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# A GEOMETRIC PROOF OF ROSENBROCK'S THEOREM ON POLE ASSIGNMENT[1]

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A new proof of the famous Rosenbrock theorem on pole placement by static state feedback is given. This proof only uses well-known basic results of the geometric approach, that are the Brunovský canonical form of controllable systems and the splitting of the state space into cyclic subspaces relatively to the invariant factors of a linear map.

## 1. INTRODUCTION

It is well-known that one can assign the poles of a controllable linear system by static state feedback [12]. As far as we are interested not only in assigning the location of the poles but also their multiplicities, it appears that the freedom in modifying the dynamics of the system is limited by the values of the controllability indices of the system. This was the famous result given by Rosenbrock [11]. The aim of the paper is to present a new geometric proof of the Rosenbrock theorem on pole assignment (RTPA).

There already exist a lot of different proofs of Rosenbrock theorem. Most of them use polynomial arguments [3], [7], [11]. Actually all the generalizations of Rosenbrock theorem that was developed are also based on polynomial arguments [13], [14], [15]. For algorithmic conveniency it would be useful to develop a geometric proof. Indeed this was already done by Flamm [4] and later by Ozçaldiran [10]. These two proofs use at key-points difficult results of linear algebra. Our aim here is to give a new proof based on results which are well-known by control theorists, namely the canonical form of controllable systems which was described by Brunovský [1] and the splitting of the state-space accordingly to the invariant factors of a given map. Actually Rosenbrock's conditions are derived from considering the dimension of certain cyclic subspaces. Our proof of their sufficiency is constructive: it effectively permits to calculate a feedback which assigns to prespecified values the invariant factors of the closed-loop system. This construction is divided in two steps. We first construct a feedback which fixes the degrees of the invariant factors. A second feedback is

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designed which set up the coefficients of the invariant factors without changing their degrees. At the difference of the previous proofs of Rosenbrock Theorem ([4], [10], [11] and [15]), we emphasize the geometric aspects of that result.

The paper is organized as follows. Rosenbrock theorem is recalled in Section 2. Sections 3 and 4 are respectively devoted to the proof of the necessity and to the proof of the sufficiency of the claim established by Rosenbrock. Finally an illustrative example is given in Section 5.[3]

## 2. ROSENBROCK THEOREM

We shall consider here classical state space descriptions:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where $A : \mathcal{X} \to \mathcal{X}$ and $B : \mathcal{U} \to \mathcal{X}$ are real matrices, $\mathcal{X} \approx R^n$ and $\mathcal{U} \approx R^m$. We assume that $(A, B)$ is controllable and that $B$ is monic [12], i.e.

$$\langle A \,|\, \operatorname{Im} B\rangle = \mathcal{X} \text{ and } \ker B = \{0\}. \tag{2}$$

One of the most classical problems of control theory is the pole-placement problem (see for example [12], [6]), which consists in finding a *static state feedback* $F : \mathcal{X} \to \mathcal{U}$ which fix at will the spectrum of the map $(A+BF)$, i.e. the zeros of its characteristic polynomial $\pi(\lambda) = \det[\lambda \mathrm{I} - (A+BF)]$, under assumption of controllability of course.

A more general problem is not only to assign the location of the zeros but also their multiplicities. In other words we want to assign the full internal structure of the system, that is the set of the *invariant factors* [11], [14] of $A + BF$.

**Definition 1.** [5] Let $D_\mu(\lambda)$, for $\mu = 1, \ldots, n$, denote the greatest common divisor of all the nonzero minors of order $\mu$ of the characteristic matrix $\lambda\mathrm{I} - A$. The *invariant factors* of $A$, say $\psi_i(\lambda)$, are the polynomials:

$$\psi_\mu(\lambda) = \frac{D_{n-\mu+1}(\lambda)}{D_{n-\mu}(\lambda)}, \quad \text{for } \mu = 1, \ldots, n \tag{3}$$

where $D_0(\lambda) = 1$.

Note the following properties of the *invariant factors*.

[3]**Notation.** Script capitals $\mathcal{V}, \mathcal{W}, \ldots,$ denote linear spaces with elements $v, w, \ldots$; span$\{x_1, x_2, \ldots, x_m\}$ is the linear space spanned by the vectors $x_1, x_2 \ldots, x_m$. The dimension of a space $\mathcal{V}$ is denoted $\dim \mathcal{V}$. $A, B, \ldots,$ is used to denote a given map or its matrix representation in a suitable base. The image of A is denoted $\operatorname{Im} A$. $A|\mathcal{V}$ denotes the restriction of $A$ to $V$. When $\mathcal{V} \subset \mathcal{W}$, $\frac{\mathcal{W}}{\mathcal{V}}$ or $\mathcal{W}/\mathcal{V}$ stands for the quotient space $\mathcal{W}$ modulo $\mathcal{V}$. The direct sum of independent spaces is written as $\oplus$. $\alpha$ and $\beta$ being two polynomials, $\alpha \,|\, \beta$ means $\alpha$ divides $\beta$; $\deg \alpha$ denotes the degree of $\alpha$. The $n \times n$ unity matrix diagonal$\{1, \ldots, 1\}$ is denoted $\mathrm{I}_n$. $\mathrm{J}_n$ will stand for the $n \times n$ matrix $\begin{bmatrix} 0 & \mathrm{I}_{n-1} \\ 0 & 0 \end{bmatrix}$. $\{\alpha_i\} \setminus \{\beta_i\}$ denotes the set $\{\gamma_i\}$ which contains the elements of the set $\{\alpha_i\}$ which are not in the set $\{\beta_i\}$. $\langle H \,|\, \mathcal{T}\rangle$ denotes $\sum_{i=1}^{n} H^{i-1}\mathcal{T}$ where $\mathcal{T} \subset \mathcal{X}$, $H : \mathcal{X} \to \mathcal{X}$ and $n = \dim \mathcal{X}$.

**Property 1.** The *invariant factors* satisfy:

(i) $\psi_2(\lambda) \mid \psi_1(\lambda), \psi_3(\lambda) \mid \psi_2(\lambda), \ldots, \psi_n(\lambda) \mid \psi_{n-1}(\lambda)$,

(ii) $\pi(\lambda) = \psi_1(\lambda)\psi_2(\lambda) \cdots \psi_n(\lambda)$,

(iii) The polynomials $\psi_i$ form a complete set of invariants of $A$ under change of basis on $\mathcal{X}$.

As it is well-known, $\mathcal{X}$ is decomposed into cyclic subspaces accordingly to the invariant factors of the map $A$. Wonham [11] restated this result in terms of controllability.

**Theorem 1.** ([12]: Ch 1.4) Let $(A, B)$ be controllable and let $\psi_1(\lambda), \ldots, \psi_r(\lambda)$ be those of the invariant factors of $A$ which are not equal to 1 ($r$ is called the cyclic index of $A$). Then $m \geq r$. There exist $A$-invariant subspaces $\mathcal{X}_i \subset \mathcal{X}$, and vectors $b_i \in \operatorname{Im} B$, for $i = 1, \ldots, n$, such that:

(i) $\mathcal{X} = \bigoplus_{j=1}^{r} \mathcal{X}_j$,

(ii) $A \mid \mathcal{X}_i$ is cyclic with minimal polynomial $\psi_i(\lambda)$, $i = 1, \ldots, r$, i.e. $\exists v_i \in \mathcal{X}_i = \operatorname{span}\{v_i, Av_i, \ldots, A^{n_i-1}v_i\}$ where: $n_i := \deg \psi_i(\lambda)$

(iii) $\langle A \mid \operatorname{span}\{b_1, \ldots, b_i\}\rangle = \bigoplus_{j=1}^{i} \mathcal{X}_j; \quad i = 1, \ldots, r$

To each nonunit invariant factor $\psi_\mu(\lambda)$ is thus associated an isolated chain of integrators, spanning $\mathcal{X}_\mu$, having $\deg\{\psi_\mu(\lambda)\}$ for size and whose dynamic is given by $\psi_\mu(\lambda)$.

We review now the Brunovský canonical form of a controllable pair $(A, B)$ [1], [12].

**Theorem 2.** [1] Let $(A, B)$ be controllable and define:

$$k_i := \operatorname{card}\{j \mid \bar{k}_j \geq i\}; \quad i = 1, \ldots, m \tag{4}$$

where

$$\begin{cases} \bar{k}_i := \dim\left\{\dfrac{\operatorname{Im} B + A\operatorname{Im} B + \cdots + A^{i-1}\operatorname{Im} B}{\operatorname{Im} B + \cdots + A^{i-2}\operatorname{Im} B}\right\}; & i = 2, \ldots, n \\ \bar{k}_1 := \dim\{\operatorname{Im} B\}. \end{cases} \tag{5}$$

Then there exists a feedback $F : \mathcal{X} \to \mathcal{U}$ and two isomorphisms $T : \mathcal{X} \to \mathcal{X}$, $G : \mathcal{U} \to \mathcal{U}$ such that (in the given basis):

$$\begin{cases} A_c = T^{-1}(A + BF)T = \text{block diagonal matrix } \{A_1, \ldots, A_m\}_{n\times n} \\ B_c = T^{-1}BG = \text{block diagonal matrix } \{b_1, \ldots, b_m\}_{n\times m} \end{cases} \tag{6}$$

with, for $i = 1, 2, \ldots, m$,

$$\left\{ \begin{array}{l} A_i = \mathrm{J}_{k_i}, \\ b_i^T = [0, 1, 0, \ldots, 0]_{1 \times k_i}. \end{array} \right. \tag{7}$$

The lists of integers $\{k_1, \ldots, k_m\}$ and $\{\bar{k}_1, \ldots, \bar{k}_n\}$ are respectively called the *controllability indices* and the *Brunovský indices* of the pair $(A, B)$. The relation (4) defines the list $\{k_i\}$ as the *conjugate* of the list $\{\bar{k}_i\}$.

This two lists have the following properties.

**Property 2.** ([1], [2]) $\{k_i\}$ and $\{\bar{k}_i\}$ satisfy:

(i) $n \geq k_1 \geq k_2 \geq \cdots \geq k_m > 0$

(ii) $m = \bar{k}_1 \geq \bar{k}_2 \geq \cdots \geq \bar{k}_n \geq 0$

(iii) $\bar{k}_1 + \bar{k}_2 + \cdots + \bar{k}_n = n$

(iv) $\sum_{j=1}^{i} \bar{k}_j = \dim \left\{ \sum_{j=1}^{i} A^{j-1} \operatorname{Im} B \right\}$

(v) $\bar{k}_i = \operatorname{card}\{j \mid k_j \geq i\}; \quad i = 1, \ldots, m.$

Let us recall another useful result related with the conjugated lists $\{k_i\}$ and $\{\bar{k}_i\}$.

**Property 3.** [8], [2] Let $\{\alpha_i\}$ and $\{\beta_i\}$ be two finite lists of non negative integers ranged in non increasing order, and let $\{\bar{\alpha}_i\}$ and $\{\bar{\beta}_i\}$ their conjugated lists:

$$\begin{aligned} \bar{\alpha}_i &= \operatorname{card}\{j \mid \alpha_j \geq i\}, \quad i \geq 1, \\ \bar{\beta}_i &= \operatorname{card}\{j \mid \beta_j \geq i\}, \quad i \geq 1. \end{aligned}$$

The two following statements are equivalent.

$$\sum_{j=1}^{i} \alpha_j \geq \sum_{j=1}^{i} \beta_j; \quad i \geq 1 \tag{8}$$

$$\sum_{j=i}^{\infty} \bar{\alpha}_j \geq \sum_{j=i}^{\infty} \bar{\beta}_j; \quad i \geq 1. \tag{9}$$

We are now in position to formulate the RTPA.

**Theorem 3.** RTPA [11] Let (1) be a controllable system with *controllability indices* $\{k_1, \ldots, k_m\}$ and $\{\psi_1(\lambda), \ldots, \psi_n(\lambda)\}$ be a set of monic polynomials satisfying the divisibility conditions

$$\psi_{i+1}(\lambda) \mid \psi_i(\lambda), \quad i = 1, \ldots, n-1.$$

Then there exists a feedback $F : \mathcal{X} \to \mathcal{U}$ such that the *invariant factors* of $(A + BF)$ are precisely the above polynomials $\psi_i(\lambda)$ if and only if:

$$\sum_{j=i}^{n} n_j \geq \sum_{j=i}^{n} k_j; \quad i = 2, \ldots, n \tag{10}$$

and

$$\sum_{j=1}^{n} n_j = \sum_{j=1}^{n} k_j \tag{11}$$

where $n_j = \deg \psi_j(\lambda)$ and $k_j = 0$ for $j \geq m + 1$.

## 3. GEOMETRIC PROOF OF THE NECESSITY

In this Section we are going to show that the *invariant factors* of $A$, $\psi_1(\lambda), \ldots, \psi_m(\lambda)$, satisfy (10) – (11).

We need for this the following Corollary of Theorem 1.

**Corollary 1.**

$$\langle A \mid \operatorname{span}\{b_1, \ldots, b_i\} \rangle = \bigoplus_{\mu=1}^{i} \bigoplus_{j=1}^{n_\mu} A^{j-1} \operatorname{span}\{b_\mu\}; \quad i = 1, \ldots, r. \tag{12}$$

Proof of Corollary 1. Note that (12) is true for $i = 1$ (remember Theorem 1). Assume that (12) is true for $i - 1$, i.e.

$$\langle A \mid \operatorname{span}\{b_1, \ldots, b_{i-1}\} \rangle = \bigoplus_{\mu=1}^{i-1} \bigoplus_{j=1}^{n_\mu} A^{j-1} \operatorname{span}\{b_\mu\}. \tag{13}$$

Defining

$$Q_\mu : \mathcal{X} \to \mathcal{X} \text{ the projection on} \bigoplus_{j=\mu}^{r} \mathcal{X}_j \text{ along } \bigoplus_{j=1}^{\mu-1} \mathcal{X}_j$$

where the $\mathcal{X}_j$ are the subspaces given in Theorem 1, we obtain (see Theorem 1)

$$\begin{aligned}
\bigoplus_{j=1}^{i} \mathcal{X}_j &= \langle A \,|\, \mathrm{span}\,\{b_1,\dots,b_{i-1}\}\rangle + \sum_{j=1}^{n} A^{j-1}\mathrm{span}\,\{b_i\} \\
&\subset \langle A \,|\, \mathrm{span}\,\{b_1,\dots,b_{i-1}\}\rangle + \sum_{j=1}^{n} A^{j-1}\mathrm{span}\,\{(\mathrm{I}-Q_i)b_i\} \\
&\qquad + \sum_{j=1}^{n} A^{j-1}\mathrm{span}\,\{Q_i b_i\} \\
&= \langle A \,|\, \mathrm{span}\,\{b_1,\dots,b_{i-1}\}\rangle + \bigoplus_{j=1}^{n_i} A^{j-1}\mathrm{span}\,\{Q_i b_i\} \\
&\subset \bigoplus_{j=1}^{i} \mathcal{X}_j .
\end{aligned} \tag{14}$$

The last equality is due to

$$\sum_{j=1}^{n} A^{j-1}\mathrm{span}\,\{(\mathrm{I}-Q_i)b_i\} \subset \langle A \,|\, \mathrm{span}\,\{b_1,\dots,b_{i-1}\}\rangle ,$$

$$Q_i b_i \in \left(\bigoplus_{j=1}^{i} \mathcal{X}_j\right) \cap \left(\bigoplus_{j=i}^{r} \mathcal{X}_j\right) = \mathcal{X}_i ,$$

and from the fact that the minimal polynomial of $\mathcal{X}_i$ has degree $n_i$. Thus the last inclusion follows since $Q_i b_i \in \mathcal{X}_i$ and $A\mathcal{X}_i \subset \mathcal{X}_i$.

From (14) and (13) we have:

$$\begin{aligned}
\bigoplus_{j=1}^{i} \mathcal{X}_j &= \bigoplus_{\mu=1}^{i-1}\bigoplus_{j=1}^{n_\mu} A^{j-1}\mathrm{span}\,\{b_\mu\} + \bigoplus_{j=1}^{n_i} A^{j-1}\mathrm{span}\,\{Q_i b_i\} \\
&= \bigoplus_{\mu=1}^{i-1}\bigoplus_{j=1}^{n_\mu} A^{j-1}\mathrm{span}\,\{b_\mu\} + \bigoplus_{j=1}^{n_i} A^{j-1}\mathrm{span}\,\{b_i\}.
\end{aligned} \tag{15}$$

Indeed, let $x$ be in $\bigoplus_{j=1}^{i} \mathcal{X}_j$ then there exists a vector $z$ in $\langle A \,|\, \mathrm{span}\,\{b_1,\dots,b_{i-1}\}\rangle$, $\alpha_1,\dots,\alpha_{n_i} \in R$, such that: $x = z + \sum_{j=1}^{n_i} A^{j-1} Q_i b_i \alpha_j = \left(z - \sum_{j=1}^{n_i} A^{j-1}(\mathrm{I}-Q_i)\, b_i \alpha_j\right)$ $+ \sum_{j=1}^{n_i} A^{j-1} b_i \alpha_j \in \langle A \,|\, \mathrm{span}\,\{b_1,\dots,b_{i-1}\}\rangle + \bigoplus_{j=1}^{n_i} A^{j-1}\mathrm{span}\,\{b_i\}$. On the other hand $\langle A \,|\, \mathrm{span}\,\{b_1,\dots,b_i\}\rangle \supset \langle A \,|\, \mathrm{span}\,\{b_1,\dots,b_{i-1}\}\rangle + \bigoplus_{j=1}^{n_i} A^{j-1}\mathrm{span}\,\{b_i\}$.

Now, since $\dim\left\{\bigoplus_{j=1}^{i} \mathcal{X}_j\right\} = \sum_{j=1}^{i} n_j$ (see Theorem 1) it appears that (15) implies (12). □

### 3.1. Proof of the necessity

Let us define:

$$\begin{cases} \lambda_\mu := \min\{i, n_\mu\} \\ \bar{n}_j := \mathrm{card}\{\mu \,|\, n_\mu \geq j\}; \quad j = 1,\dots,i. \end{cases} \tag{16}$$

We then have from (12), Theorem 1 and (16):

$$\begin{aligned}
\sum_{j=1}^{i} A^{j-1}\,\mathrm{Im}\,B \quad &\supset \sum_{j=1}^{i} A^{j-1}\left(\bigoplus_{\mu=1}^{r} \mathrm{span}\,\{b_\mu\}\right) \\
&= \sum_{\mu=1}^{r}\left(\sum_{j=1}^{i} A^{j-1}\mathrm{span}\,\{b_\mu\}\right) \\
&\supset \bigoplus_{\mu=1}^{r}\left(\bigoplus_{j=1}^{\lambda_\mu} A^{j-1}\mathrm{span}\,\{b_\mu\}\right) \\
&= \bigoplus_{j=1}^{i} A^{j-1}\left(\bigoplus_{\mu=1}^{\bar{n}_j} \mathrm{span}\,\{b_\mu\}\right).
\end{aligned} \tag{17}$$

Now, from claim (iv) of Property 2 and (17) we have:

$$\sum_{j=1}^{i} \bar{k}_j \geq \sum_{j=1}^{i} \bar{n}_j. \tag{18}$$

Hence, (18) and Property 3 imply (10).
On the other hand, from claim (iii) of Property 2 and Theorem 1, we have

$$\sum_{j=1}^{n} \bar{k}_j = n = \sum_{j=1}^{n} \deg \psi_j(\lambda) \tag{19}$$

which, together with Property 3, imply (11) and concludes the proof of necessity.

## 4. GEOMETRIC PROOF OF THE SUFFICIENCY

In this Section we are going to show that for a given set of polynomials, $\psi_1(\lambda)$, $\ldots$, $\psi_n(\lambda)$, satisfying (10)$-$(11) and the three claims of Property 1, there exists a feedback $F : \mathcal{X} \to \mathcal{U}$ such that the *invariant factors* of $(A+BF)$ be $\psi_1(\lambda), \ldots, \psi_n(\lambda)$. For this, we are first going to give some geometric interpretation of RTPA.

### 4.1. Preliminaries

Theorem 2 expresses that $\mathcal{X}$ can be decomposed as follows (cf. Theorem 5.10 of Wonham [12]):

$$\mathcal{X} = \bigoplus_{i=1}^{m} \mathcal{R}_i^* \tag{20}$$

where $\mathcal{R}_i^*$ satisfies, for $i = 1, \ldots, m$,

$$\left\{\begin{array}{rcl}
\mathcal{R}_i^* &=& \mathrm{span}\,\{b_i, A_c b_i, \ldots, A_c^{k_i-1} b_i\} \approx R^{k_i}, \\
A_c \mathcal{R}_i^* &\subset& \mathcal{R}_i^*, \\
A_c^{k_i} \mathcal{R}_i^* &=& \{0\}
\end{array}\right. \tag{21}$$

where $\{b_1, \ldots, b_m\}$ is some basis of Im $B$.

Let us establish the following result:

**Lemma 1.** Given a contrallable pair $(A, B)$ and a list of integers $n_1, \ldots, n_\mu$, $\mu \leq m$, satisfying

$$\begin{cases} n_1 \geq n_2 \geq \ldots \geq n_\mu \\ \sum_{i=1}^{j} n_i \geq \sum_{i=1}^{j} k_i & , \quad j = 1, \ldots, \mu \\ \sum_{i=1}^{\mu} n_i = \sum_{i=1}^{m} k_i = n \end{cases} \tag{22}$$

we can decompose $\mathcal{X}$ as follows

$$\mathcal{X} = \bigoplus_{i=1}^{\mu} \mathcal{I}_i \tag{23}$$

where, for $i = 1, \ldots, \mu$,

$$\begin{cases} \mathcal{I}_i &= \text{span}\,\{\bar{b}_i, \bar{A}\bar{b}_i, \ldots, \bar{A}^{n_i-1}\bar{b}_i\} \approx R^{n_i} \\ \bar{A}\mathcal{I}_i &\subset \mathcal{I}_i \\ \bar{A}^{n_i}\mathcal{I}_i &= \{0\} \end{cases} \tag{24}$$

being the $\bar{b}_i$ some projections of the $b_i$ on $\mathcal{I}_i$ and $\bar{A} = A_c + BF$.

We shall now establish the result in the case $m = 2$.

Proof. i) Let us first consider a pair $(A, B)$ having only two controllability indices, namely $k_1$ and $k_2$ with $k_1 \geq k_2$. Under these conditions the state space $\mathcal{X}$ can be decomposed as in (20)–(21) with $m = 2$.

Now, $j$ being an integer, $0 \leq j \leq k_2$, we define the feedback $F_j : \mathcal{X} \to \mathcal{U}$ and the map $\bar{A} : \mathcal{X} \to \mathcal{X}$ as follows:

$$\begin{cases} BF_j\mathcal{R}_1^* &= \{0\} \\ BF_jA_c^{j-1}b_2 &= b_1 \\ BF_jA_c^{i-1}b_2 &= 0, \quad i \in \{1, \ldots, k_2\} \setminus \{j\} \end{cases} \tag{25}$$

$$\bar{A} := A_c + BF_j, \tag{26}$$

we can construct the following subspace:

$$\mathcal{I}_1 := \text{span}\,\{b_2, \bar{A}b_2, \ldots, \bar{A}^{k_1-1+j}b_2\}. \tag{27}$$

Which has the following properties (see: (20), (21), (25) – (27)):

$$\left\{\begin{array}{ll} \mathcal{I}_1 & = \text{span}\,\{b_2, A_c b_2, \ldots, A_c^{j-1} b_2, A_c^j b_2 + b_1, \ldots, A_c^{k_2-1} b_2 + A_c^{k_2-1-j} b_1, \\ & \qquad A_c^{k_2-j} b_1, \ldots, A_c^{k_1-1} b_1\} \\ & \approx R^{k_1+j}, \\ \bar{A}\mathcal{I}_1 & \subset \mathcal{I}_1, \\ \bar{A}^{k_1+j}\mathcal{I}_1 & = \{0\}. \end{array}\right. \tag{28}$$

Remark that $\mathcal{I}_1$ is an $\bar{A}$-invariant cyclic subspace generated by $b_2$ with dimension $k_1 + j$ and minimal polynomial $\lambda^{k_1+j}$.

Let us now construct another subspace, say $\mathcal{I}_2$, as follows

$$\mathcal{I}_2 := \text{span}\,\{A_c^j b_2, \ldots, A_c^{k_2-1} b_2\}. \tag{29}$$

It is readily seen that

$$\left\{\begin{array}{ccl} \mathcal{I}_2 & \approx & R^{k_2-j}, \\ \mathcal{R}_2^* & = & (\mathcal{R}_2^* \cap \mathcal{I}_1) \oplus \mathcal{I}_2 \\ \mathcal{X} & = & \mathcal{I}_1 \oplus \mathcal{I}_2 \end{array}\right. \tag{30}$$

(first and second claim follow from (29), (20) and (28), and for the third claim, we have from this two first claims and the first claim of (28): $\dim(\mathcal{I}_1 + \mathcal{R}_2^*) = \dim\mathcal{I}_1 + \dim\mathcal{I}_2 = (k_1 + j) + (k_2 - j) = n$).

Let us now define the following projection:

$$P_j : \mathcal{X} \to \mathcal{X} \quad \text{the projection on } \mathcal{I}_2 \text{ along } \mathcal{I}_1 \tag{31}$$

we then have from (27), (28) and (30):

$$P_j b_1 = P_j \left(\bar{A}^j b_2 - A_c^j b_2\right) = -A_c^j b_2 \tag{32}$$

and from (25) we have:

$$\bar{A}\left(A_c^{i-1} b_2\right) = A_c^i b_2; \quad i = j+1, \ldots, k_2 \tag{33}$$

which implies:

$$A_c^{j+i} b_2 = \bar{A}^i \left(A_c^j b_2\right); \quad i = 1, \ldots, k_2 - j. \tag{34}$$

And thus, we have from (32), (34), (29) and (21.c):

$$\left\{\begin{array}{ccl} \mathcal{I}_2 & = & \text{span}\,\{\bar{b}_1, \bar{A}\bar{b}_1, \ldots, \bar{A}^{k_2-j-1}\bar{b}_1\} \approx R^{k_2-j}, \\ \bar{A}\mathcal{I}_2 & \subset & \mathcal{I}_2, \\ \bar{A}^{k_2-j}\mathcal{I}_2 & = & \{0\}, \end{array}\right. \tag{35}$$

with

$$\bar{b}_1 = P_j b_1. \tag{36}$$

Which implies that $\mathcal{I}_2$ is also an $\bar{A}$-invariant cyclic subspace generated by the projection of $b_1$ on $\mathcal{I}_2$ (see (31)) with dimension $k_2 - j$ and minimal polynomial $\lambda^{k_2-j}$.
□

For the case of more than two controllability indices it is enough to apply (i) successively.

### 4.2. Proof of the Sufficiency

Without any loss of generality, we can assume that the pair $(A, B)$ is already in Brunovský canonical form (we can achieve that by means of a change of bases in $\mathcal{X}$ and $\mathcal{U}$ and with a first state feedback). We next apply a second state feedback as in (25) - (26) and express the state space in terms of the subspaces $\mathcal{I}_1, \mathcal{I}_2, \ldots$ (see (27) - (28) and (35) - (36)). We thus have obtained a matrix $A$ which is block diagonal with minimal polynomials $\lambda^{n_1}, \lambda^{n_2}, \ldots, \lambda^{n_\mu}$ where the $n_i$ satisfy (22), and each block is expressed in its rational canonical form (see [12] and [5]).

Let us now go back, without any loss of generality, to the case of two controllability indices.

After having applied the above mentioned procedure, we obtain from (27) - (28) and (35) - (36)

$$[(\lambda \mathrm{I} - A_2) \mid B] = \left[ \begin{array}{cc|cc} \lambda \mathrm{I} - \mathrm{J}_{n_1} & 0 & \star & b_2 \\ 0 & \lambda \mathrm{I} - \mathrm{J}_{n_2} & \bar{b}_1 & 0 \end{array} \right] \tag{37}$$

where $b_2^T = \left[ \begin{array}{cccc} 0 & \cdots & 0 & 1 \end{array} \right]_{1\times n_1}$, $\bar{b}_1^T = \left[ \begin{array}{cccc} 0 & \cdots & 0 & 1 \end{array} \right]_{1\times n_2}$, $\star$ is a vector which value is not precised and $n_1 \geq n_2$.

Applying now the third state feedback:

$$F_3 = \left[ \begin{array}{cccccccc} 0 & 0 & \cdots & 0 & a_1^2 & a_2^2 & \cdots & a_{n_2}^2 \\ a_1^1 & a_2^1 & \cdots & a_{n_1}^1 & 0 & 0 & \cdots & 0 \end{array} \right]$$

in order to assign the coefficients corresponding to the desired invariant factors

$$\psi_i(\lambda) = \lambda^{n_i} - \left( a_1^i + a_2^i \lambda + \cdots + a_n^i \lambda^{n_i - 1} \right), \quad i = 1, 2.$$

We have (with $A_3 := A_2 + BF_3$)

$$[(\lambda \mathrm{I} - A_3) \mid B] = \left[ \begin{array}{cc|cc} \lambda I - L_1 & X & \star & b_2 \\ 0 & \lambda I - L_2 & \bar{b}_1 & 0 \end{array} \right], \tag{38}$$

where $L_i = \underline{\chi}_{n_i} \underline{a}^T + \mathrm{J}_{n_i}$, $\underline{a}^T = \left[ \begin{array}{cccc} a_1^i & a_2^i & \cdots & a_{n_i}^i \end{array} \right]$, $\underline{\chi}_{n_i}^T = \left[ \begin{array}{cccc} 0 & \cdots & 0 & 1 \end{array} \right]_{1\times n_i}$, $j = 1, \ldots, n_2$, and where $X$ is some $n_1 \times n_2$ real matrix, which entries will be denoted $x_{i,j}$ in the following.

Note that the matrices $A_2$, $A_3$ and $B$ are expressed in the basis $\{e_1, \ldots, e_{n_1}\}$ of $\mathcal{X}_1$ and $\{e_{n_1+1}, \ldots, e_{n_1+n_2}\}$ of $\mathcal{X}_2$ (see Ch. 0.10 and 1.3 of [12]) as

$$\left\{ \begin{array}{lll} e_j & = \psi_1^{(j)}(L_1) b_2, & j = 1, \ldots, n_1 \\ e_{j+n_1} & = \psi_2^{(j)}(L_2) \bar{b}_1, & j = 1, \ldots, n_2 \end{array} \right. \tag{39}$$

where for $i = 1, 2$: $\psi_i^{(0)}(\lambda) := \psi_i(\lambda)$ and $\psi_i^{(j)}(\lambda) := \frac{1}{\lambda}\left(\psi_i^{j-1}(\lambda) + a_j^i\right)$, with $j = 1, \ldots, n_i$.

The column vectors $\underline{x}_j := \begin{bmatrix} x_{1,j} & \cdots & x_{n_1,j} \end{bmatrix}^T$ of the matrix $X$ are expressed in this basis as

$$\underline{x}_j = \sum_{\kappa=1}^{n_1} x_{\kappa,j} e_\kappa = (L_1 \Delta_j + \alpha_j) b_2, \quad j = 1, \ldots, n_2 \tag{40}$$

where $\Delta_j = \sum_{\kappa=1}^{n_1-1} x_{\kappa,j} \psi_1^{\kappa+1}(L_1)$ and $\alpha_j = x_{n_1,j} - \sum_{\kappa=1}^{n_1-1} x_{\kappa,j} a_{\kappa+1}^1$, with $j = 1, \ldots, n_2$.

If we now define the new basis:

$$\begin{cases} \bar{e}_j & = e_j\,; \quad j = 1, \ldots, n_1 \\ \bar{e}_{j+n_1} & = e_{j+n_1} - \Delta_j b_2, \quad j = 1, \ldots, n_2 \end{cases} \tag{41}$$

we obtain

$$\begin{cases} \bar{e}_{n_1} & = b_2 \\ A_3 \bar{e}_j & = \bar{e}_{j-1} + a_j^1 b_2, \quad j = 1, \ldots, n_1 \end{cases} \tag{42}$$

since $A_3 e_j = L_1 e_j$, for $j = 1, \ldots, n_1$, and

$$\begin{cases} \bar{e}_{n_1+n_2} & = \bar{b}_1 - \Delta_{n_2} b_2 \\ A_3 \bar{e}_{j+n1} & = \bar{e}_{j+n_1-1} + a_j^2 \bar{b}_1 + \beta_j b_2, \quad j = 1, \ldots, n_2 \end{cases} \tag{43}$$

since $A_3 e_{j+n_1} = L_2 e_{j+n_1} + \underline{x}_j$, $\underline{x}_j = (L_1 \Delta_j + \alpha_j) b_2$ and $A_3 b_2 = L_1 b_2$.

In this new basis, (38) takes the form

$$[(\lambda \mathrm{I} - A_3) \,|\, B] = \left[\begin{array}{cc|cc} \lambda I - L_1 & \underline{\chi}_{n_1} \underline{\beta}^T & \star & b_2 \\ 0 & \lambda I - L_2 & \bar{b}_1 & 0 \end{array}\right], \tag{44}$$

where $\underline{\beta} = \begin{bmatrix} \beta_1 & \cdots & \beta_{n_2} \end{bmatrix}$.

Finally with the feedback $F_4 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -\beta_1 & -\beta_2 & \cdots & -\beta_{n_2} \end{bmatrix}$, we achieve the desired goal.

Let us finish with an illustrative example.

### 4.3. Illustrative example

Consider the system

$$\dot{x}(t) = \begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} & \mathbf{0} \\ \mathbf{0} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix} x(t) + \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} u(t),$$

to which we want to assign the following invariant polynomials

$$\begin{cases} \Psi_1(\lambda) &= (\lambda+1)^4 &= \lambda^4+4\lambda^3+6\lambda^2+4\lambda+1, \\ \Psi_2(\lambda) &= (\lambda+1)^2 &= \lambda^2+2\lambda+1. \end{cases}$$

Following the above mentioned proof, we first apply the state feedback $F_2$

$$u(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x(t) + v(t) \ ,$$

which gets the closed-loop system

$$\dot{x}(t) = \begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} & \begin{matrix} \\ \\ 0 & 0 & 1 \end{matrix} \\ \mathbf{0} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix} x(t) + \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} v(t).$$

Applying the basis change matrix, $x = T_1\xi$,

$$T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad T_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{bmatrix},$$

we obtain

$$\dot{\xi}(t) = \begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \xi(t) + \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 \\ -1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix} v(t) \ .$$

Note that we have already obtained in this way the sought dimensions of the invariant subspaces which correspond to the degrees of the desired invariant polynomials.

Apply now the state feedback $F_3$

$$v(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 2 \\ -1 & -4 & -6 & -4 & 0 & 0 \end{bmatrix} \xi(t) + w(t),$$

it comes

$$\dot{\xi}(t) = \begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -4 & -6 & -4 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 2 \\ 0 & 0 \end{bmatrix} \\ 0 & \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \end{bmatrix} \xi(t) + \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 \\ -1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix} w(t),$$

and after applying the basis change matrix, $\xi = T_2\zeta$,

$$T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

we have

$$\dot{\zeta}(t) = \begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -4 & -6 & -4 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 2 & 5 \end{bmatrix} \\ 0 & \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \end{bmatrix} \zeta(t) + \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 \\ -1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix} w(t),$$

Finally applying the following state feedback $F_4$

$$w(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & -5 \end{bmatrix} \zeta(t) + r(t) ,$$

we obtain the desired internal structure.

## 5. CONCLUSION

We give a new proof of Rosenbrock Theorem on pole assignment. This proof is done within a very geometric framework; it is based on two well-known basic results of control theory (Theorems 1 and 2).

Our proof of the necessity provides a new interpretation of Rosenbrock inequalities. Actually invariant factors are associated with cyclic subspaces generated by projections of input vectors, the system being controllable, and the dimensions of subspaces of that kind necessarily satisfy Rosenbrock's inequalities.

The proof of the sufficiency, based on the construction of such cyclic subspaces, is constructive and leads to an efficient method for the design of a feedback which assigns the invariant factors of the system. This procedure can be summarized as follows.

(i) Calculate a feedback $F_1$ which puts the system in Brunovský cannonical form.

(ii) Calculate a feedback $F_2$, which as in (25), which leads to cyclic subspaces having the desired dimensions, that are the degrees of the invariant factors.

(iii) The feedback $F_3 + F_4$ calculated as in subsection 4.2 finally permits to adjust as specified the coefficients of the invariant factors.

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