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THE INVARIANT POLYNOMIAL ASSIGNMENT PROBLEM FOR LINEAR PERIODIC DISCRETE-TIME SYSTEMS[1]

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This paper considers the problem of assigning the closed loop invariant polynomials of a feedback control system, where the plant is a linear, discrete-time, periodic system. By a matrix algebraic approach, necessary and sufficient conditions for problem solvability are established and a parameterization of all periodic output controllers assigning the desired invariant polynomials is given.

1. INTRODUCTION

Various classes of processes, such as periodically time-varying networks and filters (for example switched-capacitors circuits and multirate digital filters), chemical processes, multirate sampled-data systems, can be modeled through a linear periodic system (see, e. g., [2, 13] and references therein). Moreover, the study of linear periodic systems can be helpful even for the stabilization and control of time-invariant linear systems through a periodic controller [1, 8, 18, 19, 21, 27], and for the stabilization and control of a class of bilinear systems [10, 11, 12].

In the discrete-time case, a control theory is developing with the help of algebraic and geometric techniques and contributions on several control problem have been given, including eigenvalue assignment, state and output dead-beat control, disturbance decoupling, model matching, adaptive control, robust control and optimal H_2/H_∞ control (see, e. g., [3, 5, 7, 13, 15, 17, 22, 25, 26]).

The aim of this paper is to analyze the invariant polynomial assignment problem for the class of discrete-time linear periodic systems. This problem generalizes the characteristic polynomial assignment, which, for the same class of systems, was solved by a geometric approach in [5, 15, 17, 22]. For time-invariant plants, the invariant polynomial assignment was considered in [19, 20, 23, 27].

The paper is organized in the following way. In Section 2 preliminary definitions and results are given. The problem considered in this paper is formally stated in Section 3, and conditions for its solvability are constructively established in Section 4.

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2. PRELIMINARY RESULTS

Consider the ω-periodic discrete-time system Σ described by

$$x(k+1) = A(k)\,x(k) + B(k)\,u(k), \tag{2.1}$$
$$y(k) = C(k)\,x(t), \tag{2.2}$$

where $k \in \mathbb{Z}$, $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^p$ is the input, $y(k) \in \mathbb{R}^q$ is the output and $A(\cdot), B(\cdot), C(\cdot)$ are periodic matrices of period ω (briefly, ω-periodic). Denote also by $\Phi(k, k_0)$, $k \geq k_0$, the transition matrix associated with $A(\cdot)$.

It is well-known that, for any initial time $k_0 \in \mathbb{Z}$, the output response of system Σ for $k \geq k_0$, to given initial state $x(k_0)$ and control function $u(\cdot)$, can be obtained through the time-invariant *associated system of* Σ *at time* k_0, denoted by $\Sigma^a(k_0)$ [24]. $\Sigma^a(k)$ is represented by

$$x_k(h+1) = E_k x_k(h) + J_k u_k(h) \tag{2.3}$$
$$y_k(h) = L_k x_k(h) + M_k u_k(h) \tag{2.4}$$

where $E_k := \Phi(\omega+k, k)$, $J_k := [(J_k)_1 \ \cdots \ (J_k)_\omega]$, $(J_k)_i := \Phi(\omega+k, i+k)\,B(i-1+k)$, $i = 1, \cdots, \omega$, $L_k := [(L_k)'_1 \ \cdots \ (L_k)'_\omega]'$, $(L_k)_i := C(i-1+k)\,\Phi(i-1+k, k)$, $i = 1, \cdots, \omega$, $M_k := [(M_k)_{ij} \in \mathbb{R}^{q\times p}, \ \ i, j = 1, \cdots, \omega]$, with $(M_k)_{ij} := C(i-1+k)\,\Phi(i-1+k, j+k)\,B(j-1+k)$, if $i > j$, and $(M_k)_{ij} := 0$, if $i \leq j$.

In fact, if $x_k(0) = x(k)$ and $u_k(h) := [u'(h\omega+k) \ \ u'(h\omega+k+1) \ \cdots \ u'(h\omega+k+\omega-1)]'$ for all $h \in \mathbb{Z}^+$, then $x_k(h) = x(k+h\omega)$ and $y_k(h) = [y'(h\omega+k) \ \ y'(h\omega+k+1) \ \cdots \ y'(h\omega+k+\omega-1)]'$ for all $h \in \mathbb{Z}^+$. The notion of associated system at time k allows one to analyze structural and stability properties and pole-zero-structures of periodic systems [2, 4, 14]. For example, the subspace of reachable (unobservable) states of system Σ at time k is readily seen to coincide with that of system $\Sigma^a(k)$ if it is expressed in terms of matrices E_k, J_k, L_k and M_k [14]. Obviously, $\Sigma^a(k+\omega) = \Sigma^a(k)$ for all integer k. A simple test for the reachability (observability) of system Σ at time k was also introduced in [16] making use of the following block-diagonal matrices:

$$\mathcal{A}_k := \text{blockdiag}\{A(k),\ A(k+1), \cdots, A(\omega-1+k)\}, \tag{2.5}$$
$$\mathcal{B}_k := \text{blockdiag}\{B(k),\ B(k+1), \cdots, B(\omega-1+k)\}, \tag{2.6}$$
$$\mathcal{C}_k := \text{blockdiag}\{C(k),\ C(k+1), \cdots, C(\omega-1+k)\}, \tag{2.7}$$
$$\mathcal{R}_k(\lambda) := \begin{bmatrix} 0 & I_{(\omega-1)n} \\ \lambda I_n & 0 \end{bmatrix}, \quad \lambda \in \mathbb{C}, \tag{2.8}$$

where I_n denotes the identity matrix of dimension n.

Lemma 2.1. [16] System Σ is reachable (observable) at time k if and only if the following matrix

$$[\mathcal{A}_k - \mathcal{R}_k(\lambda) \quad \mathcal{B}_k] \qquad ([\mathcal{A}'_k - \mathcal{R}'_k(\lambda) \quad \mathcal{B}'_k]')$$

has full row-rank (column-rank) for all $\lambda \in \mathbb{C}$, or equivalently for all the eigenvalues of E_k.

The notions of invariant zero, transmission zero and pole of the ω-periodic system Σ at time k are defined with reference to the following $\omega q \times \omega p$ matrix

$$W_k(d) = L_k d(I_n - dE_k)^{-1} J_k + M_k, \tag{2.9}$$

where $d := z^{-1}$ is the backward shift operator. The rational matrix $W_k(d)$ is the transfer matrix of the associated system of Σ at time k and is called the *associated transfer matrix of* Σ *at time* k. A complete analysis of pole-zero structure of system Σ is reported in [14] and [16] making use of the associated transfer matrix characterized with the forward shift operator z. The following result, that follows from Lemma 2.1 in [14], shows the dependence of $W_k(d)$ with respect to the initial time k.

Lemma 2.2. For any integer k it holds that:

$$W_{k+1}(d) = \begin{bmatrix} 0 & I_{q(\omega-1)} \\ d^{-1}I_q & 0 \end{bmatrix} W_k(d) \begin{bmatrix} 0 & dI_p \\ I_{p(\omega-1)} & 0 \end{bmatrix}. \tag{2.10}$$

As a consequence of this result the rank m of $W_k(d)$ is independent of time k (see, e. g., [14] for a similar result with the forward shift operator z).

The transfer matrix $W_k(d)$ can be factored as

$$W_k(d) = A_k^{-1}(d)\, B_k(d) = \overline{B}_k(d)\, \overline{A}_k^{-1}(d), \tag{2.11}$$

where $A_k(d)$ and $B_k(d)$ are relatively left prime *(rlp)* polynomial matrices and $\overline{A}_k(d)$ and $\overline{B}_k(d)$ are relatively right prime *(rrp)* polynomial matrices.

Analogously to the time-invariant case [23], the invariant polynomials of $I_n - dE_k$ are called the *invariant polynomials of* Σ *at time* k. As shown in [14, 16], the product of these polynomials characterizes the stability properties of Σ.

Under the hypothesis of reachability and observability of Σ at time k, the invariant polynomials of Σ at time k are associate of the invariant polynomials of the Smith forms of $A_k(d)$ and $\overline{A}_k(d)$ [23].

Denote by $\chi(q,p,\omega)$ the class of $\omega q \times \omega p$ rational matrices

$$W(d) = \begin{bmatrix} W_{11}(d) & W_{12}(d) & \cdots & W_{1\omega}(d) \\ W_{21}(d) & W_{22}(d) & \cdots & W_{2\omega}(d) \\ \vdots & \vdots & \ddots & \vdots \\ W_{\omega 1}(d) & W_{\omega 2}(d) & \cdots & W_{\omega\omega}(d) \end{bmatrix}, \quad W_{ij}(d) \in \mathbb{C}^{q\times p},\ i, j = 1, \cdots, \omega, \tag{2.12}$$

with $W_{ij}(0) = 0$, $i < j$, $i, j = 1, \ldots, \omega$. The class $\chi(q,p,\omega)$ characterizes the transfer matrices of ω-periodic systems. In fact, the causality of ω-periodic system Σ implies that the associated transfer matrix of Σ at time k belongs to the class $\chi(q,p,\omega)$ for all $k \in \mathbb{Z}$ [6]. Then, the causality of Σ implies that the roots of the invariant polynomials of Σ at time k are different from zero for all integers k. This in turn implies that matrices $A_k(0)$ and $\overline{A}_k(0)$ are nonsingular. Foregoing considerations and Lemma 2.2 allow us to prove the following result.

Lemma 2.3. The invariant polynomials of Σ at time k are independent of k.

Remark 2.1. The choice of the backward shift operator $d = z^{-1}$ allowed us to prove the independence of pole structure of Σ of time k. The same result does not hold if the forward operator z is used [16]. In particular in [14] it is shown that the structure of null poles may depend on k.

Moreover, $\chi(q,p,\omega)$ characterizes also the class of rational matrices that can be realized by an ω-periodic system of the form (2.1),(2.2). The solution of the minimal realization problem for the periodic case is described by a system reachable and observable at any time whose matrices have generally time-varying dimensions. In general, the subspaces of reachable states and/or observable states may have time-varying dimensions. Therefore, it is natural, in order to consistently solve the minimal realization problem, to allow for state-space description having time-varying dimensions. The possibility of computing a "quasi" minimal (reachable and observable at lest in one time) uniform (fixed-dimension) realization is also available. Efficient algorithms for the computation of minimal or quasi minimal realization of a given transfer matrix are introduced in [6] and [9].

Remark 2.2 Note that, given a transfer matrix $H(d) = D^{-1}(d)\,N(d) = \overline{N}(d)\overline{D}^{-1}(d) \in \mathbb{C}^{q\omega\times p\omega}$ with $D(d)$ and $N(d)$ *rlp* polynomial matrices and $\overline{D}(d)$ and $\overline{N}(d)$ *rrp* polynomial matrices and both $D(0)$ and $\overline{D}(0)$ non singular, then a sufficient condition for $H(d)$ belong to the class $\chi(q,p,\omega)$ is that $N(0) = 0$ and $\overline{N}(0) = 0$.

3. CONTROL SYSTEM STRUCTURE AND PROBLEM STATEMENT

Assume that system Σ is minimal (reachable and observable at all times), and consider an ω-periodic minimal controller Σ_G for system Σ acting in the feedback control structure of Figure 1 and described by

$$x_G(k+1) = A_G(k)\,x_G(k) + B_G(k)\,e_2(k), \tag{3.1}$$

$$y_2(k) = C_G(k)\,x_G(k) + D_G(k)\,e_2(k), \tag{3.2}$$

where $x_G(k) \in \mathbb{R}^{n_G(k)}$ is the state, with $n_G(k+\omega) = n_G(k)$, and

$$e_1(k) := u_1(k) - y_2(k), \tag{3.3}$$

$$e_2(k) := u_2(k) + y_1(k), \tag{3.4}$$

with $y_1(k) = y(k)$ (the output of Σ), $e_1(k) = u(k)$ (the input of Σ) and $u_1(k)$ and $u_2(k)$ external inputs.

The $\omega p \times \omega q$ associated transfer matrix of Σ_G at time k is expressed by

$$W_k^G(d) = L_k^G d(I_{n_G(k)} - dE_k^G)^{-1} J_k^G + M_k^G, \tag{3.5}$$

where matrices $L_k^G \in \mathbb{R}^{\omega p\times n_G(k)}$, $E_k^G \in \mathbb{R}^{n_G(k)\times n_G(k)}$, $J_k^G \in \mathbb{R}^{n_G(k)\times\omega q}$ and $M_k^G \in \mathbb{R}^{\omega p\times\omega q}$ are defined as matrices L_k, E_k, J_k and M_k with matrices $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ substituted by matrices $A_G(\cdot)$, $B_G(\cdot)$, $C_G(\cdot)$ respectively and with $(M_k^G)_{ii} = D_G(i-1+k)$, $i = 1,\dots,\omega$.

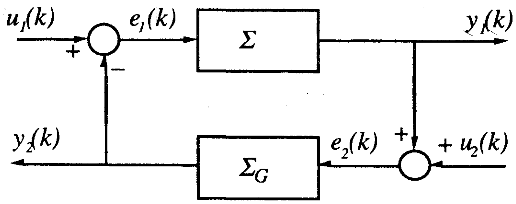

Fig. 1. The feedback control structure.

Causality of system Σ_G implies that $W_k^G(d)$ belongs to the class $\chi(p,q,\omega)$. Let $W_k^G(d)$ be factored as

$$W_k^G(d) = P_k^{-1}(d)\,Q_k(d) = \overline{Q}_k(d)\,\overline{P}_k^{-1}(d) \tag{3.6}$$

where $P_k(d)$ and $Q_k(d)$ are *rlp* polynomial matrices and $\overline{P}_k(d)$ and $\overline{Q}_k(d)$ are *rrp* polynomial matrices. The problem considered in this paper is formally stated as follows.

Problem 3.1. Given an ω-periodic system Σ reachable and observable at all times, and m causal polynomials $s_1(d), s_2(d), \ldots, s_m(d)$ such that $s_{i+1}(d)$ divides $s_i(d)$, find a minimally realized ω-periodic controller Σ_G described by (3.1), (3.2) and acting in the feedback system of Figure 1, such that the closed loop system Σ_{fb} be minimally realized and its invariant polynomials be associated of $s_i(d)$, $i = 1, 2, \ldots, m$.

4. PROBLEM SOLUTION

Denote by Σ_{fb} the ω-periodic system reported in Figure 1 and described by (2.1), (2.2), (3.1), (3.2), (3.3) and (3.4) with input $u(k)$ and output $y(k)$ of Σ equal to $e_1(k)$ and $y_1(k)$, respectively.

Define:

$$v(k) := [u_1'(k) \;\; u_2'(k)]', \quad w_1(k) := [y_1'(k) \;\; e_1'(k)]', \quad w_2(k) := [y_2'(k) \;\; e_2'(k)]', \tag{4.1}$$

the ω-periodic feedback system Σ_{fb} is described by the following equations:

$$\begin{bmatrix} x(k+1) \\ x_G(k+1) \end{bmatrix} = \begin{bmatrix} A(k) - B(k)D_G(k)C(k) & -B(k)C_G(k) \\ B_G(k)C(k) & A_G(k) \end{bmatrix} \begin{bmatrix} x(k) \\ x_G(k) \end{bmatrix} + \begin{bmatrix} B(k) & -B(k)D_G(k) \\ 0 & B_G(k) \end{bmatrix} v(k), \tag{4.2}$$

$$w_1(k) = \begin{bmatrix} C(k) & 0 \\ -D_G(k)C(k) & -C_G(k) \end{bmatrix} \begin{bmatrix} x(k) \\ x_G(k) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ I & -D_G(k) \end{bmatrix} v(k), \tag{4.3}$$

$$w_2(k) = \begin{bmatrix} D_G(k)C(k) & C_G(k) \\ C(k) & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ x_G(k) \end{bmatrix} + \begin{bmatrix} 0 & D_G(k) \\ 0 & I \end{bmatrix} v(k). \tag{4.4}$$

Denote with $W_k^1(d)$ and $W_k^2(d)$ the associated transfer matrices at time k of the ω-periodic feedback system Σ_{fb} relating input $v(\cdot)$ with outputs $w_1(\cdot)$ and $w_2(\cdot)$, respectively.

Introducing the lifted representations of inputs and outputs of Σ_{fb}:

$$u_k^1(h) := [u_1'(k+h\omega)\,u_1'(k+1+h\omega)\cdots u_1'(k+\omega-1+h\omega)]', \tag{4.5}$$
$$u_k^2(h) := [u_2'(k+h\omega)\,u_2'(k+1+h\omega)\cdots u_2'(k+\omega-1+h\omega)]', \tag{4.6}$$
$$v_k(h) := [v'(k+h\omega)\,v'(k+1+h\omega)\cdots v'(k+\omega-1+h\omega)]', \tag{4.7}$$
$$y_k^1(h) := [y_1'(k+h\omega)\,y_1'(k+1+h\omega)\cdots y_1'(k+\omega-1+h\omega)]', \tag{4.8}$$
$$e_k^1(h) := [e_1'(k+h\omega)\,e_1'(k+1+h\omega)\cdots e_1'(k+\omega-1+h\omega)]', \tag{4.9}$$
$$w_k^1(h) := [w_1'(k+h\omega)\,w_1'(k+1+h\omega)\cdots w_1'(k+\omega-1+h\omega)]', \tag{4.10}$$
$$y_k^2(h) := [y_2'(k+h\omega)\,y_2'(k+1+h\omega)\cdots y_2'(k+\omega-1+h\omega)]', \tag{4.11}$$
$$e_k^2(h) := [e_2'(k+h\omega)\,e_2'(k+1+h\omega)\cdots e_2'(k+\omega-1+h\omega)]', \tag{4.12}$$
$$w_k^2(h) := [w_2'(k+h\omega)\,w_2'(k+1+h\omega)\cdots w_2'(k+\omega-1+h\omega)]' \tag{4.13}$$

it can be verified the existence of appropriate unimodular matrices U_a and U_b such that the following relations are satisfied:

$$\begin{bmatrix} u_k^1(h) \\ u_k^2(h) \end{bmatrix} = U_a v_k(h), \tag{4.14}$$

$$\begin{bmatrix} y_k^1(h) \\ e_k^1(h) \end{bmatrix} = U_b w_k^1(h), \tag{4.15}$$

$$\begin{bmatrix} y_k^2(h) \\ e_k^2(h) \end{bmatrix} = U_a w_k^2(h). \tag{4.16}$$

Then, the associated transfer matrices $W_k^1(d)$ and $W_k^2(d)$ of Σ_{fb} at time k satisfy the following relations:

$$W_k^1(d) = U_b^{-1} \begin{bmatrix} W_k^{y_1u_1}(d) & W_k^{y_1u_2}(d) \\ W_k^{e_1u_1}(d) & W_k^{e_1u_2}(d) \end{bmatrix} U_a, \tag{4.17}$$

$$W_k^2(d) = U_a^{-1} \begin{bmatrix} W_k^{y_2u_1}(d) & W_k^{y_2u_2}(d) \\ W_k^{e_2u_1}(d) & W_k^{e_2u_2}(d) \end{bmatrix} U_a, \tag{4.18}$$

where $W_k^{y_iu_j}(d)$ and $W_k^{e_iu_j}(d)$ denote the associated transfer matrices at time k of the ω-periodic feedback system Σ_{fb} relating input $u_j(\cdot)$, $j=1,2$ with output $y_i(\cdot)$, $e_i(\cdot)$ $i=1,2$, respectively.

Denoting as

$$F_k^1(d) = P_k(d)\overline{A}_k(d) + Q_k(d)\overline{B}_k(d), \tag{4.19}$$
$$F_k^2(d) = A_k(d)\overline{P}_k(d) + B_k(d)\overline{Q}_k(d), \tag{4.20}$$

and arguing as in [23] it can be shown that

$$W_k^1(d) = U_b^{-1} \begin{bmatrix} \overline{B}_k(d) \\ \overline{A}_k(d) \end{bmatrix} (F_k^1(d))^{-1} [P_k(d) \quad -Q_k(d)]\, U_a, \tag{4.21}$$

$$W_k^2(d) \quad = \quad U_a^{-1}\begin{bmatrix}-\overline{Q}_k(d)\\ \overline{P}_k(d)\end{bmatrix}(F_k^2(d))^{-1}\,[B_k(d) \quad A_k(d)]\,U_a. \tag{4.22}$$

We are now in a position to prove the following main theorem.

Theorem 4.1 Problem 3.1 admits a solution if and only if $m \le \min(\omega p, \omega q)$.

Proof. Necessity. Under the hypothesis on reachability and observability at all times of the ω-periodic systems Σ and Σ_G, by Lemma 2.1 applied to Σ_{fb} it can be shown that the ω-periodic system Σ_{fb} is reachable at all times and observable through the outputs $w_1(\cdot)$ and $w_2(\cdot)$ at all times. Then (4.2) and (4.3) constitute a minimal realization of transfer matrix $W_k^1(d)$ and (4.2) and (4.4) constitute a minimal realization of transfer matrix $W_k^2(d)$. Moreover, for each time k, the nonunit invariant polynomials of the $(\omega p \times \omega p)$ polynomial matrix $F_k^1(d)$ are associated of the nonunit invariant polynomials of the $(\omega q \times \omega q)$ polynomial matrix $F_k^2(d)$ and both are associated of the nonunit invariant polynomials at time k of the ω-periodic feedback system Σ_{fb} [23]. This implies that the number m of the invariant polynomials at time k of the ω-periodic feedback system Σ_{fb} can not be larger than $m \le \min(\omega p, \omega q)$.

Sufficiency. As $A_k(d)$ and $B_k(d)$ are *rlp* and $\overline{A}_k(d)$ and $\overline{B}_k(d)$ are *rrp*, equations (4.19) and (4.20) can be solved for arbitrary $F_k^1(d)$ and $F_k^2(d)$. Hence, if $m \le \min(\omega p, \omega q)$, the $s_i(d), i = 1, \ldots, m$ can be assigned to Σ_{fb} as invariant polynomials choosing $F_k^1(d)$ and $F_k^2(d)$ as polynomial matrices whose nonunit invariant polynomial are associate (two polynomials are called associate if their ratio is a scalar [23]) of the $s_i(d), i = 1, \ldots, m$ and then to solve (4.19) or (4.20) with respect to the pairs $(P_k(d), Q_k(d))$ or $(\overline{P}_k(d), \overline{Q}_k(d))$ respectively. Moreover, as the invariant polynomials of Σ_{fb} are independent of k, the solutions of (4.19) and (4.20) can be found for arbitrary k.

For an arbitrary integer k, all the solutions $P_k(d)$ and $Q_k(d)$ of (4.19) are given by

$$[P_k(d) \quad Q_k(d)] = [F_k^1(d) \quad T_k(d)]\,U_k(d) \tag{4.23}$$

where $U_k(d)$ is the unimodular matrix given by

$$U_k(d) = \begin{bmatrix} G_k(d) & H_k(d) \\ -B_k(d) & A_k(d) \end{bmatrix},$$

$G_k(d)$ and $H_k(d)$ are polynomial matrices such that

$$G_k(d)\overline{A}_k(d) + H_k(d)\overline{B}_k(d) = I_{\omega p},$$

and $T_k(d)$ is an arbitrary polynomial matrix. For the solution (4.23) be adequate for Problem 3.1, $T_k(d)$ must be such that

4a) $P_k(d)$ and $Q_k(d)$ are *rlp*, 4b) $P_k^{-1}(d)\,Q_k(d) \in \chi(p, q, \omega)$.

Analogously, for an arbitrary integer k, all the solutions of (4.20) are given by

$$\begin{bmatrix} \overline{P}_k(d) \\ \overline{Q}_k(d) \end{bmatrix} = \overline{U}_k(d)\begin{bmatrix} F_k^2(d) \\ \overline{T}_k(d) \end{bmatrix}, \tag{4.24}$$

where $\overline{U}_k(d)$ is the unimodular matrix given by

$$\overline{U}_k(d) = \begin{bmatrix} \overline{G}_k(d) & -\overline{B}_k(d) \\ \overline{H}_k(d) & \overline{A}_k(d) \end{bmatrix},$$

$\overline{G}_k(d)$ and $\overline{H}_k(d)$ are polynomial matrices such that

$$A_k(d)\,\overline{G}_k(d) + B_k(d)\,\overline{H}_k(d) = I_{\omega q},$$

and $\overline{T}_k(d)$ is an arbitrary polynomial matrix. For the solution (4.24) be adequate to Problem 3.1, $\overline{T}_k(d)$ must be such that:

4$\overline{\text{a}}$) $\overline{P}_k(d)$ and $\overline{Q}_k(d)$ are rrp, 4$\overline{\text{b}}$) $\overline{Q}_k(d)\overline{P}_k(d)^{-1} \in \chi(p,q,\omega)$.

It remains to show that matrices and $T_k(d)$ and $\overline{T}_k(d)$ such that the pairs $(P_k(d), Q_k(d))$ and $(\overline{P}_k(d), \overline{Q}_k(d))$ satisfy properties 4a, 4b and 4$\overline{\text{a}}$, 4$\overline{\text{b}}$ respectively, can always be found.

With reference to solutions (4.24), matrix $\overline{T}_k(d)$ can be found as follows. By the causality of Σ, $A_k(0)$ is non singular, so that left primeness of $A_k(d)$ and $B_k(d)$ implies left primeness of $A_k(d)$ and $dB_k(d)$. This in turn implies that the equation

$$A_k(d)\,\overline{P}^a_k(d) + dB_k(d)\,\overline{Q}^a_k(d) = F^2_k(d), \tag{4.25}$$

can be solved with respect to $\overline{P}^a_k(d)$ and $\overline{Q}^a_k(d)$ for any $F^2_k(d)$. For an arbitrary integer k the general solution of (4.25) is

$$\begin{bmatrix} \overline{P}^a_k(d) \\ \overline{Q}^a_k(d) \end{bmatrix} = \overline{U}^a_k(d) \begin{bmatrix} F^2_k(d) \\ \overline{T}^a_k(d) \end{bmatrix}, \tag{4.26}$$

where $\overline{U}^a_k(d)$ is a unimodular matrix given by

$$\overline{U}^a_k(d) = \begin{bmatrix} \overline{G}^a_k(d) & -d\overline{B}_k(d) \\ \overline{H}^a_k(d) & \overline{A}_k(d) \end{bmatrix}$$

$\overline{G}^a_k(d)$ and $\overline{H}^a_k(d)$ are polynomial matrices satisfying

$$A_k(d)\,\overline{G}^a_k(d) + dB_k(d)\,\overline{H}^a_k(d) = I_{\omega q}, \tag{4.27}$$

and $\overline{T}^a_k(d)$ is an arbitrary polynomial matrix. The unimodularity of $\overline{U}^a_k(d)$ implies that if $\overline{T}^a_k(d)$ is chosen right coprime with $F^2_k(d)$, also $\overline{P}^a_k(d)$ and $\overline{Q}^a_k(d)$ are right coprime. Taking into account that by the causality of Σ_{fb} and (4.25), $\overline{P}^a_k(0)$ is nonsingular, one has that also $\overline{P}^a_k(d)$ and $d\overline{Q}^a_k(d)$ are right coprime, so that by putting $\overline{G}_k(d) = \overline{G}^a_k(d), \overline{H}_k(d) = d\overline{H}^a_k(d), \overline{T}_k(d) = d\overline{T}^a_k(d)$ one has that the pair $(\overline{P}_k(d), \overline{Q}_k(d))$ given by

$$\overline{P}_k(d) = \overline{P}^a_k(d) = \overline{G}_k(d)\,F^2_k(d) - \overline{B}_k(d)\,\overline{T}_k(d), \tag{4.28}$$
$$\overline{Q}_k(d) = d\overline{Q}^a_k(d) = \overline{H}_k(d)\,F^2_k(d) + \overline{A}_k(d)\,\overline{T}_k(d), \tag{4.29}$$

defines a class of solutions (4.24) satisfying 4$\overline{\text{a}}$ and 4$\overline{\text{b}}$ (see Remark 2.2).

By arguing in a similar way, one has that the pair

$$P_k(d) = F_k^1(d)\, G_k(d) - T_k(d)\, B_k(d), \qquad (4.30)$$

$$Q_k(d) = F_k^1(d)\, H_k(d) + T_k(d)\, A_k(d), \qquad (4.31)$$

where $G_k(d) = G_k^a(d)$, $H_k(d) = dH_k^a(d)$ with $G_k^a(d)$ and $H_k^a(d)$ such that

$$G_k^a(d)\,\overline{A}_k(d) + H_k^a(d)\, d\overline{B}_k(d) = I_{\omega p},$$

and where $T_k(d) = dT_k^a(d)$, $T_k^a(d)$ being any polynomial matrix left prime with $F_k^1(d)$, defines a class of solutions of (4.19) satisfying 4a and 4b (see Remark 2.2). Hence, under the assumption $m \leq \min(\omega p, \omega q)$, the existence of solutions of Problem 3.1 has been constructively established. □

5. CONCLUSIONS

In this paper the pole placement problem for linear discrete-time periodic systems has been considered. This problem has been formulated in the more general context of the invariant polynomial assignment, whence pole placement follows as a particular case. Necessary and sufficient conditions for problem solvability have been given in Theorem 3.1. The sufficiency proof of this theorem gives a parameterization of all controllers solving the problem in terms of causal transfer matrices that are minimally realizable with a periodic state-space representation. The proof has been performed in two steps. First, the set of all admissible solutions has been formally defined, then a procedure to effectively construct an admissible solution has been provided.

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