KYBERNETIKA — VOLUME 33 (1997), NUMBER 2, PAGES 185-201

NOTES ON A HIERARCHICAL THEORY OF SYSTEMS AND APPLICATIONS[1]

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The action of the full system equivalence transformation on a linear time-invariant system which may be considered as an interconnection of other subsystems at a lower level is considered. The generalized state space reduction problem of a linear time-invariant multivariable system is seen as a direct application of this theory.

1. INTRODUCTION

Linear time invariant multivariable systems may be considered as an interconnection of other subsystems, which may themselves be considered as an interconnection of other subsystems of lower order [7]. Within this hierarchical theory, [7] considered some of the implications the transformation of strict system equivalence [6] applied to the subsystems has on the subsystems higher up in the hierarchy. However in case the impulsive behavior of the systems is under consideration, then an extension of the known results is necessary. The reason for this extension is the inadequacy of the transformation of strict system equivalence in that it preserves only the finite and not the infinite frequency behavior of the system. One objective of this paper is therefore to extend the known results of [7], so that the impulsive behavior of the system and its subsystem is included. A second objective will be to consider as an application of these results the problem of reducing a general linear multivariable system to an equivalent generalized state space form.

2. PRELIMINARIES

Consider the set $P(p,m)$ of $(r+p)\times(r+m)$ polynomial matrices with $r\geq\max(-p,-m)$. A matrix transformation with many important systems theory implications is the following:

[1]Research is supported by EPSRC (Contract: GR/J87329) and the Greek General Secretariat of Industry, Research and Technology.

Definition 1. [2] Let $T_1(s), T_2(s) \in P(p,m)$. Then $T_1(s)$ are said to be *fully equivalent* (f. e.) if there exist polynomial matrices $M(s)$, $N(s)$ such that

$$(M(s)\ T_2(s))\begin{pmatrix} T_1(s) \\ -N(s) \end{pmatrix} = 0 \tag{2.1}$$

where the compound matrices

$$(M(s)\ T_2(s)); \quad \begin{pmatrix} T_1(s) \\ -N(s) \end{pmatrix} \tag{2.2}$$

satisfy the following:

(i) they have full normal rank,
(ii) they have no finite nor infinite zeros,
(iii) the following McMillan degree conditions hold

$$\delta_M(M(s)\ T_2(s)) = \delta_M(T_2(s)); \quad \delta_M\begin{pmatrix} T_1(s) \\ -N(s) \end{pmatrix} = \delta_M(T_1(s)). \tag{2.3}$$

A linear time invariant multivariable system Σ may be represented by an $(r+p)\times(r+m)$ (with $r > 0$) polynomial system matrix

$$P(s) = \begin{pmatrix} A(s) & B(s) \\ -C(s) & D(s) \end{pmatrix} \tag{2.4}$$

with $\det[A(s)] \neq 0$ as has been described by Rosenbrock or by the *normalized form* of $P(s)$

$$\boldsymbol{P}(s) = \begin{pmatrix} T(s) & U \\ -V & 0 \end{pmatrix} := \left(\begin{array}{cccc} A(s) & B(s) & 0 & \vdots\ 0 \\ -C(s) & D(s) & I & \vdots\ 0 \\ 0 & -I & 0 & \vdots\ I \\ \hline 0 & 0 & -I & \vdots\ 0 \end{array}\right) \tag{2.5}$$

as has been denoted by [9]. Let the set of all such matrices be denoted by $\boldsymbol{P}(p,m)$. Then we have

Definition 2. [3] $P_1(s), P_2(s) \in \boldsymbol{P}(p,m)$ are said to be *full system equivalent* (f.s.e.) if there exists polynomial matrices $M(s)$, $N(s)$, $X(s)$, $Y(s)$ such that

$$\begin{pmatrix} M(s) & 0 \\ X(s) & I \end{pmatrix}\underbrace{\begin{pmatrix} A_1(s) & B_1(s) \\ -C_1(s) & D_1(s) \end{pmatrix}}_{P_1(s)} = \underbrace{\begin{pmatrix} A_2(s) & B_2(s) \\ -C_2(s) & D_2(s) \end{pmatrix}}_{P_2(s)}\begin{pmatrix} N(s) & Y(s) \\ 0 & I \end{pmatrix} \tag{2.6}$$

is a f. e. transformation.

Some results concerning the transformation of full system equivalence, which are indicative of its importance in the generalized study of linear systems behavior, are included in the following.

Theorem 1. ([3,4])

(i) Full system equivalence is an equivalence relation.

(ii) Under full system equivalence of the following are invariant

   (a) the generalized order $\boldsymbol{f}$ and the Rosenbrock degree $\boldsymbol{d}_r$,
   (b) the transfer function and thus the finite and infinite transmission poles and zeros,
   (c) the sets of finite system poles and zeros,
   (d) the sets of finite and infinite decoupling zeros,
   (e) the controllability and observability indices

(iii) Every system matrix $P(s)$ is full system equivalent with a generalized state space system.

3. HIERARCHICAL THEORY OF SYSTEMS AND FULL SYSTEM EQUIVALENCE

We shall start this section with a review of the philosophy of the hierarchical theory of systems. The hierarchical theory of systems views that every system Σ may be considered as the interconnection of other subsystems $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$. Every subsystem Σ_i may then be considered as the interconnection of other subsystems $\Sigma_{i,1}, \Sigma_{i,2}, \ldots, \Sigma_{i,n_i}$ of lower order. Accordingly in this way we define an hierarchy of orders $0, 1, 2, \ldots, q$; the order 0 corresponds to the system Σ itself, the order 1 corresponds to the subsystems Σ_i, the order 2 corresponds to the subsystems $\Sigma_{i,j}$ etc. The order q will be considered as the level of greatest subdivision of Σ and its elements are considered as the fundamental elements of Σ. In a certain sense the level of greatest subdivision can be considered as the level at which the system becomes decoupled.

The above scheme is more theoretic rather than practical and for these reasons we shall describe the results as they related to the form of matrices in our specific field of linear, time invariant, multivariable systems.

Consider therefore a system Σ which is formed by the interconnection of subsystems $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$ which have the following form:

$$\begin{aligned} A_i(\rho)\,\beta_i(t) &= B_i(\rho)\,u_i(t) \quad i = 1, 2, \ldots, n \\ y_i(t) &= C_i(\rho)\,\beta_i(t) + D_i(\rho)\,u_i(t) \end{aligned} \tag{3.1}$$

where $A_i(\rho) \in \Re[\rho]^{r_i \times r_i}$, $B_i(\rho) \in \Re[\rho]^{r_i \times m_i}$, $C_i(\rho) \in \Re[\rho]^{p_i \times r_i}$, $D_i(\rho) \in \Re[\rho]^{p_i \times m_i}$ with corresponding system matrices:

$$P_i(s) = \begin{pmatrix} A_i(s) & B_i(s) \\ -C_i(s) & D_i(s) \end{pmatrix} \in \Re[s]^{(p_i+r_i)\times(m_i+r_i)}. \tag{3.2}$$

The system Σ corresponds to the linear multivariable system

$$\begin{aligned} A(\rho)\,\beta(t) &= B(\rho)\,u(t) \\ y(t) &= C(\rho)\,\beta(t) + D(\rho)\,u(t) \end{aligned} \tag{3.3}$$

where $A(\rho) \in \Re[\rho]^{r\times r}$, $B(\rho) \in \Re[\rho]^{r\times m}$, $C(\rho) \in \Re[\rho]^{p\times r}$, $D(\rho) \in \Re[\rho]^{p\times m}$ with Rosenbrock system matrix:

$$P(s) = \begin{pmatrix} A(s) & B(s) \\ -C(s) & D(s) \end{pmatrix} \in \Re[s]^{(p+r)\times(m+r)}. \tag{3.4}$$

We wish first to express $P(s)$ explicitly in terms of the subsystems $P_1(s)$, $P_2(s)$, ..., $P_n(s)$ and their interconnections. The specific form of the interconnection equations we consider is

$$\begin{aligned} u_i(t) &= -\sum_{j=1}^{n} F_{i,j}\, y_j(t) + K_i\, u(t) \\ y(t) &= \sum_{i=1}^{n} L_i\, y_i(t) \end{aligned} \tag{3.5}$$

with $F_{i,j} \in \Re^{m_i\times p_j}$, $K_i \in \Re^{m_i\times m}$, $L_i \in \Re^{p\times p_i}$ for $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,n$. For this purpose, first define

$$\beta_s(t) = \begin{pmatrix} \beta_1(t) \\ \beta_2(t) \\ \vdots \\ \beta_n(t) \end{pmatrix} \in \Re(t)^{\left(\sum_{i=1}^{n} r_i\right)\times 1}; \quad u_s(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix} \in \Re(t)^{\left(\sum_{i=1}^{n} m_i\right)\times 1};$$

$$y_s(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix} \in \Re(t)^{\left(\sum_{i=1}^{n} p_i\right)\times 1}$$

$$A_s(s) = \text{block diag}\,(A_1(s),\ A_2(s),\ \ldots,\ A_n(s)) \in \Re[s]^{\left(\sum_{i=1}^{n} r_i\right)\times\left(\sum_{i=1}^{n} r_i\right)}$$

$$B_s(s) = \text{block diag}\,(B_1(s),\ B_2(s),\ \ldots,\ B_n(s)) \in \Re[s]^{\left(\sum_{i=1}^{n} r_i\right)\times\left(\sum_{i=1}^{n} m_i\right)}$$

$$C_s(s) = \text{block diag}\,(C_1(s),\ C_2(s),\ \ldots,\ C_n(s)) \in \Re[s]^{\left(\sum_{i=1}^{n} p_i\right)\times\left(\sum_{i=1}^{n} r_i\right)}$$

$$D_s(s) = \text{block diag}\,(D_1(s),\ D_2(s),\ \ldots,\ D_n(s)) \in \Re[s]^{\left(\sum_{i=1}^{n} p_i\right)\times\left(\sum_{i=1}^{n} m_i\right)}$$

$$F = \begin{pmatrix} F_{1,1} & F_{1,2} & \cdots & F_{1,n} \\ F_{2,1} & F_{2,2} & \cdots & F_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n,1} & F_{n,2} & \cdots & F_{n,n} \end{pmatrix} \in \Re^{\left(\sum_{i=1}^{n} m_i\right)\times\left(\sum_{i=1}^{n} p_i\right)};$$

$$K = \begin{pmatrix} K_1 \\ K_2 \\ \vdots \\ K_n \end{pmatrix} \in \Re^{\left(\sum_{i=1}^{n} m_i\right) \times m};$$

$$L = (L_1,\ L_2,\ \cdots,\ L_n) \in \Re^{p \times \left(\sum_{i=1}^{n} p_i\right)}. \tag{3.6}$$

Then the equations $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$ can be written concisely in the form:

$$\begin{aligned} A_s(\rho)\,\beta_s(t) &= B_s(\rho)\,u_s(t) \\ y_s(t) &= C_s(\rho)\,\beta_s(t) + D_s(\rho)\,u_s(t) \end{aligned} \tag{3.7}$$

which corresponds to the subsystem matrix

$$P_s(s) = \begin{pmatrix} A_s(s) & B_s(s) \\ -C_s(s) & D_s(s) \end{pmatrix} \in \Re[s]^{\left(\sum_{i=1}^{n}(p_i+r_i)\right) \times \left(\sum_{i=1}^{n}(m_i+r_i)\right)}. \tag{3.8}$$

In the same way the composite system equations for $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$ interconnected as in (3.5) can be written as

$$\begin{aligned} \begin{pmatrix} A_s(s) & B_s(s) & 0 \\ -C_s(s) & D_s(s) & I \\ 0 & -I & F \end{pmatrix} \begin{pmatrix} \beta_s(t) \\ -u_s(t) \\ y_s(t) \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ K \end{pmatrix} u(t) \\ y(t) = \begin{pmatrix} 0 & 0 & L \end{pmatrix} &\begin{pmatrix} \beta_s(t) \\ -u_s(t) \\ y_s(t) \end{pmatrix} \end{aligned} \tag{3.9}$$

which corresponds to the system matrix

$$P_s(s) = \left(\begin{array}{ccc|c} A_s(s) & B_s(s) & 0 & 0 \\ -C_s(s) & D_s(s) & I & 0 \\ 0 & -I & F & K \\ \hline 0 & 0 & -L & 0 \end{array}\right) \tag{3.10}$$

$$\in \Re[s]^{\left(p+\sum_{i=1}^{n}(r_i+p_i+m_i)\right) \times \left(m+\sum_{i=1}^{n}(r_i+p_i+m_i)\right)}$$

under the assumption that

$$\det \begin{pmatrix} A_s(s) & B_s(s) & 0 \\ -C_s(s) & D_s(s) & I \\ 0 & -I & F \end{pmatrix} \neq 0. \tag{3.11}$$

It is important to know which properties of Σ remain invariant under transformations of a particular type applied to the subsystem Σ_i, $i = 1, 2, \ldots, n$. [7] shown that the finite pole/zero structures of Σ remains invariant under any strict system equivalence transformation of the subsystems Σ_i, $i = 1, 2, \ldots, n$. However some certain questions remain concerning the infinite pole/zero structures of Σ. Answers to these questions may be obtained as a consequence of the following

Theorem 2. Every invariant of Σ under full system equivalence is invariant under all transformations of full system equivalence applied to the systems Σ, $i = 1, 2, \ldots, n$.

Proof. Any system matrix $P_i'(s)$ which is full system equivalent to the system matrices $P_i(s)$ can be written as

$$\begin{pmatrix} M_i(s) & 0 \\ X_i(s) & \end{pmatrix} \underbrace{\begin{pmatrix} A_i(s) & B_i(s) \\ -C_i(s) & D_i(s) \end{pmatrix}}_{P_i(s)} = \underbrace{\begin{pmatrix} A_i'(s) & B_i'(s) \\ -C_i'(s) & D_i'(s) \end{pmatrix}}_{P_i'(s)} \begin{pmatrix} N_i(s) & Y_i(s) \\ 0 & I \end{pmatrix} \tag{3.12}$$

where (3.12) is a full system equivalence transformation. Consequently any set of such transformations applied to $P_1(s), P_2(s), \ldots, P_n(s)$ can be represented by the full system equivalent transformation:

$$\begin{pmatrix} M(s) & 0 \\ X(s) & I \end{pmatrix} \underbrace{\begin{pmatrix} A_s(s) & B_s(s) \\ -C_s(s) & D_s(s) \end{pmatrix}}_{P_s(s)} = \underbrace{\begin{pmatrix} A_s'(s) & B_s'(s) \\ -C_s'(s) & D_s'(s) \end{pmatrix}}_{P_s'(s)} \begin{pmatrix} N(s) & Y(s) \\ 0 & I \end{pmatrix} \tag{3.13}$$

where

$$\begin{aligned} M(s) &= \text{block diag}\left(M_1(s), \ M_2(s), \ \ldots, \ M_n(s) \right) \\ N(s) &= \text{block diag}\left(N_1(s), \ N_2(s), \ \ldots, \ N_n(s) \right) \\ X(s) &= \text{block diag}\left(X_1(s), \ X_2(s), \ \ldots, \ X_n(s) \right) \\ Y(s) &= \text{block diag}\left(Y_1(s), \ Y_2(s), \ \ldots, \ Y_n(s) \right). \end{aligned} \tag{3.14}$$

On applying the interconnection defined previously, a composite system matrix $P_\Sigma'(s)$ is obtained from $P_\Sigma(s)$ and it is readily confirmed that

$$\left(\begin{array}{ccc:c} M(s) & 0 & 0 & 0 \\ X(s) & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hdashline 0 & 0 & 0 & I \end{array}\right) \left(\begin{array}{ccc:c} A_s(s) & B_s(s) & 0 & 0 \\ -C_s(s) & D_s(s) & I & 0 \\ 0 & -I & F & K \\ \hdashline 0 & 0 & -L & 0 \end{array}\right) =$$

$$
= \left(\begin{array}{cccc}
A'_s(s) & B'_s(s) & 0 & \vdots\; 0 \\
-C'_s(s) & D'_s(s) & I & \vdots\; 0 \\
0 & -I & F & \vdots\; K \\
\hdashline
0 & 0 & -L & \vdots\; 0
\end{array}\right) = \left(\begin{array}{cccc}
N(s) & Y(s) & 0 & \vdots\; 0 \\
0 & I & 0 & \vdots\; 0 \\
0 & 0 & I & \vdots\; 0 \\
\hdashline
0 & 0 & 0 & \vdots\; I
\end{array}\right) \tag{3.15}
$$

is a full system equivalence transformation since the relevant compound matrices of the above transformation are related via strict equivalence transformations to

$$
\left(\begin{array}{cccc|cccc}
M(s) & 0 & A'_s(s) & B'_s(s) & 0 & 0 & 0 & 0 \\
X(s) & I & -C'_s(s) & D'_s(s) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0
\end{array}\right)
$$

$$
= \left(\begin{array}{cccc|cccc}
M(s) & 0 & 0 & 0 & A'_s(s) & B'_s(s) & 0 & 0 \\
X(s) & I & 0 & 0 & -C'_s(s) & D'_s(s) & I & 0 \\
0 & 0 & I & 0 & 0 & -I & F & K \\
0 & 0 & 0 & I & 0 & 0 & -L & 0
\end{array}\right) \tag{3.16}
$$

$$
\left(\begin{array}{cccccccc}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & -I & 0 \\
0 & 0 & 0 & I & I & 0 & -F & -K \\
0 & 0 & 0 & 0 & 0 & I & L & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I
\end{array}\right)
$$

and

$$
\left(\begin{array}{cccc}
A_s(s) & B_S(s) & 0 & 0 \\
-C_s(s) & D_s(s) & 0 & 0 \\
-N(s) & -Y(s) & 0 & 0 \\
0 & -I & 0 & 0 \\
\hdashline
0 & 0 & I & 0 \\
0 & 0 & 0 & I \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right) = \left(\begin{array}{cccccccc}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -I & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -I \\
0 & 0 & I & 0 & 0 & -I & F & K \\
0 & 0 & 0 & I & 0 & 0 & -L & 0
\end{array}\right) \times
$$

$$\times \begin{pmatrix} A_s(s) & B_s(s) & 0 & 0 \\ -C_s(s) & D_S(s) & I & 0 \\ 0 & -I & F & K \\ 0 & 0 & -L & 0 \\ \hdashline -N(s) & -Y(s) & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & -I \end{pmatrix}. \tag{3.17}$$

The above compound matrices are constructed from the compound matrices of the full equivalent transformation (3.13) via strict equivalence transformations and thus satisfy the conditions of full equivalence. Therefore the transformation (3.15) is a full system equivalence transformation, which verifies the theorem. □

It thus follows that any operation of full system equivalence on the subsystems Σ_i, $i = 1, 2, \ldots, n$ or equivalently to the system Σ_i corresponds to a transformation of full system equivalence on the composite system Σ. In the case where the matrices F, K and L are not constant but polynomial (respectively proper) we observe that the compound matrices of the transformation (3.15) satisfy the conditions of full equivalence ($C \cup \{\infty\}$-equivalence) provided that (3.13) is a transformation of this type and therefore Theorem 2 holds true.

This theorem shows that any property of Σ which is invariant under full system equivalence is unaffected by the particular choice of representation (within full system equivalence) of the subsystems Σ_i. Thus we can show the following

Theorem 3. Let a linear, time invariant, multivariable system Σ which arises from the interconnection of the linear, time invariant, multivariable systems Σ_i, $i = 1, 2, \ldots, n$ (see (3.9)). Then there exists a generalized state space description Σ_r of Σ which is full system equivalent with the composite system Σ, and which is an interconnection of subsystems in generalized state space form.

Proof. Let Σ_i' be a full system equivalent generalized state space system of Σ_i under the following full system equivalence transformation [4]:

$$\begin{pmatrix} M_i & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A_i(s) & B_i(s) \\ -C_i(s) & D_i(s) \end{pmatrix} = \begin{pmatrix} sE_i - A_i & B_i \\ -C_i & 0 \end{pmatrix} \begin{pmatrix} N_i(s) & Y_i(s) \\ 0 & I \end{pmatrix}. \tag{3.18}$$

then, according to relation (3.15), we shall obtain the following full system equivalence transformation

$$\begin{pmatrix} M & 0 & 0 & \vdots & 0 \\ 0 & I & 0 & \vdots & 0 \\ 0 & 0 & I & \vdots & 0 \\ \hdashline 0 & 0 & 0 & \vdots & I \end{pmatrix} \begin{pmatrix} A_s(s) & B_s(s) & 0 & \vdots & 0 \\ -C_s(s) & D_s(s) & I & \vdots & 0 \\ 0 & -I & F & \vdots & K \\ \hdashline 0 & 0 & -L & \vdots & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} sE_s - A_s & B_s & 0 & \vdots & 0 \\ -C_s & 0 & I & \vdots & 0 \\ 0 & -I & F & \vdots & K \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & -L & \vdots & 0 \end{pmatrix} = \begin{pmatrix} N(s) & Y(s) & 0 & \vdots & 0 \\ 0 & I & 0 & \vdots & 0 \\ 0 & 0 & I & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & I \end{pmatrix} \tag{3.19}$$

where

$$\begin{aligned} sE_s - A_s &= \text{block diag}\,(sE_1 - A_1,\ sE_2 - A_2,\ \ldots,\ sE_n - A_n) \\ B_s &= \text{block diag}\,(B_1,\ B_2,\ \ldots, B_n) \\ C_s &= \text{block diag}\,(C_1,\ C_2,\ \ldots, C_n) \end{aligned} \tag{3.20a}$$

and

$$\begin{aligned} M &= \text{block diag}\,(M_1,\ M_2,\ \ldots,\ M_n) \\ N(s) &= \text{block diag}\,(N_1(s),\ N_2(s),\ \ldots, N_n(s)) \\ Y(s) &= \text{block diag}\,(Y_1(s),\ Y_2(s),\ \ldots, Y_n(s)) \end{aligned} \tag{3.20b}$$

which verifies that the generalized state space description

$$\begin{aligned} \Sigma_r : \begin{pmatrix} \rho E_s - A_s & B_s & 0 \\ -C_s & 0 & I \\ 0 & -I & F \end{pmatrix} x(t) &= \begin{pmatrix} 0 \\ 0 \\ K \end{pmatrix} u(t) \\ y(t) &= \begin{pmatrix} 0 & 0 & L \end{pmatrix} x(t) \end{aligned} \tag{3.21}$$

is full system equivalent to the system Σ. □

Theorem 4. The linear, time invariant, multivariable system Σ which comes from the interconnection of the linear, time invariant, multivariable systems Σ_i (see (3.9)) is full system equivalent to the generalized state space system

$$\begin{aligned} (\rho E_s - A_s + B_s F C_s)\,\xi_s(t) &= B_s K u(t) \\ y(t) &= L C_s \xi_s(t). \end{aligned} \tag{3.22}$$

Proof. It is easily seen that the transformation:

$$\begin{pmatrix} I & -B_s F & B_s & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & L & 0 & \vdots & I \end{pmatrix} \begin{pmatrix} sE_s - A_s & B_s & 0 & \vdots & 0 \\ -C_s & 0 & I & \vdots & 0 \\ 0 & -I & F & \vdots & K \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & -L & \vdots & 0 \end{pmatrix} \tag{3.23}$$

$$= \begin{pmatrix} sE_s - A_s + B_s F C_s & \vdots & B_s K \\ \cdots & \cdots & \cdots \\ -LC_s & \vdots & 0 \end{pmatrix} \begin{pmatrix} I & 0 & 0 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & I \end{pmatrix}$$

is a full system equivalence transformation. From the full equivalent transformation (3.19) and the symmetry property of full system equivalence we finally obtain that the composite system Σ will be full system equivalent to the generalized state space system (3.22) which verifies the theorem. □

Example 1. Consider the following systems:

$$\begin{aligned} \Sigma_1 &: (\rho^2+5\rho+6)\,\beta_1(t) = (\rho+1)\,u_1(t) \\ y_1(t) &= (5-2\rho)\,\beta_1(t) + (3\rho+2)\,u_1(t) \end{aligned} \tag{E.1}$$

and

$$\begin{aligned} \Sigma_2 &: \begin{pmatrix} \rho+1 & \rho^2 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}}_{\beta_2(t)} = \begin{pmatrix} \rho+1 \\ 0 \end{pmatrix} u_2(t) \\ y_2(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}}_{\beta_2(t)} \end{aligned} \tag{E.2}$$

and consider the following equations of interconnection between Σ_1 and Σ_2:

$$\begin{aligned} u_1(t) &= -y_1(t)+y_2(t)+2u(t) \\ u_2(t) &= y_1(t)-u(t) \\ y(t) &= y_1(t)-y_2(t). \end{aligned} \tag{E.3}$$

Therefore we can define the matrices

$$F = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}; \quad K = \begin{pmatrix} 2 \\ -1 \end{pmatrix}; \quad L = \begin{pmatrix} 1 & -1 \end{pmatrix}. \tag{E.4}$$

The Rosenbrock system matrix of the composite system Σ will have the following form

$$P_s(s) = \left(\begin{array}{cccc} A_s(s) & B_s(s) & 0 & \vdots\ 0 \\ -C_s(s) & D_s(s) & I & \vdots\ 0 \\ 0 & -I & F & \vdots\ K \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & -L & \vdots\ 0 \end{array}\right) =$$

$$= \left(\begin{array}{ccccccc:c} s^2+5s+6 & 0 & 0 & s+1 & 0 & 0 & 0 & 0 \\ 0 & s+1 & s^2 & 0 & s+1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2s-5 & 0 & 0 & 3s+2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 \\ \hdashline 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{array}\right). \tag{E.5}$$

Consider also the full system equivalent generalized state space systems Σ_{R_1} and Σ_{R_2} (see [1, 4]) or Σ_1 and Σ_2 respectively,

$$P_{\Sigma_{R_1}}(s) = \left(\begin{array}{cccc:c} 5s+6 & s+1 & -s & 0 & 0 \\ 2s-5 & 3s+2 & 0 & 1 & 0 \\ s & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ \hdashline 0 & 0 & 0 & -1 & 0 \end{array}\right) \tag{E.6}$$

and

$$P_{\Sigma_{R_2}}(s) = \left(\begin{array}{ccccc:c} s+1 & 0 & s+1 & -s & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & s & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ \hdashline 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right). \tag{E.7}$$

Then the full system equivalent generalized state space system of the composite system (E.5) will be the following:

$$\begin{aligned} (\rho E_s - A_s + B_s F C_s)\,\xi_s(t) &= B_s K u(t) \\ y(t) &= L C_s \xi_s(t) \end{aligned} \tag{E.8}$$

where

$$\rho E_s - A_s + B_s F C_s =$$

$$= \begin{pmatrix} 5\rho+6 & \rho+1 & -\rho & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\rho-5 & 3\rho+2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \rho & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \rho+1 & 0 & \rho+1 & -\rho & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \rho & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}. \tag{E.9}$$

$$B_s K = \begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}^T \tag{E.10}$$

$$LC_s = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{E.11}$$

□

4. AN APPLICATION OF THE HIERARCHICAL THEORY OF SYSTEMS

A direct implication of Theorem 3 concerns the reduction of a general polynomial description to a full system equivalent generalized state space form. More specifically, consider the polynomial matrix description:

$$\begin{aligned} \Sigma : A(\rho)\,\beta(t) &= B(\rho)\,u(t) \\ y(t) &= C(\rho)\,\beta(t) + D(\rho)\,u(t) \end{aligned} \tag{4.1}$$

and let its *normalized form* [9] be the following:

$$\begin{aligned} \Sigma : \underbrace{\begin{pmatrix} A(\rho) & B(\rho) & 0 \\ -C(\rho) & D(\rho) & I \\ 0 & -I & 0 \end{pmatrix}}_{T(\rho)} \underbrace{\begin{pmatrix} \beta(t) \\ -u(t) \\ y(t) \end{pmatrix}}_{\xi(t)} &= \underbrace{\begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix}}_{U} u(t) \\ y(t) = \underbrace{\begin{pmatrix} 0 & 0 & I \end{pmatrix}}_{V} \underbrace{\begin{pmatrix} \beta(t) \\ -u(t) \\ y(t) \end{pmatrix}}_{\xi(t)}&. \end{aligned} \tag{4.2}$$

Then we have the following:

Theorem 5. Let $\{C,\, sE - A,\, B\}$ be a strongly irreducible realization of $T(s)^{-1}$. Then the generalized state space system:

$$\begin{aligned} S' : E\dot{x}(t) &= Ax(t) + BUu(t) \\ y(t) &= VCx(t) \end{aligned} \tag{4.3}$$

is full system equivalent to the system Σ of the form (4.1) - (4.2).

Proof. It is easily seen that Σ is an interconnection of the following three systems:

$$\begin{aligned} \Sigma_1 : I\beta_1(t) &= Uu_1(t) \\ y_1(t) &= I\beta_1(t) \end{aligned} \tag{4.4a}$$

$$\begin{aligned} \Sigma_2 : T(\rho)\,\beta_2(t) &= Iu_2(t) \\ y_2(t) &= I\beta_2(t) \end{aligned} \tag{4.4b}$$

$$\begin{aligned} \Sigma_3 : I\beta_3(t) &= Iu_3(t) \\ y_3(t) &= V\beta_3(t) \end{aligned} \tag{4.4c}$$

under the following interconnections:

$$\begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix} = -\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ -I & 0 & 0 \\ 0 & -I & 0 \end{pmatrix}}_{F} \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} + \underbrace{\begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix}}_{K} u(t)$$

$$y(t) = \underbrace{\begin{pmatrix} 0 & 0 & I \end{pmatrix}}_{L} \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}. \tag{4.5}$$

Σ_2 is strongly irreducible (has no finite nor infinite decoupling zeros) and thus any generalized state space realization of its transfer function i.e. $T(s)^{-1}$, will be full system equivalent to this system [4]. Thus according to our initial assumption that ($\{C,\, sE - A,\, B\}$ be a strongly irreducible realization of $T(s)^{-1}$) then S_2:

$$\begin{aligned} S_2 : E\dot{x}(t) &= Ax(t) + Bu_2(t) \\ y_2(t) &= Cx(t) \end{aligned} \tag{4.6}$$

is full system equivalent to the system Σ_2. Then according to Theorem 4, the full system equivalent generalized state space system S of the interconnected system Σ will be the following:

$$\begin{aligned} S : (\rho E_s - A_s + B_s F C_s)\,\xi_s(t) &= B_s K u(t) \\ y(t) &= L C_s \xi_s(t) \end{aligned} \tag{4.7}$$

where

$$sE_s - A_s = \begin{pmatrix} I & 0 & 0 \\ 0 & sE - A & 0 \\ 0 & 0 & I \end{pmatrix}; \quad B_s = \begin{pmatrix} U & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & I \end{pmatrix};$$

$$C_s = \begin{pmatrix} I & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & V \end{pmatrix}; \quad \xi_s = \begin{pmatrix} \beta_1(t) \\ x(t) \\ \beta_3(t) \end{pmatrix}. \tag{4.8}$$

Thus

$$sE_s - A_s + B_sFC_s$$

$$= \begin{pmatrix} I & 0 & 0 \\ 0 & sE-A & 0 \\ 0 & 0 & I \end{pmatrix} + \begin{pmatrix} U & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ -I & 0 & 0 \\ 0 & -I & 0 \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & V \end{pmatrix} .$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ -B & sE-A & 0 \\ 0 & -C & I \end{pmatrix} \tag{4.9a}$$

$$B_sK = \begin{pmatrix} U & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} U \\ 0 \\ 0 \end{pmatrix} \tag{4.9b}$$

$$LC_s = \begin{pmatrix} 0 & 0 & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C \end{pmatrix} = \begin{pmatrix} 0 & 0 & V \end{pmatrix} \tag{4.9c}$$

and therefore the Rosenbrock system matrix of S will be

$$P_s(s) = \left(\begin{array}{ccc:c} I & 0 & 0 & U \\ -B & sE-A & 0 & 0 \\ 0 & -C & I & 0 \\ \hdashline 0 & 0 & -V & 0 \end{array} \right) . \tag{4.10}$$

Note also that the following transformation:

$$\left(\begin{array}{ccc:c} B & I & 0 & 0 \\ 0 & 0 & V & I \end{array} \right) \left(\begin{array}{ccc:c} I & 0 & 0 & U \\ -B & sE-A & 0 & 0 \\ 0 & -C & I & 0 \\ \hdashline 0 & 0 & -V & 0 \end{array} \right)$$

$$= \left(\begin{array}{c:c} sE-A & BU \\ \hdashline -VC & 0 \end{array} \right) \left(\begin{array}{ccc:c} 0 & I & 0 & 0 \\ \hdashline 0 & 0 & 0 & I \end{array} \right) \tag{4.11}$$

is obviously a full system equivalence (more precisely a complete system equivalence) transformation. Thus the system:

$$\begin{aligned} S' : E\dot{x}(t) &= Ax(t) + BUu(t) \\ y(t) &= VCx(t) \end{aligned} \tag{4.12}$$

is a full system equivalent generalized state space model of Σ due to the transitivity property of full system equivalence. It is known however [5] that the full system equivalence transformation which relate the systems Σ and S' is the following:

$$\begin{pmatrix} T(s)\,C(sE-A)^{-1} & \vdots & 0 \\ \cdots & & \cdots \\ \mathbf{0} & \vdots & I \end{pmatrix} \begin{pmatrix} sE-A & \vdots & BU \\ \cdots & & \cdots \\ -VC & \vdots & 0 \end{pmatrix} = \begin{pmatrix} T(s) & \vdots & U \\ \cdots & & \cdots \\ -V & \vdots & 0 \end{pmatrix} \begin{pmatrix} C & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & I \end{pmatrix}. \tag{4.13}$$

□

An implementation of the above construction method may be found in the generalized state space reduction models presented by [8] and in the extension of Tan & VanDewalle's realization method presented by [5].

Example 2. Consider a system Σ described by the following equations:

$$\begin{aligned} \Sigma : (\rho^2+5\rho+6)\,\beta(t) &= (\rho+1)\,u(t) \\ y(t) &= (5-2\rho)\,\beta(t) + (3\rho+2)\,u(t) \end{aligned} \tag{E.1}$$

or its normalized form [9]:

$$\begin{aligned} \Sigma : \underbrace{\begin{pmatrix} \rho^2+5\rho+6 & \rho+1 & 0 \\ 2\rho-5 & 3\rho+2 & 1 \\ 0 & -1 & 0 \end{pmatrix}}_{T(\rho)} \underbrace{\begin{pmatrix} \beta(t) \\ -u(t) \\ y(t) \end{pmatrix}}_{\xi(t)} &= \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{U} = u(t) \\ y(t) = \underbrace{\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}}_{V} \underbrace{\begin{pmatrix} \beta(t) \\ -u(t) \\ y(t) \end{pmatrix}}_{\xi(t)}&. \end{aligned} \tag{E.2}$$

It is easily shown that:

$$T(s)^{-1} = \underbrace{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 9 & 11 & 3 & -3 \end{pmatrix}}_{C} \underbrace{\begin{pmatrix} s+2 & 0 & 0 & 0 \\ 0 & s+3 & 0 & 0 \\ 0 & 0 & 1 & -s \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1}}_{sE-A} \underbrace{\begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 1/3 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{B} \tag{E.3}$$

where $\{C, sE - A, B\}$ is a strongly irreducible realization. Then according to Theorem 5 the system:

$$S: \underbrace{\begin{pmatrix} \rho+2 & 0 & 0 & 0 \\ 0 & \rho+3 & 0 & 0 \\ 0 & 0 & 1 & -\rho \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{rE-a} \underbrace{\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix}}_{x(t)} = \underbrace{\begin{pmatrix} -1 \\ 2 \\ 1 \\ 1 \end{pmatrix}}_{BU} u(t)$$

$$y(t) = \underbrace{\begin{pmatrix} 9 & 11 & 3 & -3 \end{pmatrix}}_{VC} \underbrace{\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix}}_{x(t)} \tag{E.4}$$

is full system equivalent to the system Σ under the following (f. s. e.) transformation:

$$\left(\begin{array}{cccc:c} 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 \\ 0 & 1/3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 \end{array}\right) \left(\begin{array}{ccc:c} \rho^2+5\rho+6 & \rho+1 & 0 & 0 \\ 2\rho-5 & 3\rho+2 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ \hdashline 0 & 0 & -1 & 0 \end{array}\right)$$

$$= \left(\begin{array}{cccc:c} \rho+2 & 0 & 0 & 0 & -1 \\ 0 & \rho+3 & 0 & 0 & 2 \\ 0 & 0 & 1 & -\rho & 1 \\ 0 & 0 & 0 & 1 & 1 \\ \hdashline -9 & -11 & -3 & 3 & 0 \end{array}\right) \left(\begin{array}{ccc:c} \rho+3 & 1 & 0 & 0 \\ -\rho-2 & -1 & 0 & 0 \\ (2\rho-5)/3 & -1/3 & 1/3 & 0 \\ 0 & -1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right). \tag{E.5}$$

5. CONCLUSIONS

This paper presents a discussion of the role of full system equivalence transformation within the hierarchical theory of systems. More specifically we prove (Theorem 2) that every property of a linear multivariable system Σ which remains invariant under the transformation of full system equivalence, is unaffected by the particular choice (within full system equivalence) of the subsystems Σ_i, $i = 1, 2, \ldots, n$ of Σ. As a result of this conclusion we have derived a reduction algorithm which has the property to reduce any composite linear multivariable system Σ to a full system equivalent generalized state space form.

(Received June 20, 1996.)

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