{i=1}^{r} n_i = n$. Then A is said to have the block structure $\overline{n} := (n_1, \cdots, n_r)$ and is said to belong to $\Re^{\overline{n}\times\overline{n}}$ which denotes the set of all real $n\times n$ matrices with block structure $\overline{n} = (n_1, \cdots, n_r)$. Then, for $A, B \in \Re^{\overline{n}\times\overline{n}}$, $(A \otimes_b B) \in \Re^{n^2\times n^2}$ is the block Kronecker product where

$$A\otimes_b B := \begin{bmatrix} A_{11}\odot B & \cdots & A_{1r}\odot B \\ & \ddots & \\ A_{r1}\odot B & \cdots & A_{rr}\odot B \end{bmatrix} \tag{10}$$

such that $(A_{ij}\odot B) \in \Re^{n_i n\times n_j n}$ is defined as

$$A_{ij}\odot B := \begin{bmatrix} A_{ij}\otimes B_{11} & \cdots & A_{ij}\otimes B_{1r} \\ & \ddots & \\ A_{ij}\otimes B_{r1} & \cdots & A_{ij}\otimes B_{rr} \end{bmatrix}. \tag{11}$$

Thus, similar to (5), we have $(A\oplus_b B) \in \Re^{n^2\times n^2}$ is the block Kronecker sum matrix such that

$$A\oplus_b B := A\otimes_b I_n + I_n \otimes_b B. \tag{12}$$

The block Kronecker sum is just a rearrangement of the elements of the Kronecker sum and vice versa. Thus, there exists a permutation matrix $K_{\overline{n}\overline{n}} \in \Re^{n^2\times n^2}$ such that

$$A\oplus_b A = K_{\overline{n}\overline{n}}^T (A \oplus A) K_{\overline{n}\overline{n}}. \tag{13}$$

Thus, $(A\oplus_b A)$ and $(A \oplus A)$ are similar matrices, so the eigenvalues of $(A\oplus_b A)$ are also the n^2 numbers $(\lambda_i + \lambda_j)$ $[1 \le i, j \le n]$ where $\lambda_i (1 \le i \le n)$ are the eigenvalues of A.

2.4. Bialternate sum matrix

We have seen that of all the n^2 eigenvalues of $(A \oplus A)$ [or equivalently $(A \oplus_b A)$], it is necessary to know only $n(n-1)/2$ of them, namely $(\lambda_i + \lambda_j)[1 \leq j < i \leq n]$, where λ_i $(1 \leq i \leq n)$ are the eigenvalues of the matrix A, for stability analysis (Fuller [2]). For an $n \times n$ matrix X with elements x_{ij}, the n^2 vector vec(X) of the stacked columns of X is defined as (Brewer [1])

$$\text{vec}(X) \quad := \quad \left[x_{11} \cdots x_{n1} \vdots x_{12} \cdots x_{n2} \vdots \cdots \vdots x_{1n} \cdots x_{nn} \right]^T . \tag{14}$$

Let us define the column vector bivec(X), having only $n(n-1)/2$ elements of X, as

$$\text{bivec}(X) \quad := \quad \left[x_{21} \cdots x_{n1} \vdots x_{32} \cdots x_{n2} \vdots \cdots \vdots x_{n(n-1)} \right]^T . \tag{15}$$

Moreover, let X be a skew-symmetric matrix, that is

$$x_{ij} \quad = \quad \begin{cases} -x_{ji} & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases} \tag{16}$$

For such a matrix X, the elements of vec(X) are the same as those of bivec(X) with some repetitions and some additional zeros. Thus, there exists an unique full (column) rank $n^2 \times n(n-1)/2$ matrix B_n called bialternation matrix (say) satisfying

$$\text{vec}(X) = B_n \cdot \text{bivec}(X) \tag{17}$$

for all skew-symmetric $X \in \Re^{n \times n}$. It is to be noted that the only possible elements of B_n are 0, 1 or -1.
Now as B_n is of full (column) rank, its pseudo-inverse is

$$B_n^{\dagger} = (B_n^T B_n)^{-1} B_n^T \tag{18}$$

such that

$$\text{bivec}(X) = B_n^{\dagger} \cdot \text{vec}(X) \tag{19}$$

for all skew-symmetric $X \in \Re^{n \times n}$.
Then, for $A \in \Re^{n \times n}$,

$$A \overline{\overline{\oplus}} A \;=\; B_n^{\dagger}(A \oplus A) B_n \tag{20}$$

is the $n(n-1)/2$ dimensional square matrix defined as bialternate sum matrix in Fuller [2].

In order to appreciate some properties of the bialternation matrix B_n, let us define the $n^2 \times n^2$ matrix

$$N_{bn} := (I_{n^2} - K_{nn})/2. \tag{21}$$

The following lemma states some of the properties of K_{nn}, B_n and N_{bn} as well as their interrelationships which will be used quite often to develop the desired results. These properties follow analogous to those in Lemma 2.1 of Mustafa [8].

Lemma 2.1. Let B_n, K_{nn} and N_{bn} be defined as in equations (17), (8) and (21) of this paper. Then

(i) $B_n^\dagger B_n = I_{n(n-1)/2}$

(ii) $B_n B_n^\dagger = N_{bn} = (-K_{nn}) \cdot N_{bn} = N_{bn} \cdot (-K_{nn})$

(iii) $B_n = (-K_{nn}) \cdot B_n = N_{bn} \cdot B_n$

(iv) $B_n^\dagger = B_n^\dagger \cdot (-K_{nn}) = B_n^\dagger \cdot N_{bn}$

Considering matrices A and $B \in \Re^{n\times n}$, we have

(v) $(A\overline{\overline{\oplus}}A)^{-1} = B_n^\dagger (A \oplus A)^{-1} B_n$ *if* $(A \oplus A)$ *is nonsingular*

(vi) $2B_n^\dagger (A \otimes B)\, B_n = 2B_n^\dagger (B \otimes A)\, B_n = B_n^\dagger (A \otimes B + B \otimes A)\, B_n$

(vii) $(A-B)\overline{\overline{\oplus}}(A-B) = (A\overline{\overline{\oplus}}A) - (B\overline{\overline{\oplus}}B) = (A\overline{\overline{\oplus}}A) - 2B_n^\dagger (I_n \otimes B)\, B_n$

(viii) $(kA)\overline{\overline{\oplus}}(kA) = k(A\overline{\overline{\oplus}}A)\ \forall\, k \in \Re$.

Finally, for $X \in \Re^{m\times n}$,

(ix) $(X \otimes X)\, B_n = N_{bm}(X \otimes X)\, B_n$.

3. BLOCK BIALTERNATE SUM

3.1. Definition

The motivation for defining the block bialternate sum arises from the fact that the computation of $(A\overline{\overline{\oplus}}A)$ does not exploit the block-structure of the system matrix A, though it takes into account the skew-symmetry. The block bialternate sum is conceived to retain the block-structure of A while still exploiting the skew-symmetry as used in the third *critical stability criteria* by Fuller [2] so as to be of order only $n(n-1)/2$.

Proceeding on lines similar to those for defining the bialternate sum, we define the $n(n-1)/2 \times n(n-1)/2$ matrix

$$A\overline{\overline{\oplus}}_b A := B_{\overline{n}}^\dagger (A \oplus_b A) B_{\overline{n}} \tag{22}$$

which we call the block bialternate sum of A with block-structure $\overline{n} = (n_1, \cdots, n_r)$ while $B_{\overline{n}}$ is called the block bialternation matrix having elements 0, 1 and (-1). The pseudo-inverse $B_{\overline{n}}^\dagger$ is simply obtained from $B_{\overline{n}}^T$ by replacing all $B_{\overline{n_i}}^T$ blocks by $B_{\overline{n_i}}^\dagger$ and dividing all other non-zero blocks by 2.

We note that $A\overline{\overline{\oplus}}_b A$ is obtained by just a rearrangement of the rows and columns of $A\overline{\overline{\oplus}}A$. This is analogous to the case for the Kronecker and block Kronecker sums and we infer similarly that $(A\overline{\overline{\oplus}}_b A)$ and $(A\overline{\overline{\oplus}}A)$ are similar matrices having the same $n(n-1)/2$ eigenvalues, namely $(\lambda_i + \lambda_j)$ $[1 \le j < i \le n]$ where $\lambda_i (1 \le i \le n)$ are the eigenvalues of A. This allows us to restate the critical criteria of stability in (3) as

$$\left.\begin{array}{lll} \text{(i)} & (-1)^n \det(A) & > \quad 0 \\ \text{(ii)} & (-1)^{n(n-1)/2} \det(A\overline{\overline{\oplus}}_b A) & > \quad 0 \end{array}\right\} \tag{23}$$

thus exploiting the lower order of matrix $(A\overline{\overline{\oplus}}A)$ while still retaining the block-structure.

3.2. The 2–block case

Several practical systems are those with a 2-block structure as is the case with singularly perturbed systems also. For the general 2-block case, $\overline{n} := (n_1, n_2)$ is the block-structure of A so that

$$A := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{24}$$

where $A_{11} \in \Re^{n_1 \times n_1}, A_{22} \in \Re^{n_2 \times n_2}$.
Then,

$$B_{\overline{n}} := \begin{bmatrix} B_{n_1} & 0 & 0 \\ 0 & I_{n_1 n_2} & 0 \\ 0 & (-K_{n_2 n_1}) & 0 \\ 0 & 0 & B_{n_2} \end{bmatrix} \tag{25}$$

and

$$B^{\dagger}_{\overline{n}} := \begin{bmatrix} B^{\dagger}_{n_1} & 0 & 0 & 0 \\ 0 & (1/2)I_{n_1 n_2} & (1/2)(-K_{n_2 n_1}) & 0 \\ 0 & 0 & 0 & B^{\dagger}_{n_2} \end{bmatrix}. \tag{26}$$

From Hyland and Collins [3],

$$A \oplus_b A := \begin{bmatrix} A_{11} \oplus A_{11} & I_{n_1} \otimes A_{12} & A_{12} \otimes I_{n_1} & 0 \\ I_{n_1} \otimes A_{21} & A_{11} \oplus A_{22} & 0 & A_{12} \otimes I_{n_2} \\ A_{21} \otimes I_{n_1} & 0 & A_{22} \oplus A_{11} & I_{n_2} \otimes A_{12} \\ 0 & A_{21} \otimes I_{n_2} & I_{n_2} \otimes A_{21} & A_{22} \oplus A_{22} \end{bmatrix}. \tag{27}$$

Using (25), (26) and (27) in (22) and simplifying using the properties as in Lemma 2.1, we have

$$A \overline{\overline{\oplus}}_b A := \begin{bmatrix} A_{11} \overline{\overline{\oplus}} A_{11} & 2B^{\dagger}_{n_1}(I_{n_1} \otimes A_{12}) & 0 \\ (I_{n_1} \otimes A_{21})B_{n_1} & A_{11} \oplus A_{22} & (A_{12} \otimes I_{n_2})B_{n_2} \\ 0 & 2B^{\dagger}_{n_2}(A_{21} \otimes I_{n_2}) & A_{22} \overline{\overline{\oplus}} A_{22} \end{bmatrix}. \tag{28}$$

4. APPLICATIONS

The block bialternate sum has been used to solve the general problem of integral controllability. This approach has then been adapted to solve for stability bound of the standard singularly perturbed system. The high-gain feedback system has also been considered, in which the lower bound of the high-gain parameter has been evaluated using this method. In all these cases, the use of block bialternate sum results in a lower-order formula while also providing certain other computational advantages over other methods.

4.1. Radius of integral controllability

Let us consider the negative feedback connection of the $m \times m$ stable n-state system $G(s) = D + C(sI - A)^{-1}B$ and an integral controller kI_m/s. Lunze [5] and Morari [7] showed that there exists $k^* > 0$ such that (kI_m/s) stabilizes $G(s) \forall\, k \in (0, k^*)$ iff $G(0)$ has eigenvalues only in the open RHP, in which case $G(s)$ is said to be integral controllable. Mustafa [8] used the concept of block Lyapunov sum to evaluate the largest possible k^*, denoted as the radius of integral controllability $k^*_{\max}$ using a mn dimensional eigenvalue formula. However, this formula requires that two block matrices of dimension $m(m+1)/2$ and $n(n+1)/2$ be inverted. The use of the block bialternate sum reduces the order of these block matrices to $m(m-1)/2$ and $n(n-1)/2$ respectively. This provides considerable computational advantage, particularly when n and m are of higher orders, while still retaining the overall dimension of the eigenvalue formula at mn.

Some notations used for stating the following results are:

$$\lambda^+_{\min}(\cdot) = \text{smallest positive real eigenvalue or } +\infty \text{ (if no positive real eigenvalues exist)}$$

and

$$\lambda^+_{\max}(\cdot) = \text{largest positive real eigenvalue or } 0 \text{ (if no positive real eigenvalues exist).}$$

Theorem 4.1. Let $G(s) = D + C(sI - A)^{-1}B$ be the transfer function of an n-state stable $(m \times m)$ system that is integral controllable. Assuming that D and $-D$ as well as $\tilde{A}$ and $(-\tilde{A})$ have no common eigenvalues where $\tilde{A} := A - BD^{-1}C$, the radius of integral controllability is

$$k^*_{\max} = \lambda^+_{\min}(Y), \quad Y \in \Re^{mn \times mn} \tag{29}$$

where

$$\begin{aligned} Y := & (A \otimes D^{-1}) + 2[(\tilde{A} \otimes D^{-1}C)\, B_n \quad (B \otimes I_m)\, B_m] \\ & \times \begin{bmatrix} (\tilde{A} \overline{\overline{\oplus}} \tilde{A})^{-1} & 0 \\ 0 & (D \overline{\overline{\oplus}} D)^{-1} \end{bmatrix} \times \begin{bmatrix} B_n^\dagger (I_n \otimes BD^{-1}) \\ -B_m^\dagger (C \otimes D^{-1}) \end{bmatrix}. \end{aligned}$$

Proof. See Appendix A.1.

4.2. Stability bound of singularly perturbed systems

The standard singularly perturbed system

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 \\ \varepsilon \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 \end{aligned}$$

where $x_1 \in \Re^{n_1}, x_2 \in \Re^{n_2}$ is stable $\forall \varepsilon \in (0, \varepsilon^*)$ where there exists an $\varepsilon^* > 0$ if A_{22}^{-1} exists and both A_{22} and $A_0 (= A_{11} - A_{12}A_{22}^{-1}A_{21})$ are asymptotically stable (Kokotović et al [4]), ε being the singular perturbation parameter. Mustafa [8] transformed the problem to one of radius of integral controllability for which an $n_1 n_2$ dimensional eigenvalue formula was derived using the block Lyapunov sum approach. Sen and Datta [9] had solved the same problem using the bialternate sum approach to give a $(n_1 + n_2)(n_1 + n_2 - 1)/2$ dimensional eigenvalue formula. Here the block bialternate sum is used to obtain an $n_1 n_2$ dimensional formula which also incorporates the advantage of reducing the order of the two block matrices to be inverted to $n_i(n_i - 1)/2$ from $n_i(n_i + 1)/2$ for $i = 1, 2$ as required in the block Lyapunov sum approach. Moreover, no transformation of the original system matrix (as required in Mustafa's [8] approach) is necessary for this method. This ensures that one of the two extra assumptions made by Mustafa [8] becomes redundant. The maximal stability bound $\varepsilon^*_{\max}$ is thus obtained from the formula as stated in the following theorem.

Theorem 4.2. Consider the following singularly perturbed system

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 \\ \varepsilon \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 \end{aligned}$$

where $x_1 \in \Re^{n_1}, x_2 \in \Re^{n_2}$. Assume that A_{22} and $A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21}$ are asymptotically stable and A_{11} *and* $(-A_{11})$ have no common eigenvalues. Then, the maximal $\varepsilon^* > 0$ such that the system is stable $\forall \varepsilon \in (0, \varepsilon^*)$ is given by

$$\varepsilon^*_{\max} = 1/\lambda^+_{\max}(Z) \tag{30}$$

where Z is the $n_1 n_2 \times n_1 n_2$ matrix

$$\begin{aligned} Z \; := \; & (A_{11} \otimes A_{22}^{-1}) + 2[(A_0 \otimes A_{22}^{-1}A_{21})\, B_{n_1} \quad (A_{12} \otimes I_{n_2})\, B_{n_2}] \\ & \times \begin{bmatrix} (A_0 \overline{\overline{\oplus}} A_0)^{-1} & 0 \\ 0 & (A_{22} \overline{\overline{\oplus}} A_{22})^{-1} \end{bmatrix} \times \begin{bmatrix} B^{\dagger}_{n_1}(I_{n_1} \otimes A_{12}A_{22}^{-1}) \\ -B^{\dagger}_{n_2}(A_{21} \otimes A_{22}^{-1}) \end{bmatrix}. \end{aligned}$$

Proof. Considering the critical criteria of stability as stated in eqn(3), we have

$$\left.\begin{array}{ll} \text{(i)} & (-1)^n \det(A) > 0 \\ \text{(ii)} & (-1)^{n(n-1)/2} \det(A \overline{\overline{\oplus}}_b A) > 0 \end{array}\right\} \tag{31}$$

where

$$A \; := \; \begin{pmatrix} A_{11} & A_{12} \\ A_{21}/\varepsilon & A_{22}/\varepsilon \end{pmatrix}.$$

The first condition is satisfied as A_0 and A_{22} are assumed to be asymptotically stable. The eigenvalue formula is obtained from the second condition in a manner similar to that in Appendix A.1.

4.3. Minimal stability bound for high–gain feedback systems

The high-gain feedback system can be looked upon as a particular case of singularly perturbed systems. Introduce a high-gain feedback system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u \tag{32}$$

$$u = g \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{33}$$

Young et al [12] have shown that under certain assumptions the closed loop system can be transformed into the singularly perturbed form

$$\left.\begin{aligned} \dot{\tilde{x}}_1 &= H_{11}\tilde{x}_1 + H_{12}\tilde{x}_2 \\ \mu\dot{\tilde{x}}_2 &= \mu H_{21}\tilde{x}_1 + (\mu H_{22} + C_2B_2)\tilde{x}_2 \end{aligned}\right\} \tag{34}$$

where $\mu = 1/g$; g being the high-gain parameter. $H_{ij}\ \forall\ i, j = 1,2$ are the transformed block matrices along with C_2B_2. As $\mu \rightarrow 0 (g \rightarrow \infty)$, the system eigenvalues in (34) decompose into the fast and slow modes as shown in Young et al [12] as

$$\left.\begin{aligned} \lambda_i^f &= (1/\mu)[\lambda_i(C_2B_2) + O(\mu)] & i &= 1,2,\ldots,n_2 \\ \lambda_j^s &= \lambda_j(H_{11}) + O(\mu) & j &= 1,2,\ldots,n_1. \end{aligned}\right\} \tag{35}$$

Kokotović et al [4] states the condition for the existence of a lower bound $g_0 > 0$ for which the system as in (34) remains stable $\forall\, g \in (g_0, \infty)$, as

Lemma 4.1. If (C_2B_2) and H_{11} are Hurwitz matrices, then there exists a $g_0 > 0$ such that the resulting closed loop system (34) is asymptotically stable $\forall\, g \in (g_0, \infty)$.

Let us now represent the closed loop system (34) as

$$\dot{\tilde{x}} = H(g)\tilde{x} \tag{36}$$

where

$$H(g) := \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & (H_{22} + gC_2B_2) \end{bmatrix}$$

which is Hurwitz as $g \rightarrow \infty$ owing to the aforementioned conditions in the Lemma. The investigation of the stability of the system matrices when subjected to parameter variations leads to the critical criteria as in (31). Of the two criteria, the first criterion, as shown in Sen, Ghosh and Datta [10], reduces to $\det[gI + P \cdot (C_2B_2)^{-1}] > 0$ where $P = H_{22} - H_{21}H_{11}^{-1}H_{12}$.

The second criterion can also be transformed into an equivalent form, details of which are given in Appendix A.2.

The minimal stability bound for the high-gain feedback system can thus be obtained from the following theorem in terms of an $n_2[n_1 + (n_2 - 1)/2]$ dimensional eigenvalue formula which is lower than the $(n_1 + n_2)(n_1 + n_2 - 1)/2$ order formula given by Sen, Ghosh and Datta [10] particularly as $(n_1 + n_2)$ becomes greater than 3. It is to be noted that the order of the matrix cannot be reduced in this case to n_1n_2, as for singularly perturbed systems.

Theorem 4.3. Consider the linear time-invariant high-gain feedback system with its singularly perturbed representation (34), with $\mu = 1/g > 0$, and assume H_{11} and C_2B_2 are Hurwitz. The necessary and sufficient conditions for the stability of the system are

$$\left.\begin{array}{ll}(a) & \det[gI + P\cdot(C_2B_2)^{-1}] > 0 \\ (b) & \det[gI + QR^{-1}] > 0\end{array}\right\} \qquad (37)$$

where $P = H_{22} - H_{21}H_{11}^{-1}H_{12}$ and

$$QR^{-1} := \begin{bmatrix} \begin{array}{c}(H_{11}\oplus H_{22})\cdot(I_{n_1}\otimes(C_2B_2)^{-1}) \\ -\{2(I_{n_1}\otimes H_{21})B_{n_1}\cdot(H_{11}\overline{\overline{\oplus}}H_{11})^{-1} \\ \cdot B_{n_1}^{\dagger}\cdot(I_{n_1}\otimes H_{12}(C_2B_2)^{-1})\}\end{array} & (H_{12}\otimes I_{n_2})\cdot B_{n_2}(C_2B_2\overline{\overline{\oplus}}C_2B_2)^{-1} \\ 2B_{n_2}^{\dagger}\cdot(H_{21}\otimes(C_2B_2)^{-1}) & (H_{22}\overline{\overline{\oplus}}H_{22})(C_2B_2\overline{\overline{\oplus}}C_2B_2)^{-1} \end{bmatrix}.$$

Under these conditions, the minimal $g_0 > 0$ such that the system is stable $\forall\, g \in (g_0, \infty)$ is given by

$$g_0 = \max(g_1, g_2) \qquad (38)$$

where

$$\begin{aligned} g_1 &= \lambda_{\max}^{+}[-P\cdot(C_2B_2)^{-1}] \\ g_2 &= \lambda_{\max}^{+}[-QR^{-1}]. \end{aligned}$$

Proof. As given in Appendix A.3.

4.4. Examples

4.4.1. *Singu'arly perturbed system*

Consider the example of a 4 × 4 dimensional singularly perturbed system as given in Sen and Datta [9] and Mustafa [8]. The block matrices for this system are

$$A_{11} = \begin{bmatrix} -3 & 4 \\ 0 & 2 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -3 & 4 \\ -1 & -2 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -2 & 3 \\ 0 & -3 \end{bmatrix}.$$

Using Theorem 4.2,

$$Z = \begin{pmatrix} -0.7 & 0.1 & 0.4 & 2.0 \\ 0.2571 & -0.6 & -0.1714 & 1.4476 \\ 0.0286 & 0.2 & 0.5143 & 0.8571 \\ 0.0286 & 0 & 0.1143 & 0.7238 \end{pmatrix}$$

whose eigenvalues are the same as the non-zero eigenvalues of $(-FE^{-1})$ as computed in Sen and Datta [9]. The matrices E and F (both of order 6) were constructed

using the bialternate sum concept as defined in Fuller [2].
The eigenvalues of Z are -0.7976, -0.4865, 0.2021 and 1.0201.
So, $\varepsilon^*_{\max} = 1/1.0201 = 0.9803$. Mustafa [8], using the block Lyapunov sum approach, also obtains a 4×4 matrix Z' whose eigenvalues are -2.0555, -1.2538, 0.9803 and 4.9480, such that $\varepsilon^*_{\max}$ is the minimum real positive eigenvalue.
However, while computing Z', his method requires the inversion of two 3×3 matrices. The two matrices, which are the Lyapunov sums of $\tilde{A}$ $(:= A_{22} - A_{21}A_{11}^{-1}A_{12})$ and A_{11} with themselves, are

$$\tilde{A}\overline{\oplus}\tilde{A} = \begin{bmatrix} -8/3 & 46/3 & 0 \\ 1 & -7/3 & 23/3 \\ 0 & 2 & -2 \end{bmatrix}; \quad A_{11}\overline{\oplus}A_{11} = \begin{bmatrix} -6 & 8 & 0 \\ 0 & -1 & 4 \\ 0 & 0 & 4 \end{bmatrix}$$

which have to be inverted in order to evaluate Z'. The block bialternate sum approach requires the inversion of $(A_0\overline{\overline{\oplus}}A_0)$ and $(A_{22}\overline{\overline{\oplus}}A_{22})$, which in this case are

$$(A_0\overline{\overline{\oplus}}A_0) = (-35/6) \quad \text{and} \quad (A_{22}\overline{\overline{\oplus}}A_{22}) = -5.$$

Thus, in addition to an overall saving in dimension as compared to Sen and Datta's [9] method, we also achieve a considerable saving in computation of the two inverses. These advantages are also obtained while evaluating the radius of integral controllability of an integral control problem.

4.4.2. *High-gain feedback*

The eigenvalue formula for the minimal stability bound g_0 for a high-gain feedback system is slightly different from the other two cases owing to the particular structure of the closed loop high-gain feedback system.

Considering the feedback system studied by Young et al [12], we have

$$H_{11} = [-3], \quad H_{12} = \begin{bmatrix} 1 & -0.5 \end{bmatrix}$$

$$H_{21} = \begin{bmatrix} 0 \\ 60 \end{bmatrix}, \quad H_{22} = \begin{bmatrix} 3 & 0 \\ -12 & 7 \end{bmatrix}, \quad C_2B_2 = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}.$$

Using these in the formula given in Theorem 4.3, we obtain

$$P = \begin{bmatrix} 3 & 0 \\ 8 & -3 \end{bmatrix}, \quad \text{so} \quad P(C_2B_2)^{-1} = \begin{bmatrix} -3 & -1.5 \\ -11 & -4 \end{bmatrix}.$$

while

$$QR^{-1} = \begin{pmatrix} 0 & 0 & -0.25 \\ 16 & 6 & -0.50 \\ 60 & 30 & -5.00 \end{pmatrix}.$$

Thus

$$g_1 = 7.593,$$
$$g_2 = 2.807$$

and

$$\begin{aligned} g_0 &= \max(7.593, 2.807) \\ &= 7.593. \end{aligned}$$

For the 3×3 system, the dimension of QR^{-1} is the same as that of FE^{-1} as evaluated in Sen, Ghosh and Datta [10] using the bialternate sum. However, as the system order increases, the saving in dimension is considerable. For $n_1 = 4, n_2 = 2$ (namely, for a 6×6 system), E and F are 15×15 matrices while QR^{-1} is of order 8×8 only. Moreover, matrix E requires to be inverted in the earlier case, while the maximum order of inversion required in the present method (in order to compute QR^{-1}) is $n_i.(n_i - 1)/2$.

5. CONCLUSION

In this paper, the block bialternate sum has been defined and its properties have been explored. This has then been used to solve various stability problems, namely those of integral controllability, singularly perturbed systems and high-gain feedback systems. Exact bounds for these problems have been evaluated using this method as for example, in the high-gain feedback case, the minimum value of the high-gain parameter g has been evaluated such that the system shows instability if any value of g lower than or equal to that value is used for feedback purposes. Mustafa [8] uses the block Lyapunov sum approach to compute the exact bounds for both the integral control as well as the singularly perturbed problems. Though the dimensions of the eigenvalue formulae in both cases (block Lyapunov sum approach and block bialternate sum approach) are the same, yet the dimensions of the matrices required to be inverted to compute these formulae are lower using the present method. This gives a considerable saving in computation. Moreover, for the singularly perturbed case, two additional assumptions have been made by Mustafa [8] of which one becomes redundant using the present approach.

The approaches used by Sen and Datta [9] and Sen, Ghosh and Datta [10] yield exact results for the singularly perturbed and high-gain feedback systems respectively. The present method is, however, computationally superior to both, being of lower dimension, as it exploits the block-structure inherent in the systems.

A. APPENDICES

A.1. Proof of Theorem 4.1

The closed loop A-matrix for the feedback connection of kI_m/s to G(s) is

$$A_{\text{cl}} \;:=\; \begin{bmatrix} A & -Bk \\ C & -Dk \end{bmatrix}.$$

Let us define

$$\nu(k) := \det(A_{\text{cl}} \overline{\overline{\oplus}}_b A_{\text{cl}}).$$

Then

$$\nu(k) = \prod_{i>j} (\lambda_i + \lambda_j)$$

where the upper limits of the product are $n+m$ and λ_i $(i = 1 : n+m)$ are the eigenvalues of A_{cl}. Thus, if k varies from a value at which A_{cl} is Hurwitz, then $\nu(k) = 0$ iff $\exists$ a pair of purely imaginary eigenvalues. The condition that D and $-D$ as well as $\tilde{A}$ and $-\tilde{A}$ have no common eigenvalues takes care of the fact that no eigenvalues of A_{cl} crosses into the RHP through the origin. Thus, integral controllability of G(s) ensures that all the eigenvalues of A_{cl} belong in the open LHP $\forall\, k \in (0, k^*_{\text{max}})$ where k^*_{max} is the smallest positive real root of $\nu(k) = 0$ (or $+\infty$ if there are no positive real roots).

Now, to consider $\nu(k) = 0$ as an eigenvalue problem, we have from (28),

$$A_{\text{cl}} \overline{\overline{\oplus}}_b A_{\text{cl}} = \begin{bmatrix} A \overline{\overline{\oplus}} A & -2k B_n^\dagger (I_n \otimes B) & 0 \\ (I_n \otimes C) B_n & A \oplus (-kD) & -k(B \otimes I_m) B_m \\ 0 & 2B_m^\dagger (C \otimes I_m) & -k(D \overline{\overline{\oplus}} D) \end{bmatrix}. \tag{39}$$

Note that the assumptions made for this problem are

(i) A is asymptotically stable, so $\det(A \overline{\overline{\oplus}} A) \neq 0$.

(ii) D and $-D$ have no common eigenvalues, so $\det(D \overline{\overline{\oplus}} D) \neq 0$.

and (iii) $\tilde{A}$ and $-\tilde{A}$ have no common eigenvalues, so $\det(\tilde{A} \overline{\overline{\oplus}} \tilde{A}) \neq 0$.

Applying the Schur formula for partitioned determinants twice yields

$$\nu(k) = \det(A_{\text{cl}} \overline{\overline{\oplus}}_b A_{\text{cl}}) = (-k)^{m(m-1)/2} \cdot \det(A \overline{\overline{\oplus}} A) \cdot \det(D \overline{\overline{\oplus}} D) \cdot \det(\widehat{A} - k\widehat{D})$$

where

$$\begin{aligned} \widehat{A} &:= (A \otimes I_m) - 2(B \otimes I_m) B_m (D \overline{\overline{\oplus}} D)^{-1} B_m^\dagger (C \otimes I_m) \\ \widehat{D} &:= (I_n \otimes D) - 2(I_n \otimes C) B_n (A \overline{\overline{\oplus}} A)^{-1} B_n^\dagger (I_n \otimes B). \end{aligned}$$

Thus

$$k^*_{\text{max}} = \lambda^+_{\text{min}} (\widehat{A} \widehat{D}^{-1})$$

which exists only if $\widehat{D}^{-1}$ exists.

But using the properties of Kronecker products, we find

$$\det \widehat{D} = \det(D)^n \cdot \det(A \oplus A)^{-1} . \det(\tilde{A} \oplus \tilde{A})$$

which is nonsingular as all the determinants are nonsingular. Thus $\widehat{D}^{-1}$ exists.

Now to show that $Y = \widehat{A}\widehat{D}^{-1}$, we make use of the properties in Lemma 2.1. We start by using the matrix inversion lemma to obtain

$$\widehat{D}^{-1} = (I_n \otimes D^{-1}) + 2(I_n \otimes D^{-1}C) B_n (\tilde{A} \overline{\overline{\oplus}} \tilde{A})^{-1} B_n^\dagger (I_n \otimes BD^{-1}).$$

Then

$$\widehat{A}\widehat{D}^{-1} = (A \otimes D^{-1}) - 2(B \otimes I_m) B_m (D \overline{\overline{\oplus}} D)^{-1} B_m^\dagger (C \otimes D^{-1}) + \alpha_1 + \alpha_2 \tag{40}$$

where

$$\begin{aligned}\alpha_1 &:= 2(A\otimes D^{-1}C)B_n(\tilde{A}\overline{\overline{\oplus}}\tilde{A})^{-1}B_n^\dagger(I_n\otimes BD^{-1})\\ \alpha_2 &:= -4(B\otimes I_m)\,B_m(D\overline{\overline{\oplus}}D)^{-1}B_m^\dagger(C\otimes D^{-1}C)B_n(\tilde{A}\overline{\overline{\oplus}}\tilde{A})^{-1}B_n^\dagger(I_n\otimes BD^{-1}).\end{aligned}$$

Examining part of α_2, we have

$$\begin{aligned}2B_m^\dagger(C\otimes D^{-1}C)\,B_n &= 2B_m^\dagger(D\otimes I_m)\,B_mB_m^\dagger(D^{-1}C\otimes D^{-1}C)\,B_n\\ &= (D\overline{\overline{\oplus}}D)\,B_m^\dagger(D^{-1}C\otimes D^{-1}C)\,B_n. \end{aligned} \tag{41}$$

Substituting (41) into the expression for α_2, we have

$$\alpha_2=-2(BD^{-1}C\otimes D^{-1}C)B_n(\tilde{A}\overline{\overline{\oplus}}\tilde{A})^{-1}B_n^\dagger(I_n\otimes BD^{-1}).$$

So,

$$\alpha_1+\alpha_2=2(\tilde{A}\otimes D^{-1}C)B_n(\tilde{A}\overline{\overline{\oplus}}\tilde{A})^{-1}B_n^\dagger(I_n\otimes BD^{-1}) \tag{42}$$

Substituting (42) into (40), we have

$$\begin{aligned}\widehat{A}\widehat{D}^{-1} &= (A\otimes D^{-1})-2(B\otimes I_m)\,B_m(D\overline{\overline{\oplus}}D)^{-1}B_m^\dagger(C\otimes D^{-1})\\ &\quad +2(\tilde{A}\otimes D^{-1}C)B_n(\tilde{A}\overline{\overline{\oplus}}\tilde{A})^{-1}B_n^\dagger(I_n\otimes BD^{-1})\\ &= Y\end{aligned}$$

as claimed.

A.2. Proof of Theorem 4.3

For the high-gain feedback closed loop system matrix

$$H(g) \;:=\; \begin{bmatrix} H_{11} & H_{12}\\ H_{21} & (H_{22}+gC_2B_2)\end{bmatrix}$$

the second critical criterion for stability in terms of (31) is

$$(-1)^{n(n-1)/2}\det[H(g)\overline{\overline{\oplus}}_bH(g)]>0.$$

From (28), we have

$$H(g)\overline{\overline{\oplus}}_bH(g):=\begin{bmatrix} H_{11}\overline{\overline{\oplus}}H_{11} & 2B_{n_1}^\dagger(I_{n_1}\otimes H_{12}) & 0\\ (I_{n_1}\otimes H_{21})B_{n_1} & H_{11}\oplus(H_{22}+gC_2B_2) & (H_{12}\otimes I_{n_2})B_{n_2}\\ 0 & 2B_{n_2}^\dagger(H_{21}\otimes I_{n_2}) & (H_{22}+gC_2B_2)\overline{\overline{\oplus}}(H_{22}+gC_2B_2)\end{bmatrix}. \tag{43}$$

Using Schur's partitioned determinant once yields

$$\det(H(g)\overline{\overline{\oplus}}_bH(g))=\det(H_{11}\overline{\overline{\oplus}}H_{11})\cdot\det S \tag{44}$$

where

$$
\begin{aligned}
S &= \begin{bmatrix} \begin{array}{l} (H_{11} \oplus (H_{22} + gC_2B_2)) \\ -\{2(I_{n_1} \otimes H_{21}).B_{n_1}. \\ (H_{11} \overline{\overline{\oplus}} H_{11})^{-1}.B^{\dagger}_{n_1}.(I_{n_1} \otimes H_{12})\} \end{array} & (H_{12} \otimes I_{n_2})B_{n_2} \\ 2B^{\dagger}_{n_2}(H_{21} \otimes I_{n_2}) & (H_{22} + gC_2B_2) \overline{\overline{\oplus}} (H_{22} + gC_2B_2) \end{bmatrix} \\
&= Q + gR
\end{aligned}
$$

where

$$
Q = \begin{bmatrix} \begin{array}{l} (H_{11} \oplus H_{22}) \\ -\{2(I_{n_1} \otimes H_{21}).B_{n_1}. \\ (H_{11} \overline{\overline{\oplus}} H_{11})^{-1}.B^{\dagger}_{n_1}.(I_{n_1} \otimes H_{12})\} \end{array} & (H_{12} \otimes I_{n_2})B_{n_2} \\ 2B^{\dagger}_{n_2}(H_{21} \otimes I_{n_2}) & (H_{22} \overline{\overline{\oplus}} H_{22}) \end{bmatrix}
$$

and

$$
R = \begin{bmatrix} (I_{n_1} \otimes C_2B_2) & 0 \\ 0 & (C_2B_2 \overline{\overline{\oplus}} C_2B_2) \end{bmatrix}.
$$

Now

$$
\begin{aligned}
\det S &= \det(Q + gR) \\
&= \det(QR^{-1} + gI) \cdot \det R
\end{aligned}
$$

because $\det R$ is nonsingular as C_2B_2 is Hurwitz.

Similarly, $\det(H_{11} \overline{\overline{\oplus}} H_{11})$ is also nonsingular, H_{11} being Hurwitz, so the eigenvalue problem reduces to the solution of

$$
\det(gI + QR^{-1}) = 0
$$

which yields the critical value of g for which $H(g)$ becomes unstable due to the existence of a pair of purely imaginary eigenvalues of the system.

(Received February 14, 1995.)

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