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BALANCING OF SYSTEMS WITH PERIODIC JUMPS[1]

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In this paper we study the balancing and model-reduction of linear systems with discrete jumps at periodic time instants. These systems arise in the study of linear systems with sampled data control and filtering problems. We study the balancing for the case of fixed and infinite intervals. We show that the system balancing can be used to obtain a reduced order-model of the system with the properties of the original system. An example is provided to illustrate the procedure.

1. INTRODUCTION

In the recent past considerable interest has been shown in the design of controllers for continuous systems with sampled data measurements. This is largely due to the ease of implementation of the controller with the use of a digital computer. The emphasis has been on direct design of the digital controllers without resorting to discretization of the plant or the controller to take into consideration the intersample behavior of the system [1, 2, 14]. The study of these problems has given rise to linear systems with finite discrete jumps at periodic instants [14]. The systems with jumps arise naturally in H_2 and H_∞ optimization problems. These systems have the properties of both continuous and discrete systems and in fact both the continuous time and discrete time systems can be derived as special cases [14].

On the other hand the notion of balanced realizations of systems was introduced by Moore in [6] and studied extensively in [6, 15]. The balanced systems were used to study the concept of model reduction by [6] and connections to the Hankel-norm approximations were drawn in [3]. It was shown that the internal realization of the system in certain co-ordinates has minimum sensitivity with regard to parameter variations in [7, 16]. Applications to filter design with minimum sensitivity were also explored. The concept was extended to time varying systems in [9] and [17]. In fact [17] explores the possibility of input-output balancing for various gramians and gives necessary and sufficient conditions for the balancing transformations to exist in terms of the parameters of the system. While [9] explored the balanced realizations for uniform realizations, [17] considered balancing over a general class of intervals.

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These include the Fixed Interval Balancing (FIB) and Infinite Interval Balancing (IIB). The main idea of balancing is that the "degree" of reachability (or controllability) and observability (or constructability) of states of the system is quantified in some manner. Thus the effect of the individual states on the input-output map is quantified. By eliminating the least effective states a "good" approximation of a lower order is obtained. Therefore, this leads to a method of model reduction based on truncating the least reachable and observable states of the system which describe the dynamics. The balanced co-ordinates are chosen to make the reachability and observability gramians equal and diagonal. When the system is time varying these transformations are also time varying [9].

In this paper we extend the notion of balancing to linear continuous systems with periodic discrete jumps and study the notion of model reduction for these systems. In Section 2 we present some preliminary results on linear systems with periodic finite discrete jumps. In Section 3 we present the conditions under which the balancing co-ordinates for the system exist and study their properties. In Section 4 we provide an example to illustrate the procedure. Finally, Section 5 concludes the paper.

2. PRELIMINARIES AND BACKGROUND

In this section we introduce the linear system with discrete jumps and present some preliminary results related to these systems. The linear system with discrete jumps, represented by Σ, is described by the following equations:

$$\begin{aligned} \dot{x}(t) &= A(t)\,x(t) + B(t)\,u(t), \quad t \neq kh && (2.1a)\\ x(kh) &= A_d[k]\,x(kh^-) + B_d[k]\,u(kh), && (2.1b)\\ y(t) &= C(t)\,x(t), \quad t \neq kh && (2.1c)\\ y(kh) &= C_d[k]\,x(kh^-) && (2.1d) \end{aligned}$$

where the matrices $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ are of compatible dimensions with the state and the input vectors are piecewise continuous and bounded. The matrices $A_d[\cdot]$, $B_d[\cdot]$, $C_d[\cdot]$ are assumed to be bounded. The discrete jumps occur at periodic intervals of period h. The solution $x(t)$ is unique and piecewise right continuous. That is, the solution is such that $\lim_{t\to kh^+} x(t) = x(kh)$ and $x(t)$ may be left discontinuous. The unique solution $x(t)$ to the *unforced system*, i.e., when the input $u(t) \equiv 0$, $\forall t$ is given by

$$x(t) = \Phi(t,s)\,x(s), \quad t \geq s \tag{2.2}$$

where $\Phi(t,s)$ is the state-transition matrix which is piecewise continuous with possible discontinuities at $t = kh$. The state-transition matrix satisfies the following conditions [14]:

$$\begin{aligned} \frac{\partial}{\partial t}\Phi(t,s) &= A(t)\,\Phi(t,s), \quad t > s, \quad t \neq kh, && (2.3a)\\ \Phi(kh,s) &= A_d[k]\,\Phi(kh^-,s), \quad kh > s && (2.3b)\\ \Phi(s,s) &= I. && (2.3c) \end{aligned}$$

The internal stability of the system is expressed in terms of the exponential stability of the state-transition matrix and is defined as follows:

Definition 2.1. The system Σ described by equations (2.1a) – (2.1d) is said to be exponentially uniformly stable if there are positive constants c_1, c_2 such that

$$||\Phi(t,s)|| \leq c_1 e^{-c_2(t-s)}, \quad t \geq s. \tag{2.4}$$

The notions of reachability and observability for these systems are standard. That is, the reachable subspace in the time interval $[t_0, t_f]$ is the subspace of all reachable states with a finite energy input $u(t), t \in [t_0, t_f]$ and a finite energy sequence $u(kh), kh \in [t_0, t_f]$. In other words, the reachability subspace at time t_f, denoted by $X_r(t_f)$ is given by

$$X_r(t_f) = \{x(t_f) : x(t_0) = 0,\ u(\cdot) \in \mathcal{L}_2[t_0, t_f],\ u(kh) \in l_2\} \tag{2.5}$$

where $\mathcal{L}_2$ denotes the space of all square integrable functions and l_2 denotes the space of all square summable sequences. The unobservable subspace at time t_0 for an observation in the time interval $[t_0, t_f]$ is the subspace of all states such that the output is identically zero in $[t_0, t_f]$ when the initial state belongs to the unobservable subspace with $u(t) \equiv 0$. Let $\overline{X}_0(t_0)$ denote the unobservable subspace. Then,

$$\overline{X}_0(t_0) = \{x(t_0) : y(t) \equiv 0,\ u(t) \equiv 0,\ t \in [t_0, t_f]\}. \tag{2.6}$$

We now define the reachability and observability gramians which characterize the reachable and observable subspaces for these systems. The reachability gramian in the interval $[t_0, t]$ is given by the positive-semidefinite matrix $Q(t_0, t)$ which satisfies the following differential Lyapunov equation with jumps.

$$\frac{\partial}{\partial t} Q(t_0, t) = A(t)\, Q(t_0, t) + Q(t_0, t)\, A^T(t) + B(t)\, B^T(t), \quad t \neq kh, \tag{2.7}$$

$$Q(t_0, kh) = A_d[k]\, Q(t_0, kh^-) A_d^T[k] + B_d[k]\, B_d^T[k]. \tag{2.8}$$

The observability gramian in the interval $[t, t_f]$ is a positive-semidefinite matrix $P(t, t_f)$ satisfying the following equations

$$-\frac{\partial}{\partial t} P(t, t_f) = A^T(t)\, B(t, t_f) + P(t, t_f)\, A(t) + C^T(t)\, C(t), \quad t \neq kh, \tag{2.9}$$

$$P(kh^-, t_f) = A_d^T[k]\, P(kh, t_f)\, A_d[k] + C_d^T[k]\, C_d[k]. \tag{2.10}$$

The "balancing" co-ordinates of the realization are the co-ordinates for which the reachability and observability gramians are equal and diagonal over some time interval. We will elaborate on the nature of the time interval later in the paper.

The balancing of a system is achieved by a transformation of the state vector. In the case of time-varying systems, the transformation is also time-varying. Now, consider the transformation of the state vector. Let $T(t)$ be a non-singular, piecewise right continuous matrix with bounded jumps at periodic instants. Then, under the transformation

$$x(t) = T(t)\,\widehat{x}(t), \tag{2.11}$$

the state equations of (2.1a) – (2.1d) are transformed as follows: We will represent the transformed system by $\widehat{\Sigma}$.

$$\begin{aligned} \dot{\widehat{x}}(t) &= \widehat{A}(t)\,\widehat{x}(t) + \widehat{B}(t)\,u(t), \quad t \neq kh & (2.12a)\\ \widehat{x}(kh) &= \widehat{A}_d[k]\,\widehat{x}(kh^-) + \widehat{B}_d[k]\,u(kh), & (2.12b)\\ y(t) &= \widehat{C}(t)\,\widehat{x}(t), \quad t \neq kh & (2.12c)\\ y(kh) &= \widehat{C}_d[k]\,\widehat{x}(kh^-), & (2.12d) \end{aligned}$$

where the transformed system matrices are given as

$$\begin{aligned} \widehat{A}(t) &= T^{-1}(t)\left[A(t)\,T(t) - \dot{T}(t)\right], & (2.13a)\\ \widehat{B}(t) &= T^{-1}(t)\,B(t), & (2.13b)\\ \widehat{C}(t) &= C(t)\,T(t), \quad t \neq kh; & (2.13c)\\ \widehat{A}_d[k] &= T^{-1}(kh)\,A_d[k]\,T(kh^-), & (2.13d)\\ \widehat{B}_d[k] &= T^{-1}(kh)\,B_d[k], & (2.13e)\\ \widehat{C}_d[k] &= C_d[k]\,T(kh^-). & (2.13f) \end{aligned}$$

Note that the transformation in the continuous part of the system is identical to the transformation for continuous systems and the discrete part to the discrete systems. The state-transition matrix is transformed to

$$\widehat{\Phi}(t,s) = T^{-1}(t)\,\Phi(t,s)\,T(s), \quad \forall\, t,\, s. \tag{2.14}$$

Transformations that preserve internal stability of the systems are called Lyapunov transformations. Clearly, the requirements on the transformations are that the inverse and the derivative at all times except kh be well defined, continuous and bounded. This leads to the definition of equivalence of systems. We formalize the notion in the following definition.

Definition 2.2. The systems Σ and $\widehat{\Sigma}$ defined by equations (2.1a) – (2.1d) and (2.12a) – (2.12d) respectively are topologically equivalent if they can be transformed into the other by the transformation $T(t)$, where $T(t)$ is Lyapunov. That is, $T(t)$, $T^{-1}(t)$ and $\dot{T}(t)$, when it exists, are bounded.

We note that the derivative of $T(t)$ is not defined at $t = kh$. However, it is well defined in the neighborhood of kh. This ensures that the matrices $\widehat{A}(t)$, $\widehat{B}(t)$, $\widehat{C}(t)$ are well defined in the neighborhood of kh. Further, the boundedness of the transformed matrices follows from the boundedness of $T(kh^-)$ and $T(kh^+)$. It is easy to see that the exponential stability remains invariant under Lyapunov transformations. Further, it is easy to verify that the reachability and observability gramians are transformed to (see Appendix A.1.)

$$\widehat{Q}(t_0,t_f) = T^{-1}(t_f)\,Q(t_0,t_f)\,T^{-T}(t_f), \tag{2.15}$$

$$\widehat{P}(t_0,t_f) = T^T(t_0)\,P(t_0,t_f)\,T(t_0). \tag{2.16}$$

Clearly, the input-output gramian $P(t,t_f)Q(t_0,t)$ with $t \in [t_0,t_f]$, undergoes a similarity transformation given by

$$\widehat{P}(t,t_f)\,\widehat{Q}(t_0,t) = T^T(t)\,B(t,t_f)\,Q(t_0,t)\,T^{-T}(t). \tag{2.17}$$

We now proceed to study the conditions under which the reachability and observability gramians can be diagonalized by Lyapunov transformations. We say that the system Σ is in balanced co-ordinates if $Q(t_0,t)$ and $P(t,t_f)$ are diagonal and equal.

3. BALANCING AND MODEL REDUCTION

In this section we establish the conditions for the existence of a balancing Lyapunov transformation. Further, we show how the system can be reduced to obtain a lower order model for the input-output description of the system. In particular, we will explore the stability properties of the reduced order model.

3.1. Finite interval balancing

We will consider the input-output balancing with respect to the reachability and observability gramians defined over specific time intervals. We first consider the fixed interval balancing (FIB) [17]. Here, no assumptions need to be made with regard to the internally stability of the system. Consider the time intervals for the reachability and observability gramians with fixed initial and final time periods i.e., t_0 and t_f are fixed and given. We want to find a Lyapunov transformation so that $Q(t_0,t)$ and $P(t,t_f)$ are equal and diagonal for $t \in [t_0,t_f]$. To simplify the notation we will denote $Q(t_0,t)$ and $P(t,t_f)$ as $Q(t)$, $P(t)$ respectively. We assume that the system is "totally" reachable and observable in the time intervals of definition [12]. Therefore, $Q(t)$ and $P(t)$ are positive-definite in $t \in [t_0,t_f]$. We now characterize the nature of balancing in terms of the gramians.

Theorem 3.1. The realization of the system Σ described by equations (2.1a)–(2.1d) is FIB over $[t_0,t_f]$ iff the following equations are satisfied:

$$Q(t) = P(t) = \Lambda(t), \tag{3.1}$$

$$\Lambda(t_0) = \Lambda(t_f) = 0. \tag{3.2}$$

Further, the matrix $\Lambda(t)$ is diagonal and satisfies

$$\dot{\Lambda}(t) = A(t)\,\Lambda(t) + \Lambda(t)\,A^T(t) + B(t)\,B^T(t),\ t \neq kh, \tag{3.3a}$$

$$\begin{aligned}
\Lambda(kh) &= A_d[k]\,\Lambda(kh^-)A_d^T[k] + B_d[k]\,B_d^T[k], & (3.3b)\\
-\dot{\Lambda}(t) &= A^T(t)\,\Lambda(t) + \Lambda(t)\,A(t) + C^T(t)\,C(t),\ t \neq kh, & (3.3c)\\
\Lambda(kh^-) &= A_d^T[k]\,\Lambda(kh)\,A_d[k] + C_d^T[k]\,C_d[k]. & (3.3d)
\end{aligned}$$

Proof. Necessity: Let the system Σ be FIB over $[t_0, t_f]$. Then, $Q(t)$ and $P(t)$ are diagonal and equal. Furthermore, $Q(t_0) = 0$ and $P(t_f) = 0$.

Sufficiency: Since the solution to (2.1a) – (2.1b) is unique, the gramians are unique and hence, the system is FIB in $[t_0, t_f]$. □

We now give the balancing transformation that can be constructed from the system. The existential conditions of the transformation are derived from the construction of the balancing transformation. This approach is on the lines of [9]. The drawback of this approach is that the conditions are obtained in terms of the gramians and not in terms of the system parameters. While [17] gives conditions in terms of the system parameters, the assumptions on the system matrices are also very strong. However, we do not assume analyticity of the system matrices here. We first note the following.

Lemma 3.1. Let Σ described by (2.1a) – (2.1d) be totally reachable and observable in $t \in [t_0, t_f]$. Then, $Q(t)$ and $P(t)$ are Lyapunov.

Proof. We prove that $Q(t)$ is Lyapunov in $t \in [t_0, t_f]$. The proof for $P(t)$ follows similarly. Consider the equations (2.7) and (2.8) with jumps. A closed form solution for $Q(t)$ is given by

$$\begin{aligned}
Q(t) &= \Phi(t,kh)\,Q(kh)\,\Phi^T(t,kh) + \int_{kh}^{t} \Phi(t,\tau)\,B(\tau)\,B^T(\tau)\,\Phi(t,\tau)\,\mathrm{d}\tau, & (3.4)\\
Q(kh) &= A_d[k]\,Q(kh^-)\,A_d^T[k] + B_d[k]\,B_d^T[k], & (3.5)
\end{aligned}$$

with $Q(t_0) = 0$. Clearly, the matrix $Q(t)$ is bounded in the finite interval with finite jumps at time periods $t = kh$. Further, since the system is assumed to be totally reachable in $[t_0, t_f]$, the inverse exists and is also bounded. The derivative of $Q(t)$ for $t \neq kh$ is given by (2.7) and is bounded whenever $Q(t)$ is bounded. Therefore, $Q(t)$ is Lyapunov. □

Now, consider the transformation $T(t) = Q^{\frac{1}{2}}(t)$. Then,

$$\begin{aligned}
\overline{Q}(t) &= I, & (3.6a)\\
\overline{P}(t) &= S(t) & (3.6b)
\end{aligned}$$

where $S(t)$ is given by

$$S(t) = Q^{\frac{T}{2}}(t)\,B(t)\,Q^{\frac{1}{2}}(t). \quad (3.7)$$

This transformation is referred to as the "preinput normalizing transformation" [17]. Note that $S(t)$ is symmetric and positive definite on the interval $[t_0, t_f]$.

Clearly, $S(t)$ is Lyapunov. It is well known that there may not be a eigenvalue decomposition of $S(t)$ with the unitary matrix being Lyapunov [9]. Moreover, if there are no conditions on the system matrices (Analyticity), then the conditions for the existence of such decomposition are obtained in terms of the gramians of the system and not in terms of the parameters of the system [17, 9]. We therefore, provide the conditions for the existence of a decomposition with Lyapunov factors. Let $S(t)$ satisfy the following properties:

Property I. The eigenvalues of $S(t)$, $\sigma_i^2(t), \sigma_j^2(t)$, $i \neq j$ cross only at isolated points on the interval. Furthermore, the set of points, denoted by Ω, does not contain the points $kh \in [t_0, t_f]$.

Property II. The eigenvalues $\sigma_i^2(t), \sigma_j^2(t)$, $i \neq j$ do not have common derivatives on the set Ω.

Property III. $S(t)$ has continuous second derivatives on the neighborhood of all $t \in \Omega$.

Remark. The conditions on $S(t)$ are sufficient only. We note that the continuity of $U(t)$ and $\dot{U}(t)$ is essential only for $t \neq kh$ and discontinuities are allowed at $t = kh$. Therefore, the set Ω is modified to accommodate for the discontinuities at $t = kh$.

If $S(t)$ satisfies the above conditions, then by Lemma A.2.2, there is a unitary and $U(t)$ and a $\Lambda(t)$ so that,

$$S(t) = U(t)\,\Lambda^2(t)\,U^T(t) \tag{3.8}$$

where $U(t)$ and $\Lambda(t)$ are Lyapunov. Then, we use the transformation $\widehat{T}(t) = U(t)\,\Lambda^{-\frac{1}{2}}$. Clearly, $\widehat{T}(t)$ is Lyapunov as it is the product of two Lyapunov matrices by Lemma A.2.1. Therefore,

$$\begin{aligned} \widehat{Q}(t) &= \Lambda^{\frac{1}{2}}(t)\,U^T(t)\,I\,U(t)\,\Lambda^{\frac{1}{2}}(t), \\ \widehat{P}(t) &= \Lambda^{-\frac{1}{2}}(t)\,U^T(t)\,U(t)\,\Lambda^2(t)\,U^T(t)\,U(t)\,\Lambda^{-\frac{1}{2}}(t), \end{aligned} \tag{3.9}$$

which gives

$$\widehat{Q}(t) = \widehat{P}(t) = \Lambda(t). \tag{3.10}$$

The overall transformation to transform the system Σ to balanced co-ordinates is given by

$$T(t) = U(t)\,\Lambda^{\frac{1}{2}}(t)\,Q^{\frac{1}{2}}(t). \tag{3.11}$$

We now study some of the properties of the FIB realizations. We shall henceforth assume that the system is in balanced co-ordinates. Since the system is essentially a continuous system with discrete behavior at periodic times, we expect the balanced realization to exhibit properties of both continuous and discrete systems. This is indeed the case as we show in our next result. It is well known that in the case of continuous systems, the matrix $A(t)$ is negative-definite in the interval $[t_0, t_f]$ [17]. We see that this property extends to the continuous part of the system. Furthermore, the discrete part of the system also satisfies the contractive property [15].

Lemma 3.2. Let the system Σ be FIB in $[t_0, t_f]$. Then, $A(t)$ is non-positive definite in the interval $[t_0, t_f]$, $t \neq kh$. Furthermore, $A_d[k]$ is contractive, i.e.,

$$||A_d[k]|| \leq 1. \tag{3.12}$$

Proof. Consider equations (3.3a)–(3.3b). Adding (3.3a) and (3.3b) we have

$$A_s(t)\,\Lambda(t) + \Lambda(t)\,A_s(t) = -(C^TC + BB^T)\,(t) \tag{3.13}$$

where $A_s(t) = A(t) + A^T(t)$ with $\Lambda(t) > 0$ for all $t \in [t_0, t_f]$. Therefore,

$$A_s(t) = -\int_0^{\infty} e^{-\Lambda(t)\tau}(C^TC + BB^T)\,(t)e^{-\Lambda(t)\tau}\,\mathrm{d}\tau, \quad t \neq kh. \tag{3.14}$$

Hence, $A_s(t)$ is negative semi-definite. Now, substituting (3.3d) in (3.3c) we have,

$$\Lambda(kh) = A_dA_d^T\Lambda(kh)\,A_dA_d^T + A_dC_d^TC_dA_d^T + B_dB_d^T. \tag{3.15}$$

Following the argument in [15] (Theorem 3.1), we have

$$\lambda_{\max}(A_dA_d^T[k]) \leq 1 \tag{3.16}$$

and therefore, $||A_d[k]|| < 1$. □

Note that the system with jumps possesses the properties of the continuous and discrete time systems. That is, the continuous part satisfies the property of the continuous system and the discrete part, that of the discrete system. As a consequence of the above Lemma we have the following result:

Lemma 3.3. Let the system Σ be FIB in $[t_0, t_f]$. Then, the system is dissipative in the time interval. That is, $||x(t)||$ is non-increasing for $t \neq kh$ and $||x(kh)|| \leq ||x(kh^-)||$ for $u \equiv 0$.

Proof. Let $f(t) = ||x(t)||^2$. From (2.2) we have

$$x(t) = \Phi(t, t_0)\,x(t_0). \tag{3.17}$$

Therefore,

$$f(t) = x^T(t_0)\,\Phi^T(t, t_0)\,\Phi(t, t_0)\,x(t_0), \quad t \in [t_0, t_f]. \tag{3.18}$$

Now,

$$\begin{aligned} \dot{f}(t) &= x^T(t_0)\,\dot{\Phi}^T(t, t_0)\,\Phi(t, t_0)\,x(t_0) + x^T(t_0)\,\Phi^T(t, t_0)\dot{\Phi}(t, t_0)\,x(t_0), \quad t \neq kh \\ &= x^T(t)\,A^T(t)\,x(t) + x^T(t)\,A(t)\,x(t) \;\leq\; 0. \end{aligned} \tag{3.19}$$

Therefore, for $t \neq kh$, $||x(t)||$ is non-increasing. Now, consider $f(kh)$.

$$\begin{aligned} f(kh) &= x^T(t_0)\,\Phi^T(kh, t_0)\,\Phi(kh, t_0)\,x(t_0) \\ &= x^T(kh^-)\,A_d^T[k]A_d[k]\,x(kh^-) \;\leq\; f(kh^-). \end{aligned} \tag{3.20}$$

□

Therefore, when the system is in balanced co-ordinates the state is *dissipative* in the interval $[t_0, t_f]$ as it is the case with both the continuous and discrete systems.

We now explore the notion of model reduction for the jump systems over the finite interval. It is well known that the reduced order model is dissipative in both the continuous and discrete time systems. We show that these properties carry over. However, it is to be noted that the lower order model may not be balanced as the discrete part of the system does not result in a balanced reduced order system.

Lemma 3.4. Let the system Σ be FIB in $[t_0, t_f]$. Let the system be partitioned as $x = [\,x_1^T \quad x_2^T\,]^T$ and the system matrices be partitioned conformably. Then, the reduced order model described by

$$\begin{aligned}
\dot{x}_1(t) &= A_{11}(t)\,x(t) + B_1(t)\,u(t), \quad t \neq kh && (3.22a)\\
x_1(kh) &= A_{11d}[k]\,x_1(kh^-) + B_{1d}[k]\,u(kh), && (3.22b)\\
y(t) &= C_1(t)\,x_1(t), \quad t \neq kh && (3.22c)\\
y(kh) &= C_{1d}[k]\,x_1(kh^-) && (3.22d)
\end{aligned}$$

satisfies the equations

$$\begin{aligned}
\dot{\Lambda}_1(t) &= A_{11}(t)\,\Lambda_1(t) + \Lambda_1(t)\,A_{11}^T(t) + B_1(t)\,B_1^T(t), \quad t \neq kh, && (3.23a)\\
\Lambda_1(kh) &= A_{11d}[k]\,\Lambda_1(kh^-)\,A_{11d}^T[k] + A_{12d}[k]\,\Lambda_2(kh^-)\,A_{12d}^T[k] &&\\
&\quad + B_{1d}[k]\,B_{1d}^T[k], && (3.23b)\\
-\dot{\Lambda}_1(t) &= A_{11}^T(t)\,\Lambda_1(t) + \Lambda_1(t)\,A_{11}(t) + C_1^T(t)\,C_1(t), \quad t \neq kh, && (3.23c)\\
\Lambda_1(kh^-) &= A_{11d}^T[k]\,\Lambda_1(kh)\,A_{11d}[k] + A_{21d}^T[k]\,\Lambda_1(kh)\,A_{21d}[k] &&\\
&\quad + C_{1d}^T[k]\,C_{1d}[k] && (3.23d)
\end{aligned}$$

where

$$\Lambda(t) = \begin{bmatrix} \Lambda_1(t) & 0 \\ 0 & \Lambda_2(t) \end{bmatrix}. \tag{3.24}$$

Proof. Follows directly from substitution and simplification. □

Clearly, the system is not balanced as the discrete portion of the system is not balanced. However, the continuous part of the system is balanced and it is clear that the subsystem has the non-positive property for $A_{11}(t)$. The system is "approximately" balanced if $\Lambda_1 \gg \Lambda_2$. Furthermore, it is easy to verify that the contractive property extends to the discrete part of the system. We show this in the following Lemma.

Lemma 3.5. For the reduced order system described by equations (2.12a) – (2.12d), A_{11d} is contractive in $[t_0, t_f]$.

Proof. Consider the partitioned equations (3.3b) and (3.3d). We have

$$\begin{aligned}\Lambda_1(kh) &= A_{11d}[k]\,\Lambda_1(kh^-)\,A_{11d}^T[k] + A_{12d}[k]\,\Lambda_2(kh^-)\,A_{12d}^T[k] \\ &\quad + B_{1d}B_{1d}^T[k], &(3.25a)\\ \Lambda_1(kh^-) &= A_{11d}^T[k]\,\Lambda_1(kh)\,A_{11d}[k] + A_{21d}^T[k]\,\Lambda_2(kh)\,A_{21d}[k] \\ &\quad + C_{1d}^T C_{1d}[k], &(3.25b)\\ \Lambda_2(kh^-) &= A_{12d}^T[k]\,\Lambda_1(kh)\,A_{12d}[k] + A_{22d}^T[k]\,\Lambda_2(kh)\,A_{22d}[k] \\ &\quad + C_{2d}^T C_{2d}[k]. &(3.25c)\end{aligned}$$

Substituting (3.25b) and (3.25c) in (3.25a), we have

$$\begin{aligned}\Lambda_1(kh) &= A_{11d}A_{11d}^T[k]\,\Lambda_1(kh)\,A_{11d}A_{11d}^T[k] + A_{12d}A_{22d}^T[k]\,\Lambda_1(kh)\,A_{22d}A_{12d}^T[k] \\ &\quad + A_{11d}A_{21d}^T[k]\,\Lambda_2(kh)\,A_{21d}A_{11d}^T[k] + A_{12d}A_{22d}^T[k]\,\Lambda_2(kh)\,A_{22d}A_{12d}^T[k] \\ &\quad + A_{11}C_1^T C_1 A_{11}^T[k] + A_{12}C_2^T C_2 A_{12}^T + B_1 B_1^T[k]. \qquad (3.26)\end{aligned}$$

By noting that the second and subsequent terms on the right hand side constitute a positive semi-definite matrix, the result follows by an argument similar to the one in Lemma 3.2. □

A direct consequence of the above result is that the reduced order system is once again *dissipative*. Therefore, in FIB co-ordinates, the system and all sub-systems are *dissipative*. We formalize this by the following Lemma.

Lemma 3.6. The reduced system described by equations (3.23a) – (3.23d) is dissipative in the time interval $[t_0, t_f]$.

In this section we have studied the properties of finite interval balancing. We have provide the conditions for the existence of the balancing transformation. Furthermore, we have shown that the dissipative property of the balanced system extends to the lower order system. We are now ready to study the case of infinite interval balancing.

3.2. Infinite interval balancing

In this section we explore the case of infinite interval balancing (IIB). In this case, we let $t_0 \to -\infty$ and $t_f \to +\infty$. We will denote the gramians by $P_\infty(t)$ and $Q_\infty(t)$. We do not provide a rigorous solution to the problem but provide an outline of the procedure highlighting the differences from the case of FIB.

The first question that needs to be resolved is the existence of the gramians in the asymptotic case when t_0 and t_f tend to infinity. In this case, a sufficient condition for the existence of the gramians is the square integrability of $\|\Phi(\tau, t)\, C(\tau)\|$ and $\|\Phi(t, \tau)\, B(\tau)\|$ and square summability of $\|\Phi(kh, t)\, C_d[k]\|$ and $\|\Phi(t, kh)\, B_d[k]\|$. This may be obtained by considering the closed form of the gramians (3.4) – (3.5). This in turn is guaranteed by the asymptotic stability of $\Phi(t, s)$ and the continuity and boundedness of the system matrices. Therefore, the next question that needs to be answered is the time period over which the balancing is done. If t varies in finite intervals, then the results of the previous section follow with minor modifications [17]. In the case that $t \to \pm\infty$ we need additional conditions which guarantee that the transformation in the limiting case is well defined and Lyapunov. However, we note that in the limit when $t \to \infty$, the gramian converges to that of the continuous time gramian. And therefore, it is reasonable to expect that the conditions of the continuous time case carry over to these systems too.

Therefore, we give the results without proofs. The sufficient condition for the existence of a well defined transformation in the limit as $t \to \pm\infty$ is the eigenvalues of the gramians be disjoint [17]. Before we present the results, we recall the notion of disjoint eigenvalues [17]. A matrix $M(t)$ is said to be disjoint iff for any two eigenvalues $\sigma_i(t)$ and $\sigma_j(t)$, $i \neq j$ of the matrix, there is a τ_{ij} and ϵ_{ij} so that

$$|\sigma_i(t) - \sigma_j(t)| > \epsilon_{ij} \quad \forall\, t > \tau_{ij}. \tag{3.27}$$

Lemma 3.7. Let the system Σ be such that $Q_\infty(t)$ and $P_\infty(t)$ are well defined. Then, the pre-input normalizing transformation is well defined and Lyapunov as $t \to \pm\infty$ if $Q_\infty(t)$ is disjoint.

Now, a similar characterization is necessary for the rest of the transformation to be well defined and Lyapunov in the limit as t tends to infinity.

Lemma 3.8. If the system Σ is asymptotically stable and $Q_\infty(t)$, $Q_\infty P_\infty(t)$ are disjoint, then the system is topologically equivalent to a balanced realization for $t \in (-\infty, \infty)$.

Now we characterize the balanced system in terms of the Lyapunov equations. We assume that the system is IIB. Therefore, we have

Theorem 3.2. Let the system Σ be IIB. Then, gramians $Q_\infty(t)$ and $P_\infty(t)$ satisfy the following conditions:

$$Q_\infty(t) = P_\infty(t) = \Lambda(t). \tag{3.28}$$

Further, the matrix $\Lambda(t)$ is diagonal and satisfies

$$\begin{aligned}
\dot{\Lambda}(t) &= A(t)\,\Lambda(t) + \Lambda(t)\,A^T(t) + B(t)\,B^T(t),\ t \neq kh, && \text{(3.29a)}\\
\Lambda(kh) &= A_d[k]\,\Lambda(kh^-)\,A_d^T[k] + B_d[k]\,B_d^T[k], && \text{(3.29b)}\\
-\dot{\Lambda}(t) &= A^T(t)\,\Lambda(t) + \Lambda(t)\,A(t) + C^T(t)\,C(t),\ t \neq kh, && \text{(3.29c)}\\
\Lambda(kh^-) &= A_d^T[k]\,\Lambda(kh)\,A_d[k] + C_d^T[k]\,C_d[k]. && \text{(3.29d)}
\end{aligned}$$

Note that the conditions are only necessary since for the balancing at infinity, additional conditions on the eigenvalues of the gramians are needed. When the system is in balanced co-ordinates, the notion of model reduction can be explored. In the case of infinite interval balancing, the properties of interest are the asymptotic stability of the reduced system. These results are currently under investigation.

4. ILLUSTRATIVE EXAMPLE

In this section we consider an example to illustrate the balancing of the system with jumps. The system is a double integrator with discrete jumps and is defined as follows:

$$
\begin{aligned}
\dot{x}(t) &= A(t)\,x(t) + B(t)\,u(t), \quad t \neq kh && (4.1a)\\
x(kh) &= A_d[k]\,x(kh^-) + B_d[k]\,u(kh), && (4.1b)\\
y(t) &= C(t)\,x(t), \quad t \neq kh && (4.1c)\\
y(kh) &= C_d[k]\,x(kh^-) && (4.1d)
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [\,1 \quad 0\,], && (4.2a)\\
A_d &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad B_d = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C_d = [\,1 \quad 1\,]. && (4.2b)
\end{aligned}
$$

The system has periodic jumps which occur at $t = k$, where k is an integer. The discrete jumps are characterized as rotations effected by A_d and translations effected by B_d. The balancing of this system is considered over the time interval $[0, 1]$. Here $\theta = \frac{\pi}{4}$ when $k = 1$. The computations were performed by using the *symbolic toolbox* of MATLAB [5]. The solution to the reachability and observability gramians are given by the following equations

$$
\begin{aligned}
Q(t) &= \begin{bmatrix} \frac{t^3}{3} & \frac{t^2}{2} \\ \frac{t^2}{2} & t \end{bmatrix}, \quad t \in [0, 1), && (4.3a)\\
Q(t) &= \begin{bmatrix} \frac{7}{6} & \frac{2}{3} \\ \frac{2}{3} & \frac{13}{6} \end{bmatrix}, \quad t = 1. && (4.3b)
\end{aligned}
$$

and

$$
\begin{aligned}
P(t) &= \begin{bmatrix} 2 - t & \frac{3}{2} - 2t + \frac{t^2}{2} \\ \frac{3}{2} - 2t + \frac{t^2}{2} & \frac{7}{3} - 3t + 2t^2 - \frac{t^3}{3} \end{bmatrix}, \quad t \in [0, 1), && (4.4a)\\
P(t) &= 0, \quad t = 1. && (4.4b)
\end{aligned}
$$

Clearly, the 'pre-input normalizing' transformation is given by

$$
\begin{aligned}
Q^{\frac{1}{2}}(t) &= \begin{bmatrix} \frac{t^{3/2}\sqrt{3}}{3} & 0 \\ \frac{\sqrt{t}\sqrt{3}}{2} & \frac{\sqrt{t}}{2} \end{bmatrix}, \quad t \in [0, 1), && (4.5a)\\
Q^{\frac{1}{2}}(t) &= \begin{bmatrix} 1.0801 & 0 \\ 0.6172 & 1.3363 \end{bmatrix}, \quad t = 1. && (4.5b)
\end{aligned}
$$

The observability gramian is transformed to $S(t)$ and is given by equation (3.7) as

$$S(t) = \begin{bmatrix} \frac{t^3}{6} - \frac{t^4}{12} - \frac{3t^2}{4} + \frac{7t}{4} & -\frac{\sqrt{3}t^2}{2} + \frac{\sqrt{3}t^3}{6} + \frac{7t\sqrt{3}}{12} \\ -\frac{\sqrt{3}t^2}{2} + \frac{\sqrt{3}t^3}{6} + \frac{7t\sqrt{3}}{12} & \frac{7t}{12} - \frac{3t^2}{4} + \frac{t^3}{2} - \frac{t^4}{12} \end{bmatrix}, \quad t \in [0,1) \qquad (4.6a)$$

$$S(t) = 0, \quad t = 1. \qquad (4.6b)$$

The singular value decomposition of $S(t)$ is given by equation (3.8) where $\Lambda^2(t)$ is given by

$$\Sigma^2(t) = \begin{bmatrix} \alpha + \frac{t\beta}{6} & 0 \\ 0 & \alpha - \frac{t\beta}{6} \end{bmatrix}, \quad t \in [0.1) \qquad (4.7a)$$

$$\Sigma^2(t) = 0, \quad t = 1 \qquad (4.7b)$$

where

$$\alpha = \frac{7t}{6} - \frac{3t^2}{4} + \frac{t^3}{3} - \frac{t^4}{12}, \qquad (4.8)$$

$$\beta = \sqrt{49 - 63t + 41t^2 - 18t^3 + 4t^4}. \qquad (4.9)$$

The singular vectors are given by

$$U(t) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}, \quad t \in [0,1), \qquad (4.10a)$$

$$U(t) = 0, \quad t = 1. \qquad (4.10b)$$

where

$$\phi = \sin^{-1}\sqrt{\frac{\frac{t^3}{2} - \frac{7t}{4} + \frac{t\beta}{2}}{t\beta}}. \qquad (4.11)$$

Therefore, the overall balancing transformation is given by equation (3.11) as

$$T(t) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \frac{t^{5/4}\sqrt{2}}{\sqrt[4]{3}\sqrt[4]{\gamma - 2\beta}} & 0 \\ \frac{\sqrt[4]{t}\,3^{3/4}}{\sqrt{2}\sqrt[4]{\gamma + 2\beta}} & \frac{\sqrt[4]{3t}}{\sqrt{2}\sqrt[4]{\gamma + 2\beta}} \end{bmatrix}, \quad t \in (0,1) \qquad (4.12)$$

where

$$\gamma = -14 + 9t - 4t^2 + t^3. \qquad (4.13)$$

It can be verified easily that the transformation $T(t)$ is Lyapunov in the interval $(0,1)$. And the system is in balanced co-ordinates with the reachability and observability gramians given by $\Lambda(t)$ in the interval $[0,1]$.

5. CONCLUSIONS

In this paper we have studied balancing for continuous systems with finite jumps at periodic time instants. We provide sufficient conditions for the existence of balancing transformations over a finite interval and showed that the properties of both the continuous and discrete time systems extend to these systems. We have considered the case of infinite interval balancing and provided the conditions under which balancing transformations exist by heuristic arguments. Formal proofs of these results are under investigation. The notion of model reduction and the properties of the reduced model are under investigation.

APPENDIX A.1

In this appendix we carry out the calculations for the transformation of the reachability gramian. The calculations for the observability gramian are similar. We will denote $Q(t_0, t)$ by $Q(t)$ to simplify the notation.

For the system $\widehat{\Sigma}$ described by equations (2.12a)–(2.12d) we have,

$$\begin{aligned} \dot{\widehat{Q}}(t) &= \widehat{A}(t)\,\widehat{Q}(t) + \widehat{Q}(t)\,\widehat{A}^T(t) + \widehat{B}(t)\,\widehat{B}^T(t), \quad t \neq kh, & \text{(A.1.1)} \\ \widehat{Q}(kh) &= \widehat{A}_d[k]\,\widehat{Q}(kh^-)\widehat{A}_d^T[k] + \widehat{B}_d[k]\,\widehat{B}_d^T[k]. & \text{(A.1.2)} \end{aligned}$$

simplifying (A.1.1),

$$\begin{aligned} \dot{\widehat{Q}}(t) &= T^{-1}\left[AT - \dot{T}\right]\widehat{Q}(t) + \widehat{Q}(t)\left[T^T A^T - \dot{T}^T\right]T^{-T} + T^{-1}BB^T T^{-T}, \\ T\dot{\widehat{Q}}(t)\,T^T &= \left[AT - \dot{T}\right]\widehat{Q}(t)T^T + T\widehat{Q}(t)\left[T^T A^T - \dot{T}^T\right] + BB^T, \\ \dot{\overline{T\widehat{Q}(t)\,T^T}} &= A\left(T\widehat{Q}T^T\right) + \left(T\widehat{Q}T^T\right)A^T + BB^T, & \text{(A.1.3)} \end{aligned}$$

we have,

$$Q(t) = T\widehat{Q}(t)T^T, \quad t \neq kh. \tag{A.1.4}$$

Similarly (A.1.2) is

$$\begin{aligned} \widehat{Q}(kh) &= \widehat{A}_d[k]\,\widehat{Q}(kh^-)\widehat{A}_d^T[k] + \widehat{B}_d[k]\,\widehat{B}_d^T[k], \\ &= T^{-1}(kh)\,A_d[k]\,T(kh^-)\widehat{Q}(kh^-)T^T(kh^-)\,A_d^T[k]\,T^{-T}(kh) \\ &\quad + T^{-1}(kh)\,B_d[k]\,B_d^T[k]\,T^{-T}(kh), & \text{(A.1.5)} \\ T(kh)\widehat{Q}(kh)T^T(kh) &= A_d[k]\,T(kh^-)\widehat{Q}(kh^-)T^T(kh^-)\,A_d^T[k] + B_d[k]\,B_d^T[k]. & \text{(A.1.6)} \end{aligned}$$

Therefore,

$$\widehat{Q}(t) = T^{-1}(t)\,Q(t)\,T^{-T}(t), \quad \forall t. \tag{A.1.7}$$

APPENDIX A.2

In this appendix we present some ancillary results on Lyapunov transformations. The proofs are straightforward and are modifications of the continuous time case.

Lemma A.2.1. Let $P(t)$ and $Q(t)$ be two Lyapunov matrices with periodic finite jumps. Then $P(t)\,Q(t)$ is Lyapunov.

Lemma A.2.2. Let $S(t)$ be a symmetric positive-definite and Lyapunov over a finite time interval with periodic finite jumps. Furthermore, let $S(t)$ satisfy the Properties I – III. Then, there is a eigenvalue decomposition

$$S(t) = U(t)\,\Lambda^2(t)\,U^T(t) \tag{A.2.8}$$

where $U(t)$ is unitary and $\Lambda(t)$ is positive-definite with $U(t)$ piecewise continuous with bounded discontinuities at periodic time instants. Furthermore, $U(t)$ and $\Lambda(t)$ are Lyapunov.

Proof. It is clear that the argument is exactly identical to the continuous time case when $t \neq kh$. For $t = kh$, we have that the jump in $S(t)$ is finite and hence $U(kh)$ and $\Lambda(kh)$ are Lyapunov. □

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