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PERIODIC TRANSFORMATIONS OF THE SAMPLE AVERAGE RECIPROCAL VALUE[1]

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The paper presents results on the convergence $\exp\left(i2\pi\frac{\alpha_n}{S_n}\right) \xrightarrow[n\to+\infty]{\mathcal{D}} \exp(i2\pi U)$, where S_n is a random walk with zero mean and a positive finite variance. The positive real numbers α_n fulfill $n^{-\frac{1}{2}}\alpha_n \to +\infty$ and U is a random variable uniformly distributed on the interval $[0,1)$. The asymptotics is derived in a more general setting for a sequence of random variables S_n that have either absolutely continuous distributions or distribution functions which satisfy a Berry–Esseen type condition.

1. INTRODUCTION

We are looking for sufficient conditions for

$$f\left(\frac{\alpha_n}{\overline{X}_n} I\left[\overline{X}_n \neq 0\right]\right) \xrightarrow[n\to+\infty]{\mathcal{D}} f(U_K)\,,$$

where f is a periodic function with the period K, $\overline{X}_n$ denotes the average of i.i.d. random variables with zero mean and a positive finite variance and the random variable U_K is uniformly distributed on the interval $[0,K)$. For example, that convergence is fulfilled if the random variables possess bounded density and $\alpha_n\sqrt{n} \to +\infty$. But there are three other cases summed up in Corollary 3.1.

The research is motivated by the following problem in the robust estimation theory. Let X_i be i.i.d. random variables and ψ be a given function. We consider the function $f(t) = \mathsf{E}\,\psi(X-t)$ and assume unique solution θ_0 of $f(\theta_0) = 0$. The point θ_0 is the unknown location parameter of the random sample and we estimate it by the M-estimator $\hat{\theta}_n$; i. e. fulfilling the equation $\sum_{i=1}^n \psi(X_i - \hat{\theta}_n) = 0$. The difference $\sqrt{n}(\hat{\theta}_n - \theta_0)$ can be expressed as a sum of i.i.d. random variables $\frac{1}{\sqrt{n}}\sum_{i=1}^n \psi(X_i - \theta_0)$ with remainder of $o_P(n^{-\frac{1}{2}})$, typically $O_P(n^{-1})$, see for that e. g. [2], [3] or [4]. The properties of the M-estimator $\hat{\theta}_n$ are closely related to the behaviour of $f(\hat{\theta}_n)$. The aim of the paper is to see what happens if the function f is discontinuous.

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2. THE GENERAL RESULTS

Let us start with the necessary and sufficient condition for the weak convergence in the unit circle. To avoid misunderstandings, we recall that if x is a real number then $\lfloor x \rfloor$ denotes the integer part of x, i.e. the integer fulfilling $n \le x < n+1$, and $\{x\}$ is its fractional part, i.e. $\{x\} = x - \lfloor x \rfloor$.

Lemma 2.1. Let Z_n be a sequence of random variables. Then the following two statements are equivalent:

i) There exists a random variable Z with values in the interval $[0,1)$ such that

$$\exp(i2\pi Z_n) \xrightarrow[n\to+\infty]{\mathcal{D}} \exp(i2\pi Z). \tag{1}$$

ii) There exists a dense subset S of the interval $(0,1)$ such that

$$\lim_{n\to+\infty} \operatorname{Prob}(x < \{Z_n\} \le y) \text{ exists for each pair of points } x, y \in S. \tag{2}$$

If the statement ii) is valid then

$$\lim_{n\to+\infty} \operatorname{Prob}(x < \{Z_n\} \le y) = \operatorname{Prob}(x < Z \le y) \tag{3}$$

for each $0 < x < y < 1$ such that $\operatorname{Prob}(Z = x) = \operatorname{Prob}(Z = y) = 0$.

Proof. i) Let $\exp(i2\pi Z_n) \xrightarrow[n\to+\infty]{\mathcal{D}} \exp(i2\pi Z)$.
Define $S = \{x : 0 < x < 1, \operatorname{Prob}(Z = x) = 0\}$. This set is dense in the interval $(0,1)$. We take a pair $x, y \in S$, $x < y$ and consider the open set $G = \{e^{i2\pi\alpha} : x < \alpha < y\}$ and the closed set $F = \{e^{i2\pi\alpha} : x \le \alpha \le y\}$. Then we have

$$\begin{aligned}\operatorname{Prob}(\exp(i2\pi Z) \in G) &\le \liminf_{n\to+\infty} \operatorname{Prob}(\exp(i2\pi Z_n) \in G) \\ &\le \limsup_{n\to+\infty} \operatorname{Prob}(\exp(i2\pi Z_n) \in F) \le \operatorname{Prob}(\exp(i2\pi Z) \in F).\end{aligned}$$

Therefore,

$$\begin{aligned}\operatorname{Prob}(x < Z < y) &\le \liminf_{n\to+\infty} \operatorname{Prob}(x < \{Z_n\} < y) \le \liminf_{n\to+\infty} \operatorname{Prob}(x < \{Z_n\} \le y) \\ &\le \limsup_{n\to+\infty} \operatorname{Prob}(x < \{Z_n\} \le y) \le \limsup_{n\to+\infty} \operatorname{Prob}(x \le \{Z_n\} \le y) \\ &\le \operatorname{Prob}(x \le Z \le y) = \operatorname{Prob}(x < Z < y) \quad \text{since } x, y \in S.\end{aligned}$$

Consequently, the limit $\lim_{n\to+\infty} \operatorname{Prob}(x < \{Z_n\} \le y)$ exists for each pair $x, y \in S$.

ii) Let S be a dense subset of the interval $(0,1)$ such that the limit $\lim_{n\to+\infty} \operatorname{Prob}(x < \{Z_n\} \le y)$ exists for each pair $x, y \in S$. We define the function H on $[0,1)$ by

$$H(0) = 1 - \lim_{\Delta\to 0+} \liminf_{n\to+\infty} \operatorname{Prob}(\Delta < \{Z_n\} \le 1 - \Delta)$$

and

$$H(y) = H(0) + \lim_{\Delta \to 0+} \liminf_{n \to +\infty} \text{Prob}\,(\Delta < \{Z_n\} \leq y) \quad \text{if } 0 < y < 1.$$

The function H is non-decreasing and $H(1-) = 1$. Then there is a random variable Z with values in the interval $[0, 1)$ such that $\text{Prob}\,(Z < y) = H(y-)$.
Let G be an open subset of the unit circle.

ii1) Let $1 \notin G$.

Then the set G can be expressed as at most countable union of disjoint open sets

$$G = \bigcup_{k \in \mathcal{N}} \{\exp(i2\pi\alpha) : a_k < \alpha < b_k\} \quad \text{where } 0 \leq a_k < b_k \leq 1\,.$$

The set S is dense in the interval $[0, 1]$ and one can find points $a_{k,j}, b_{k,j} \in S$ such that $a_k \leq a_{k,j} < b_{k,j} < b_k$, $a_{k,j}$ tends to a_k and $b_{k,j}$ tends to b_k. Then we have

$$\liminf_{n \to +\infty} \text{Prob}\,(\exp(i2\pi Z_n) \in G) \geq \liminf_{n \to +\infty} \sum_{k=1}^{j} \text{Prob}\,(a_{k,j} < \{Z_n\} \leq b_{k,j})$$

$$= \sum_{k=1}^{j} (H(b_{k,j}) - H(a_{k,j})) \geq \sum_{k=1}^{j} \text{Prob}\,(a_{k,j} + \varepsilon < Z < b_{k,j} - \varepsilon)$$

for every $\varepsilon > 0$ and every natural number j. Letting $\varepsilon \to 0+$ and $j \to +\infty$, we get

$$\liminf_{n \to +\infty} \text{Prob}\,(\exp(i2\pi Z_n) \in G) \geq \text{Prob}\,(\exp(i2\pi Z) \in G)\,.$$

ii2) Let $1 \in G$.

Then the set G can be written as an at most countable union of disjoint open sets

$$G = \{\exp(i2\pi\alpha) : 0 \leq \alpha < b\} \cup \{\exp(i2\pi\alpha) : a < \alpha < 1\} \cup \bigcup_{k \in \mathcal{N}} \{\exp(i2\pi\alpha) : a_k < \alpha < b_k\}$$

where $0 < a < 1$, $0 < b < 1$ and $0 < a_k < b_k < 1$. One can find points $a_{0,j}, b_{0,j}, a_{k,j}, b_{k,j} \in S$ such that $a < a_{0,j}$, $b_{0,j} < b, a_k \leq a_{k,j} < b_{k,j} < b_k$, $a_{0,j}$ tends to a, $b_{0,j}$ tends to b, $a_{k,j}$ tends to a_k and $b_{k,j}$ tends to b_k. Then we have

$$\liminf_{n \to +\infty} \text{Prob}\,(\exp(i2\pi Z_n) \in G)$$

$$\geq \liminf_{n \to +\infty} \left[(1 - \text{Prob}\,(b_{0,j} < \{Z_n\} \leq a_{0,j})) + \sum_{k=1}^{j} \text{Prob}\,(a_{k,j} < \{Z_n\} \leq b_{k,j}) \right]$$

$$= (1 - H(a_{0,j}) + H(b_{0,j})) + \sum_{k=1}^{j} (H(b_{k,j}) - H(a_{k,j}))$$

$$\geq 1 - \text{Prob}\,(b_{0,j} - \varepsilon \leq Z \leq a_{0,j} + \varepsilon) + \sum_{k=1}^{j} \text{Prob}\,(a_{k,j} + \varepsilon < Z < b_{k,j} - \varepsilon)$$

for every $\varepsilon > 0$ and every natural number j. Letting $\varepsilon \to 0+$ and $j \to +\infty$, we again get

$$\liminf_{n \to +\infty} \text{Prob}\,(\exp(i2\pi Z_n) \in G) \geq \text{Prob}\,(\exp(i2\pi Z) \in G)\,. \qquad \square$$

There is a helpful criterion verifying (2).

Lemma 2.2. Let S be a dense subset of $(0,1)$ such that

$$\liminf_{n\to+\infty} \text{Prob}\,(x < Z_n \le y) \ge h(y) - h(x) \tag{4}$$

for every pair $x, y \in S$, $x < y$ and some non-decreasing function h with property $h(1-) - h(0+) = 1$. Then,

$$\lim_{n\to+\infty} \text{Prob}\,(x < Z_n \le y) = h(y) - h(x) \tag{5}$$

for every pair $x, y \in S$.

Proof. Fix a pair $x, y \in S$, $x < y$.Taking $u, v \in S$, $u < x < y < v$, we have

$$\begin{aligned} h(y) - h(x) &\le \liminf_{n\to+\infty} \text{Prob}\,(x < Z_n \le y) \le \limsup_{n\to+\infty} \text{Prob}\,(x < Z_n \le y) \\ &\le \limsup_{n\to+\infty} \text{Prob}\,(u < Z_n \le v) - \liminf_{n\to+\infty} \text{Prob}\,(u < Z_n \le x) \\ &- \liminf_{n\to+\infty} \text{Prob}\,(y < Z_n \le v) \le 1 - h(x) + h(u) - h(v) + h(y)\,. \end{aligned}$$

Because the difference $1 - h(v) + h(u)$ can be made arbitrarily small, we conclude

$$\lim_{n\to+\infty} \text{Prob}\,(x < Z_n \le y) = h(y) - h(x)\,. \qquad \square$$

Assuming the existence of densities and their "left uniform" convergence to another density, we get the following result.

Theorem 2.1. Let Y_n be random variables with distribution functions F_n having a density h_n. Assume that there is a density h such that for almost all x, $h_n(x_n) \to h(x)$ whenever $x_n < x$, $x_n \to x$. Then for each sequence $\alpha_n > 0$, $\alpha_n \to +\infty$, we have

$$\exp\left(i2\pi\frac{\alpha_n}{Y_n}\right) \xrightarrow[n\to+\infty]{\mathcal{D}} \exp(i2\pi U)\;,$$

where U is an random variable uniformly distributed on the interval $[0,1)$.

Proof. Fix $0 < x < y < 1$.We can immediately compute the criterion (2),

$$\begin{aligned} \text{Prob}\left(x < \left\{\frac{\alpha_n}{Y_n}\right\} \le y\right) &= \sum_{k=-\infty}^{+\infty}\left(F_n\left(\frac{\alpha_n}{x+k}\right) - F_n\left(\frac{\alpha_n}{y+k}\right)\right) \\ &= \int_{-\infty}^{+\infty}\left(F_n\left(\frac{\alpha_n}{\lfloor z\rfloor + 1 + x}\right) - F_n\left(\frac{\alpha_n}{\lfloor z\rfloor + 1 + y}\right)\right)\mathrm{d}z \\ &= \int_{-\infty}^{+\infty}\alpha_n\left(F_n\left(\frac{\alpha_n}{\lfloor \alpha_n z\rfloor + 1 + x}\right) - F_n\left(\frac{\alpha_n}{\lfloor \alpha_n z\rfloor + 1 + y}\right)\right)\mathrm{d}z \\ &= \int_{-\infty}^{+\infty}\left(\int_x^y h_n\left(\frac{\alpha_n}{\lfloor \alpha_n z\rfloor + 1 + u}\right)\left(\frac{\alpha_n}{\lfloor \alpha_n z\rfloor + 1 + u}\right)^2 \mathrm{d}u\right)\mathrm{d}z\,. \end{aligned}$$

Since $\frac{\alpha_n}{\lfloor \alpha_n z \rfloor + 1 + u} < \frac{1}{z}$ and $\frac{\alpha_n}{\lfloor \alpha_n z \rfloor + 1 + u} \to \frac{1}{z}$, we may apply Fatou's lemma and obtain

$$\begin{aligned} &\liminf_{n\to+\infty} \mathrm{Prob}\left(x < \left\{\frac{\alpha_n}{Y_n}\right\} \le y\right) \\ \ge\ &\int_{-\infty}^{+\infty} h\left(\frac{1}{z}\right) \frac{y-x}{z^2}\,\mathrm{d}z = (y-x)\int_{-\infty}^{+\infty} h(z)\,\mathrm{d}z = y-x\,. \end{aligned}$$

Finally, for every $0 < x < y < 1$, we get

$$\lim_{n\to+\infty} \mathrm{Prob}\left(x < \left\{\frac{\alpha_n}{Y_n}\right\} \le y\right) = y - x\,,$$

according to Lemma 2.2. The theorem follows from Lemma 2.1. □

Another possibility is to assume a Berry–Esseen type condition.

Theorem 2.2. Let Y_n be random variables with distribution functions F_n and $\alpha_n > 0$, $\alpha_n \to +\infty$, $\beta_n > 0$, $\beta_n \to 0$ such that

$$\alpha_n \sup_{|t|\le K} |F_n(t) - F(t) - \beta_n G(t)| \xrightarrow[n\to+\infty]{} 0 \quad \text{for every } K \in \mathcal{R}\,, \tag{6}$$

where F is a distribution function with a density which is continuous almost everywhere and G is a left-continuous function with finite variation on every compact interval. Then

$$\exp\left(i2\pi\frac{\alpha_n}{Y_n} I[Y_n \ne 0]\right) \xrightarrow[n\to+\infty]{\mathcal{D}} \exp(i2\pi U)\,, \tag{7}$$

where U is a random variable uniformly distributed on the interval $[0,1)$.

P r o o f. The proof is based on the integration by parts which is valid in the form

$$\int_{[a,b]} g(x)\,\mathrm{d}q(x) = g(b)\,q(b) - g(a)\,q(a) - \int_{[a,b]} q(x)\,\mathrm{d}g(x)$$

if g, q have finite variation on the interval $[a,b]$, g is continuous and q is left-continuous. A more general statement can be found in [6].

The set of all continuous functions with finite variation on the unit circle is dense in the set of all continuous functions on the unit circle. Thus, to show the convergence in distribution it is sufficient to verify the convergence

$$\mathsf{E}\int h\left(\exp\left(i2\pi\frac{\alpha_n}{Y_n} I[Y_n \ne 0]\right)\right) \xrightarrow[n\to+\infty]{} \mathsf{E}\int h(\exp(2\pi U))$$

for every continuous function with finite variation.

Fix such a function h. We need to show that the quantity

$$Q_n = \left|\int h\left(\exp\left(i2\pi\frac{\alpha_n}{x} I[x \ne 0]\right)\right)\mathrm{d}F_n(x) - \int h\left(\exp\left(i2\pi\frac{\alpha_n}{x}\right)\right)\mathrm{d}F(x)\right|$$

is vanishing.

Fix $\varepsilon > 0$. One can find points $0 < a < b < +\infty$ such that

$$\max\{F_n(-b), F_n(a) - F_n(-a), 1 - F_n(b), F(-b), F(a) - F(-a), 1 - F(b)\} < \varepsilon\,.$$

The function $h(\exp(i2\pi \frac{\alpha_n}{x}))$ is continuous with finite variation on the interval $[a, b]$ as well as on the interval $[-b, -a]$.

For simplicity, we denote $\tilde{h}(t) := h(\exp(i2\pi t))$ and $H = \max\{|\tilde{h}(t)| : t \in R\}$. The bound H is always finite since h is a continuous function. We derive

$$\begin{aligned}
Q_n \le & \left| \int_{[a,b]} \tilde{h}\left(\frac{\alpha_n}{x}\right) \mathrm{d}F_n(x) - \int_{[a,b]} \tilde{h}\left(\frac{\alpha_n}{x}\right) \mathrm{d}F(x) \right| \\
+ & \left| \int_{[-b,-a]} \tilde{h}\left(\frac{\alpha_n}{x}\right) \mathrm{d}F_n(x) - \int_{[-b,-a]} \tilde{h}\left(\frac{\alpha_n}{x}\right) \mathrm{d}F(x) \right| + 6H\varepsilon \\
\le & \left| \int_{[a,b]} \tilde{h}\left(\frac{\alpha_n}{x}\right) \mathrm{d}F_n(x) - \int_{[a,b]} \tilde{h}\left(\frac{\alpha_n}{x}\right) \mathrm{d}F(x) - \beta_n \int_{[a,b]} \tilde{h}\left(\frac{\alpha_n}{x}\right) \mathrm{d}G(x) \right| \\
+ & \left| \int_{[-b,-a]} \tilde{h}\left(\frac{\alpha_n}{x}\right) \mathrm{d}F_n(x) - \int_{[-b,-a]} \tilde{h}\left(\frac{\alpha_n}{x}\right) \mathrm{d}F(x) - \beta_n \int_{[-b,-a]} \tilde{h}\left(\frac{\alpha_n}{x}\right) \mathrm{d}G(x) \right| \\
+ & \beta_n \left| \int_{[a,b]} \tilde{h}\left(\frac{\alpha_n}{x}\right) \mathrm{d}G(x) \right| + \beta_n \left| \int_{[-b,-a]} \tilde{h}\left(\frac{\alpha_n}{x}\right) \mathrm{d}G(x) \right| + 6H\varepsilon \\
\le & \left| \tilde{h}\left(\frac{\alpha_n}{b}\right) (F_n(b) - F(b) - \beta_n G(b)) - \tilde{h}\left(\frac{\alpha_n}{a}\right) (F_n(a) - F(a) - \beta_n G(a)) \right. \\
- & \left. \int_{[a,b]} (F_n(x) - F(x) - \beta_n G(x))\, \mathrm{d}\tilde{h}\left(\frac{\alpha_n}{x}\right) \right| \\
+ & \left| \tilde{h}\left(\frac{\alpha_n}{-a}\right) (F_n(-a) - F(-a) - \beta_n G(-a)) - \tilde{h}\left(\frac{\alpha_n}{-b}\right) (F_n(-b) - F(-b) - \beta_n G(-b)) \right. \\
- & \left. \int_{[-b,-a]} (F_n(x) - F(x) - \beta_n G(x))\, \mathrm{d}\tilde{h}\left(\frac{\alpha_n}{x}\right) \right| + \beta_n H \bigvee_{-b}^{b} G + 6H\varepsilon \\
\le & \; 2 \sup_{|t| \le b} |F_n(t) - F(t) - \beta_n G(t)| \left\{ \left(\alpha_n \left(\frac{1}{a} - \frac{1}{b} \right) + 1 \right) \bigvee_{0}^{1} \tilde{h} + 2H \right\} \\
+ & \beta_n H \bigvee_{-b}^{b} G + 6H\varepsilon\,.
\end{aligned}$$

Therefore Q_n is vanishing since the variations $\bigvee_0^1 \tilde{h}$ and $\bigvee_{-b}^{b} G$ are bounded and the ε may be arbitrarily small.

This completes the proof since the random variable X with the distribution function F fulfills

$$\exp\left(i2\pi \frac{\alpha_n}{X}\right) \xrightarrow[n \to +\infty]{\mathcal{D}} \exp(i2\pi U) \quad \text{according to Theorem 2.1}\,. \qquad \square$$

3. RESULTS FOR RANDOM SAMPLES

This section presents results for random samples. In the sequel, we will assume $X_1, X_2, X_3, \dots$ to be i.i.d. random variables with the distribution function H and the characteristic function ψ. Denote $S_n = \sum_{i=1}^n X_i$ and $\overline{X}_n = \frac{1}{n} S_n$.

Theorem 3.1. Let $\mathsf{E}\, X_1 = 0$, $0 < \mathsf{var}\, X_1 < +\infty$ and let H^{*k} have a bounded density for some k. If $\alpha_n > 0$, $\frac{\alpha_n}{\sqrt{n}} \to +\infty$ then

$$\exp\left(i2\pi \frac{\alpha_n}{S_n}\right) \xrightarrow[n\to+\infty]{\mathcal{D}} \exp(i2\pi U) , \tag{8}$$

where U is a random variable uniformly distributed on the interval $[0,1)$.

Proof. We have

$$\exp\left(i2\pi \frac{\alpha_n}{S_n}\right) = \exp\left(i2\pi \frac{\frac{\alpha_n}{\sqrt{n}}}{\frac{S_n}{\sqrt{n}}}\right) \xrightarrow[n\to+\infty]{\mathcal{D}} \exp(i2\pi U) ,$$

according to Theorem 2.1 as the densities of $\frac{1}{\sqrt{n}} S_n$ converge uniformly to the density of standard Gaussian random variable, see [7], theorem VII.2.8, p. 244. □

Another possibility is assuming a finite absolute moment.

Theorem 3.2. Let $\mathsf{E}\, X_1 = 0$, $0 < \mathsf{var}\, X_1$ and $\mathsf{E}\, |X_1|^{2+\delta} < +\infty$ for some $0 < \delta \le 1$. Then

$$\exp\left(i2\pi \frac{\alpha_n}{S_n} I[S_n \neq 0]\right) \xrightarrow[n\to+\infty]{\mathcal{D}} \exp(i2\pi U) , \tag{9}$$

whenever $\alpha_n > 0$, $\frac{\alpha_n}{\sqrt{n}} \to +\infty$ and $n^{-\frac{1+\delta}{2}} \alpha_n \to 0$. The random variable U is uniformly distributed on the interval $[0,1)$.

Proof. The proof immediately follows from Theorem 2.2 and the Berry–Esseen inequality

$$\sup_{t\in\mathcal{R}} |H_n(\sqrt{n}t) - \Phi(t)| = O_P(n^{-\frac{\delta}{2}}) \quad \text{see [7], theorem V.3.4, p.140.}$$ □

Theorem 3.3. Let $\mathsf{E}\, X_1 = 0$, $0 < \mathsf{var}\, X_1$, $\mathsf{E}\, |X_1|^3 < +\infty$ and $\limsup_{|t|\to+\infty} |\psi(t)| < < 1$. Then

$$\exp\left(i2\pi \frac{\alpha_n}{S_n} I[S_n \neq 0]\right) \xrightarrow[n\to+\infty]{\mathcal{D}} \exp(i2\pi U) , \tag{10}$$

whenever $\alpha_n > 0$, $\sup_{n\in\mathcal{N}} \frac{\alpha_n}{n} < +\infty$ and $\frac{\alpha_n}{\sqrt{n}} \to +\infty$. The random variable U is uniformly distributed on the interval $[0,1)$.

Proof. The proof immediately follows from Theorem 2.2 and the improved Berry–Esseen inequality

$$\sup_{t\in\mathcal{R}} \left| H_n(\sqrt{n}t) - \Phi(t) - \frac{\mathsf{E}\, X_1^3}{\sqrt{2\pi n}} (t^2 - 1) e^{-t^2} \right| = o_P\left(\frac{1}{\sqrt{n}}\right) ,$$

see [7], theorem VI.3.5, p. 209. □

In the case of lattice distribution, we can utilize a local limit theorem. For that the finite variance is sufficient but the result is surprisingly weaker.

Theorem 3.4. Let $\mathsf{E}\,X_1 = 0$, $0 < \mathsf{var}\,X_1 < +\infty$ and $\mathrm{Prob}\,(X_1 \in a + d\mathcal{Z}) = 1$ ($\mathcal{Z}$ denotes the set of all integers) for some origin a and step d. If $\alpha_n > 0$, $\frac{\alpha_n}{\sqrt{n}} \to +\infty$ and $\frac{\alpha_n}{n} \to 0$ then

$$\exp\left(i2\pi\frac{\alpha_n}{S_n}I[S_n \neq 0]\right) \xrightarrow[n\to+\infty]{\mathcal{D}} \exp(i2\pi U)\ , \tag{11}$$

where U is a random variable uniformly distributed on the interval $[0,1)$.

Proof. Denote $\sigma^2 := \mathsf{var}\,X_1$ and assume that the step d is the largest possible. Fix $0 < x < y < 1$ and consider (2):

$$\mathrm{Prob}\left(x < \left\{\frac{\alpha_n}{S_n}\right\} \leq y\right) = \sum_{k=-\infty}^{+\infty} \mathrm{Prob}\,(S_n = na + dk)\ I\left[x < \left\{\frac{\alpha_n}{na+dk}\right\} \leq y\right]$$

$$= \int_{-\infty}^{+\infty} \mathrm{Prob}\,(S_n = na + d\lfloor z\rfloor)\ I\left[x < \left\{\frac{\alpha_n}{na + d\lfloor z\rfloor}\right\} \leq y\right] \mathrm{d}z$$

$$= \frac{\sigma\sqrt{n}}{d}\int_{-\infty}^{+\infty} \mathrm{Prob}\left(S_n = na + d\left\lfloor\frac{\sqrt{n}\sigma z - na}{d}\right\rfloor\right) I\left[x < \left\{\frac{\alpha_n}{na + d\lfloor\frac{\sqrt{n}\sigma z - na}{d}\rfloor}\right\} \leq y\right] \mathrm{d}z$$

$$\geq \frac{\sigma\sqrt{n}}{d}\int_{A\leq|z|\leq B} \mathrm{Prob}\left(S_n = na + d\left\lfloor\frac{\sqrt{n}\sigma z - na}{d}\right\rfloor\right)$$

$$I\left[x < \left\{\frac{\alpha_n}{\sqrt{n}\sigma z}\right\} \leq y - \frac{\alpha_n d}{\sigma^2 nA(A - \frac{d}{\sqrt{n}\sigma})}\right] \mathrm{d}z$$

$$\geq \int_{-\infty}^{+\infty} \varphi(z)\ I\left[x < \left\{\frac{\alpha_n}{\sqrt{n}\sigma z}\right\} \leq y\right] \mathrm{d}z - {}^1W_n + {}^2W_n + {}^3W_n$$

$$-\Phi(z : |z| < A) - \Phi(z : |z| > B)\ ,$$

where $0 < A < B < +\infty$, Φ denotes the standard Gaussian distribution with the density $\varphi(x) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2})$ and

$$0 \leq {}^1W_n = \int_{A\leq|z|\leq B} \varphi(z)\ I\left[y - \frac{\alpha_n d}{\sigma^2 nA(A - \frac{d}{\sqrt{n}\sigma})} < \left\{\frac{\alpha_n}{\sqrt{n}\sigma z}\right\} \leq y\right] \mathrm{d}z\ ,$$

$${}^2W_n = \int_{A\leq|z|\leq B} \left(\varphi\left(\frac{na + d\lfloor\frac{\sqrt{n}\sigma z - na}{d}\rfloor}{\sqrt{n}\sigma}\right) - \varphi(z)\right)$$

$$I\left[x < \left\{\frac{\alpha_n}{\sqrt{n}\sigma z}\right\} \leq y - \frac{\alpha_n d}{\sigma^2 nA(A - \frac{d}{\sqrt{n}\sigma})}\right] \mathrm{d}z\ ,$$

$$
{}^3W_n = \int_{A\le|z|\le B}\left(\frac{\sigma\sqrt{n}}{d}\text{Prob}\left(S_n = na + d\left\lfloor\frac{\sqrt{n}\sigma z - na}{d}\right\rfloor\right)\right.
$$
$$
\left.-\varphi\left(\frac{na + d\lfloor\frac{\sqrt{n}\sigma z - na}{d}\rfloor}{\sqrt{n}\sigma}\right)\right)
$$
$$
I\left[x < \left\{\frac{\alpha_n}{\sqrt{n}\sigma z}\right\} \le y - \frac{\alpha_n d}{\sigma^2 nA(A-\frac{d}{\sqrt{n}\sigma})}\right]\,\mathrm{d}z\,.
$$

All these members are vanishing for $n \to +\infty$: ${}^1W_n$ because $\frac{\alpha_n d}{\sigma^2 nA(A-\frac{d}{\sqrt{n}\sigma})} \to 0$, ${}^2W_n$ because φ is continuous, and ${}^3W_n$ because of the local limit theorem for lattice distributions, see [7], theorem VII.1.1, p. 231. Therefore, we have shown that

$$
\liminf_{n\to+\infty}\text{Prob}\left(x < \left\{\frac{\alpha_n}{X_n}\right\} \le y\right)
$$
$$
\ge \liminf_{n\to+\infty}\int_{-\infty}^{+\infty}\varphi(z)I\left[x < \left\{\frac{\alpha_n}{\sqrt{n}\sigma z}\right\} \le y\right]\mathrm{d}z - \Phi(z:|z|<A) - \Phi(z:|z|>B)
$$

for any $0 < A < B < +\infty$. According to the previous Theorem 3.1 or Theorem 2.1, we have

$$
\liminf_{n\to+\infty}\text{Prob}\left(x < \left\{\frac{\alpha_n}{X_n}\right\} \le y\right) \ge \liminf_{n\to+\infty}\text{Prob}\left(x < \left\{\frac{\frac{\alpha_n}{\sigma\sqrt{n}}}{W}\right\} \le y\right) = y - x\,,
$$

where W is a standard Gaussian random variable. Lemma 2.2 and Lemma 2.1 conclude the proof, again. □

These theorems imply a corollary for the sample average (denoted by $\overline{X}_n$).

Corollary 3.1. Let $\mathsf{E}X_1 = 0$, $0 < \mathsf{var}\,X_1 < +\infty$, f be a periodic function with the period K, the set of discontinuity points of f is negligible w.r.t. Lebesgue measure, and $\alpha_n > 0$, $\alpha_n\sqrt{n} \to +\infty$. Let at least one of the following conditions be fulfilled

- $X_1 + X_2 + \ldots + X_k$ has a bounded density for some k;
- $\mathsf{E}|X_1|^3 < +\infty$, $\limsup_{|t|\to+\infty}|\psi(t)| < 1$ and $\sup_{n\in\mathcal{N}}\alpha_n < +\infty$;
- for some $0 < \delta \le 1$, $\mathsf{E}|X_1|^{2+\delta} < +\infty$ and $\alpha_n n^{\frac{1-\delta}{2}} \to 0$;
- the distribution of X_1 is the lattice distribution and $\alpha_n \to 0$.

Then

$$
f\left(\frac{\alpha_n}{\overline{X}_n}I[\overline{X}_n \ne 0]\right) \xrightarrow[n\to+\infty]{\mathcal{D}} f(U_K)\,,
$$

where the random variable U_K is uniformly distributed on the interval $[0, K)$.

Proof. Each of these assumptions implies the convergence

$$
\exp\left(i\frac{2\pi\alpha_n}{K\overline{X}_n}I[\overline{X}_n \ne 0]\right) = \exp\left(i\frac{2\pi n\alpha_n}{KS_n}I[S_n \ne 0]\right) \xrightarrow[n\to+\infty]{\mathcal{D}} \exp(i2\pi U)\,,
$$

where U is a random variable uniformly distributed on the interval $[0,1)$, cf. Theorem 3.1 Theorem 3.2, Theorem 3.3 or Theorem 3.4.

The function f is periodic with the period K, therefore we can write $f(x) = \tilde{f}(\exp(i\frac{2\pi}{K}x))$ where $\tilde{f}$ is defined on the unit circle and the set of its discontinuity points is negligible w.r.t. Haar measure on the unit circle. Therefore $\tilde{f}$ preserves the weak convergence and we receive

$$f\left(\frac{\alpha_n}{\overline{X}_n} I[\overline{X}_n \neq 0]\right) = \tilde{f}\left(\exp\left(i\frac{2\pi\alpha_n}{K\overline{X}_n} I[\overline{X}_n \neq 0]\right)\right) \xrightarrow[n\to+\infty]{\mathcal{D}} \tilde{f}(\exp(i2\pi U)) = f(KU),$$

and $U_K = KU$ is a random variable uniformly distributed on the interval $[0,K)$. □

The corollary solves the problem of asymptotic behaviour of $\sin(\frac{1}{\overline{X}_n})$ for absolutely continuous distribution with bounded density and for distribution with the third finite moment and fulfilling Cramer's condition; i.e. $\limsup_{|t|\to+\infty} |\psi(t)| < 1$. However, the behaviour is still unknown for the other cases. Especially, the treatment fails for discrete and singular distributions.

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