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ON A METHOD OF ESTIMATING PARAMETERS IN NON–NEGATIVE ARMA MODELS

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The purpose of this paper is to introduce a method of estimating parameters in non-negative ARMA processes. The method is a generalization of the procedures which were derived for autoregressive and moving-average processes. The estimates are constructed in the form of minima of certain fractions or some functions of these minima. A theorem concerning the strong consistence of these estimates is proved and its applications to the models ARMA(1,1), ARMA(2,1) and ARMA(p,1), $p > 2$ are demonstrated.

1. INTRODUCTION

Non-negative ARMA processes are investigated in this paper. A method of estimating parameters of these processes is introduced. This method is a generalization of a procedure proposed by Bell and Smith in [3] and used by Anděl in [2].

Bell and Smith considered an AR(1) process

$$X_t = bX_{t-1} + e_t, \quad t = 1, \dots, n,$$

where $0 \le b < 1$, $e_t \ge 0$ is a strict white noise, this means a sequence of independent identically distributed random variables, and $X_0 \ge 0$ is a given variable independent of $e_1, \dots, e_n$. They constructed the following simple estimate for the parameter b

$$b^* = \min\left(\frac{X_1}{X_0}, \frac{X_2}{X_1}, \dots, \frac{X_n}{X_{n-1}}\right).$$

They proved that b^* is strongly consistent if and only if the distribution function F of the white noise e_t satisfies the condition

$$F(d) - F(c) < 1 \quad \text{for all } 0 < c < d < \infty. \tag{1.1}$$

A natural way to generalize the above estimate to some other time series models is to derive the estimates in the form of minima of certain fractions or some functions of these minima. Anděl proposed such estimates in non-negative first and second order moving average processes. In the MA(1) model which is defined by the equation

$$X_t = e_t + ae_{t-1}, \quad t = 1, 2, \dots, n,$$

where $e_t \geq 0$ is a strict white noise satisfying (1.1) and $0 \leq a \leq 1$, the estimate

$$a^* = \min_{2 \leq t \leq n-1} \frac{X_{t+1} + X_{t-1}}{X_t}$$

was proposed.

In the MA(2) process

$$X_t = e_t + a_1 e_{t-1} + a_2 e_{t-2}, \quad t = 1, 2, \ldots, n,$$

where e_t is the same as in the previous model, $a_1 \geq 0$, $a_2 \geq 0$ and all roots of the polynomial $z^2 + a_1 z + a_2$ lie inside the unit circle, the following estimates were derived

$$\begin{aligned} a_1^* &= \min_{2 \leq t \leq n-1} \frac{X_{t+1} + 3X_{t-1}}{X_t}, \\ a_2^* &= \min_{3 \leq t \leq n-2} \frac{X_{t+2} + 2X_{t+1} + X_{t-2}}{X_t}. \end{aligned}$$

The strong consistence of the estimates a^*, a_1^*, a_2^* was proved and some approximations of their distribution functions and means were constructed in [2].

Similar estimates were also found in non-negative second order autoregressive processes, but their convergence is slower than in the case of estimates obtained by other methods.

The subject of this paper is estimating parameters in non-negative ARMA processes using a generalization of the method published by Anděl in [1] and [2]. If the ARMA process X_t is stationary it can be written as a linear process. Some appropriate fractions are chosen such that their minima are strongly consistent estimates of the coefficients of the linear process X_t. The estimates of the parameters of the ARMA process X_t are then found as a solution of a system of linear equations.

2. ASSUMPTIONS

We shall consider a non-negative ARMA model which satisfies the following assumptions through the remaining part of our paper.

A.1 Let $e_t > 0$ be a strict white noise with $\mu_e = Ee_t < \infty$ and $E{e_t}^2 < \infty$.

A.2 Let $F(y) = P(e_t \leq y)$ be the distribution function of the random variable e_t.

A.3 Let $0 < F(y) < 1$ for all $y > 0$.

A.4 Let $a_1, \ldots, a_q$, $b_1, \ldots, b_p$ be real numbers such that $0 \leq a_i < 1$, $i = 1, \ldots, q$, $0 \leq b_j$, $j = 1, \ldots, p$.

A.5 Let

$$\sum_{i=1}^{q} a_i + \sum_{j=1}^{p} b_j \neq 0.$$

A.6 Let the polynomials

$$\Theta(z) = z^q + \sum_{k=0}^{q-1} a_{q-k} z^k, \quad \Psi(z) = z^p - \sum_{k=0}^{p-1} b_{p-k} z^k$$

have all roots inside the unit circle and let they have no common roots.

A.7 Let X_t be ARMA(p,q) process defined by the equation

$$X_t = b_1 X_{t-1} + \ldots + b_p X_{t-p} + e_t + a_1 e_{t-1} + \ldots + a_q e_{t-q}. \tag{2.1}$$

3. REMARKS

Remark 3.1. Define $a_i = 0$ for $i > q$. The process X_t can be written in the following form

$$X_t = \sum_{k=0}^{\infty} c_k e_{t-k},$$

where

$$c_0 = 1, \quad c_j = a_j + \sum_{i=1}^{\min(j,p)} b_i c_{j-i} \quad \text{for } j \geq 1.$$

Remark 3.2. Let n be a positive integer and let $\beta_j \geq 0,\ j = 1, \ldots, n$. Denote

$$\beta = \sum_{j=0}^{n-1} \beta_{n-j}.$$

If $\beta \geq 1$ then there exists $x \geq 1$ such that

$$x^n = \sum_{j=0}^{n-1} \beta_{n-j} x^j.$$

Proof. If $\beta = 1$, we put $x = 1$. Consider $\beta > 1$. When we define a polynomial

$$f(t) = t^n - \sum_{j=0}^{n-1} \beta_{n-j} t^j, \quad t \in R,$$

we have $f(1) = 1 - \beta < 0$ and $f(t) \to \infty$ for $t \to \infty$. Thus there exists $x > 1$ such that $f(x) = 0$. □

Remark 3.3. The parameters $b_j,\ j = 1, 2, \ldots, p$ are less than one.

Proof. The roots of the polynomial $\Psi(z)$ lie inside the unit circle. It follows from Remark 3.2 that $b_1 + \ldots + b_p < 1$. Since $b_j \geq 0,\ j = 1, \ldots, p$ we get $b_j < 1, j = 1, \ldots, p$. □

4. MAIN RESULTS

Let a realization $X_1, X_2, \dots, X_n$ of the process (2.1) be given. Let $k, m < n$ be non-negative integers, $m+1 \le n-k$, $r_i = r_i(k,m) \ge 0$, $i = 1, \dots, m$. Consider the following variables

$$V_t = V_t(k,m) = \frac{1}{X_t}\left(X_{t+k} + \sum_{i=1}^{m} r_i X_{t-i}\right). \tag{4.1}$$

We can write according to Remark 3.1

$$V_t = \frac{1}{X_t}\sum_{j=0}^{\infty} c_j \left(e_{t+k-j} + \sum_{i=1}^{m} r_i e_{t-i-j}\right) = \frac{1}{X_t}\sum_{j=-k}^{\infty} \alpha_j e_{t-j},$$

where

$$\begin{aligned} \alpha_j &= \alpha_j(k,m), \\ \alpha_j &= c_{k+j} \text{ for } j = -k, -k+1, \dots, 0, & (4.2) \\ \alpha_j &= c_{k+j} + \sum_{i=1}^{\min(j,m)} c_{j-i} r_i \quad \text{for } j = 1, 2, \dots & (4.3) \end{aligned}$$

Denote

$$P_j = P_j(k,m) = \frac{\alpha_j}{c_j} \quad \text{for } j \ge 0, c_j \ne 0 \tag{4.4}$$

and define

$$P_j = \infty \text{ for negative } j \text{ and for } c_j = 0. \tag{4.5}$$

Let the coefficients r_i be chosen in such a way that

$$\min_{j \ge -k} P_j = P_s \ne 0 \text{ exists for some } s \ge 0,\ c_s \ne 0. \tag{4.6}$$

Such numbers r_i can be found in ARMA(p,q) models defined by (2.1) with $a_i > 0$, $i = 1, 2, \dots, q$ and $b_j > 0$, $j = 1, 2 \dots, p$ as we can see in some models of this type in [6] and in the case of ARMA(p,1) models, $p > 2$ in the next section.

Theorem 4.1. Denote

$$M_k = M_k(k,m) = \min_{m+1 \le t \le n-k} V_t \quad \text{for } m+1 \le n-k,$$

where V_t is defined in (4.1). Then

$$M_k = P_s + \min_{m+1 \le t \le n-k} z_t,$$

where P_s is defined in (4.6),

$$\begin{aligned} z_t &= z_t(k,m) = (X_t)^{-1} \sum_{j=-k}^{\infty} \lambda_j e_{t-j}, \\ \lambda_j &= \lambda_j(k,m) \ge 0 \text{ for } j \ge -k \text{ and } \lambda_s = 0 \end{aligned}$$

and

$$\min_{m+1\le t\le n-k} z_t \to 0 \text{ a.s. for } n\to\infty.$$

Proof. It is not difficult to prove that

$$\sum_{j=0}^{\infty} c_j < \infty \quad \text{and} \quad \sum_{j=-k}^{\infty} \alpha_j < \infty.$$

Using the condition (4.6) we can apply Theorem 6.1 to the sequences $\{\alpha_j\}$ and $\{c_j\}$ and we obtain

$$M_k = P_s + \min_{m+1\le t\le n-k} z_t,$$

where

$$z_t = (X_t)^{-1} \sum_{j=-k}^{\infty} \lambda_j e_{t-j}, \quad \lambda_j \ge 0 \text{ for } j\ge -k \text{ and } \lambda_s = 0.$$

Here s is the index defined in (4.6). It is easy to show that

$$\lambda_j = c_{k+j} \text{ for } j=-k,\dots,-1, \tag{4.7}$$

$$\lambda_0 = 0 \text{ for } s=0, \tag{4.8}$$

$$\lambda_0 = c_k - \frac{1}{c_s}\left(c_{k+s} + \sum_{i=1}^{\min(s,m)} c_{s-i} r_i\right) \text{ for } s>0,$$

$$\lambda_j = c_{k+j} + \sum_{i=1}^{\min(j,m)} c_{j-i} r_i - c_j c_k \qquad \text{for } s=0,\ j=1,2,\dots, \tag{4.9}$$

$$\lambda_j = c_{k+j} + \sum_{i=1}^{\min(j,m)} c_{j-i} r_i - \frac{c_j}{c_s}\left(c_{k+s} + \sum_{i=1}^{\min(s,m)} c_{s-i} r_i\right) \text{ for } s>0,\ j=1,2,\dots \text{and } \lambda_s=0.$$

Define random variables $w_t = w_t(k,m)$ a $R_u = R_u(k,m)$ as follows

$$w_t = (e_t)^{-1} \sum_{j=-k}^{\infty} \lambda_j e_{t-j},$$

$$R_u = \sum_{j=u+1}^{\infty} \lambda_j e_{t-j} \quad \text{for } u\ge -k.$$

Clearly,

$$ER_u^2 = Ee_{t-j}^2 \sum_{j=u+1}^{\infty} \lambda_j^2 + \mu_e^2 \sum_{\substack{i,j=u+1\\ i\ne j}}^{\infty} \lambda_i\lambda_j.$$

Since

$$c_s \ne 0, \quad \sum_{j=0}^{\infty} c_j^2 < \infty,$$

we have with respect to (4.7) – (4.9)

$$\sum_{j=-k}^{\infty} \lambda_j^2 < \infty.$$

Therefore $ER_u^2 \to 0$ for $u \to \infty$ from which we immediately get $P(R_u > \delta) \to 0$ for arbitrary $\delta > 0$. This means $P(R_u \leq \delta) > 0$ for sufficiently large u. Let $\varepsilon > 0,\ \gamma > 0$ be given numbers and choose u such that $P[R_u \leq \varepsilon\gamma(k+u+2)^{-1}] > 0$. Then

$$\begin{aligned} P(w_t < \varepsilon) &= P\left(\sum_{j=-k}^{u} \lambda_j e_{t-j} + R_u < \varepsilon e_t\right) \\ &\geq \prod_{j=-k}^{u} P\left(\lambda_j e_{t-j} \leq \frac{\varepsilon\gamma}{k+u+2}\right) \cdot P\left(R_u \leq \frac{\varepsilon\gamma}{k+u+2}\right) \cdot P\left(e_t > \gamma\right) \\ &\geq \prod_{\substack{j=-k \\ \lambda_j \neq 0}}^{u} F\left[\frac{\varepsilon\gamma}{(k+u+2)\lambda_j}\right] \cdot P\left(R_u \leq \frac{\varepsilon\gamma}{k+u+2}\right) \cdot P\left(e_t > \gamma\right) > 0. \end{aligned}$$

Since $z_t \leq w_t$, we also have $P(z_t < \varepsilon) > 0$.

Denote $\{y_t, t = 1, 2, \ldots\}$ the sequence of indicators of the events $\{z_t < \varepsilon\}$. The independent identically distributed random variables e_t represent a strictly stationary and ergodic sequence. Therefore the sequences $\{z_t\}$, $\{y_t\}$ are according to Theorem VI.6.3, p. 394 in [5] strictly stationary and ergodic. Thus

$$\frac{1}{n}\sum_{t=1}^{n} y_t \to Ey_1 = P(z_t < \varepsilon) > 0 \ \text{a.s.} \quad \text{for } n \to \infty$$

and infinitely many events $\{z_t < \varepsilon\}$ occur with probability 1. This implies

$$\min_{m+1 \leq t \leq n-k} z_t \to 0 \ \text{a.s.} \quad \text{for } n \to \infty. \qquad \square$$

Corollary 4.2. The random variable M_k from Theorem 4.1 is a strongly consistent estimate for P_s.

Remark 4.3. The fractions P_j are functions of the parameters $a_1, \ldots, a_q, b_1, \ldots, b_p$. The function $P_s = P_s(a_1, \ldots, a_q, b_1, \ldots, b_p)$ has a simple form when $r_1, \ldots, r_m$ are such that

$$\min_{j \geq -k} P_j = P_0.$$

Indeed, we have $P_0 = \alpha_0 = c_k$, but $\alpha_j,\ j = 1, 2 \ldots$, are linear combinations of the coefficients c_i.

5. ESTIMATING PARAMETERS IN ARMA MODELS

Applying Theorem 4.1 to the models AR(1), AR(2), MA(1) and MA(2), we obtain the results published in [1] and [2]. Strongly consistent estimates of the parameters in ARMA(1,1) and ARMA(2,1) models were derived using Theorem 4.1 in [6] and we show them in a brief review. Then we derive strongly consistent estimates of the parameters in ARMA(p,1) model with $p > 2$.

5.1. Model ARMA(1,1)

The process X_t is defined by the equation

$$X_t = b_1 X_{t-1} + e_t + a_1 e_{t-1}, \quad 0 < a_1,\ b_1 < 1.$$

We have the following strongly consistent estimates for a_1, b_1:

$$\begin{aligned} \hat{a}_{1n} &= \min_{2 \le t \le n-1} \frac{X_{t+1} + 2X_{t-1}}{X_t} - \hat{b}_{1n}, \\ \hat{b}_{1n} &= M_1 = \min_{1 \le t \le n-1} \frac{X_{t+1}}{X_t}. \end{aligned}$$

5.2. Model ARMA(2,1)

The process is defined as follows

$$X_t = b_1 X_{t-1} + b_2 X_{t-2} + e_t + a_1 e_{t-1}, \quad 0 < a_1,\ b_1,\ b_2 < 1.$$

The strongly consistent estimates for a_1, b_1, b_2 are

$$\begin{aligned} \hat{a}_{1n} &= M_1 - \frac{M_3 - M_1 M_2}{M_2 - M_1^2}, \\ \hat{b}_{1n} &= \frac{M_3 - M_1 M_2}{M_2 - M_1^2}, \\ \hat{b}_{2n} &= \frac{M_2^2 - M_1 M_3}{M_2 - M_1^2}. \end{aligned}$$

They represent a unique solution of the system of equations

$$\begin{aligned} M_1 &= \hat{a}_1 + \hat{b}_1, \\ M_2 &= \hat{b}_1(\hat{a}_1 + \hat{b}_1) + \hat{b}_2, \\ M_3 &= (\hat{b}_1^2 + \hat{b}_2)(\hat{a}_1 + \hat{b}_1) + \hat{b}_1 \hat{b}_2, \end{aligned}$$

where M_1, M_2, M_3 are defined in Theorem 4.1.

5.3. Model ARMA(p,1)

The process X_t is defined by the equation

$$X_t = b_1 X_{t-1} + b_2 X_{t-2} + \ldots + b_p X_{t-p} + e_t + a_1 e_{t-1}, \quad p > 2,\ 0 < a_1, b_1, \ldots b_p < 1.$$

Another expression of X_t is according to Remark 3.1

$$X_t = \sum_{k=0}^{\infty} c_k e_{t-k},$$

where

$$\begin{aligned} &c_0 = 1, \quad c_1 = a_1 + b_1, \quad c_2 = b_1 c_1 + b_2, \ \ldots, \\ &c_{p-1} = b_1 c_{p-2} + b_2 c_{p-3} + \ldots + b_{p-1}, \\ &c_j = b_1 c_{j-1} + b_2 c_{j-2} + \ldots + b_p c_{j-p} \ \text{ for } j = p,\, p+1, \ldots \end{aligned}$$

Put $m = p$ in (4.1) and consider the variables

$$V_t = \frac{1}{X_t}\left(X_{t+k} + \sum_{i=1}^{p} r_i X_{t-i} \right).$$

They can be written in the form

$$V_t = \frac{1}{X_t} \sum_{j=-k}^{\infty} \alpha_j e_{t-j}.$$

Here we have with respect to (4.2) and (4.3)

$$\begin{aligned} \alpha_j &= c_{k+j} \ \text{ for } j = -k, -k+1, \ldots, 0, \\ \alpha_j &= c_{k+j} + \sum_{i=1}^{\min(j,p)} c_{j-i} r_i \ \text{ for } j = 1, 2, \ldots \end{aligned}$$

and according to (4.4) a (4.5)

$$\begin{aligned} P_j &= \frac{\alpha_j}{c_j} \ \text{ for } j = 0, 1, 2, \ldots, c_j \neq 0, \\ P_j &= \infty \ \text{ for } j = -k, -k+1, \ldots, -1 \text{ and for } c_j = 0. \end{aligned}$$

Since $P_0 = c_k$ we get the condition

$$\min_{j \geq 0} P_j = P_0 \text{ if and only if } c_k \leq P_j \quad \text{for all } j > 0$$

which leads to the following system of inequalities for a fixed k

$$c_k c_j \leq c_{k+j} + r_1 c_{j-1} + r_2 c_{j-2} + \ldots + r_j, \qquad j = 1, 2, \ldots, p-1, \tag{5.3.1}$$

$$c_k c_j \leq c_{k+j} + r_1 c_{j-1} + r_2 c_{j-2} + \ldots + r_p c_{j-p}, \quad j = p, p+1, \ldots \tag{5.3.2}$$

We can easily prove some properties of the coefficients c_k by complete induction. These properties are summarized in Lemmas 5.3.1 – 5.3.3.

Lemma 5.3.1. Let $p \geq 2$. Define $A_0 = 0$, $A_1 = 1$, $B_0 = 1$, $B_1 = 0$. Then $c_k = A_k c_1 + B_k$ for $k \geq 2$, where

$$A_k = b_1 A_{k-1} + B_{k-1}, \quad A_k = \sum_{j=1}^{\min(k,p)} b_j A_{k-j}, \quad B_k = \sum_{j=1}^{\min(k,p)} b_j B_{k-j}.$$

Lemma 5.3.2. The numbers A_k, B_k from Lemma 5.3.1 satisfy

$$A_k \leq 2^{k-2}, \quad B_k \leq 2^{k-2} \text{ for } k \geq 2.$$

Lemma 5.3.3. The coefficients c_k satisfy

$$\begin{aligned} c_k &\leq 3 \cdot 2^{k-2} && \text{for } k = 2, \ldots p, \\ c_{p+1} &\leq 3 \cdot 2^{p-1} - 1. \end{aligned}$$

The solutions of the system of inequalities (5.3.1), (5.3.2) are introduced in the following three Lemmas.

Lemma 5.3.4. Let $k = 1$. If the numbers $r_1, r_2, \ldots, r_p$ satisfy the condiotions

$$r_1 \geq 2, \tag{5.3.3}$$

$$r_j \geq 3 \cdot 2^{j-1}, \quad j = 2, \ldots, p-1, \tag{5.3.4}$$

$$r_p \geq 2 \tag{5.3.5}$$

then they represent a solution of the system of inequalities (5.3.1), (5.3.2).

P r o o f. Consider $j = 1$. The first inequality from the system (5.3.1) can be written in the form

$$c_1^2 - c_2 \leq r_1. \tag{5.3.6}$$

We have with respect to the assumption A.4 and Remark 3.3

$$c_1^2 - c_2 = a_1^2 + a_1 b_1 - b_2 \leq 2.$$

Thus $r_1 \geq 2$ can be chosen as a solution of the inequality (5.3.6).

Let $j \geq 2$. The inequalities from the system (5.3.1) can be written in the form

$$c_1 c_j - c_{j+1} - r_1 c_{j-1} - \ldots - r_{j-1} c_1 \leq r_j, \quad j = 2, \ldots p-1. \tag{5.3.7}$$

The left side of the inequality (5.3.7) can be rewritten as

$$\begin{aligned} & c_1 \sum_{i=1}^{j} b_i c_{j-i} - \sum_{i=1}^{j+1} b_i c_{j+1-i} - \sum_{i=1}^{j-1} r_i c_{j-i} \\ = & \sum_{i=1}^{j-1} (b_i c_1 - b_{i+1} - r_i) c_{j-i} + c_1 b_j - b_1 c_j - b_{j+1} \end{aligned}$$

$$\leq \quad c_1 \sum_{i=1}^{j-2} b_i c_{j-i} + c_1^2 b_{j-1} + c_1 b_j$$
$$\leq \quad \sum_{i=1}^{j-2} 6 \cdot 2^{j-i-2} + 6 = 3 \cdot 2^{j-1}, \quad j = 2, \dots, p-1.$$

The last upper bound was found using Lemma 5.3.3. Therefore a solution of the system (5.3.1) is $r_j \geq 3 \cdot 2^{j-1}$, $j = 2, \dots, p+1$.

The inequalities from the system (5.3.2) can be written in the form

$$c_1 c_j \leq c_{j+1} + r_1 c_{j-1} + \dots + r_p c_{j-p}, \quad j = p, p+1, \dots$$

or equivalently as

$$c_1 \sum_{i=1}^{p-1} b_i c_{j-i} + c_1 b_p c_{j-p} \leq \sum_{i=1}^{p-1} (b_{i+1} + r_i)\, c_{j-i} + b_1 c_j + r_p c_{j-p}, \; j = p, p+1, \dots \quad (5.3.8)$$

The inequality (5.3.8) is satisfied if $r_i \geq c_1 b_i - b_{i+1}$, $i = 1, \dots, p-1$ and $r_p \geq c_1 b_p$. With respect to the assumption A.4 and Remark 3.3, it suffices to chose $r_i \geq 2$, $i = 1, \dots, p$. □

Lemma 5.3.5. Let $2 \leq k \leq p$. If the numbers $r_1, r_2, \dots, r_p$ satisfy the conditions

$$r_1 \geq 3 \cdot 2^{k-2}, \quad (5.3.9)$$
$$r_j \geq 9 \cdot 2^{k+j-4}, \quad j = 2, \dots, p-1, \quad (5.3.10)$$
$$r_p \geq 3 \cdot 2^{k-2} \quad (5.3.11)$$

then they represent a solution of the system of inequalities (5.3.1), (5.3.2).

Proof. The assertion can be proved by a similar procedure as in the case of Lemma 5.3.4. □

Lemma 5.3.6. Let $k = p+1$. If the numbers $r_1, r_2, \dots, r_p$ satisfy the conditions

$$r_1 \geq 3 \cdot 2^{p-1} - 1, \quad (5.3.12)$$
$$r_j \geq (3 \cdot 2^{p-1} - 1) \cdot 3 \cdot 2^{j-2}, \quad j = 2, \dots, p-1, \quad (5.3.13)$$
$$r_p \geq 3 \cdot 2^{p-1} - 1 \quad (5.3.14)$$

then they represent a solution of the system of equations (5.3.1), (5.3.2).

Proof. The procedure is similar as in Lemma 5.3.4. □

Remark 5.3.7. According to Lemma 5.3.4, the sequence $r_1, r_2, \dots, r_p$ which satisfies (5.3.3)–(5.3.5) is a solution of the system of inequalities (5.3.1), (5.3.2). On the other hand, there are solutions of the system (5.3.1), (5.3.2) for which the conditions (5.3.3)–(5.3.5) are not fulfilled. This is demonstrated in the following example.

Consider the ARMA(3,1) model with the parameters $a_1 = 0.5$, $b_1 = 0.5$, $b_2 = 0.125$, $b_3 = 0.0625$. This process is stationary and invertible. We have $c_0 = 1$, $c_1 = 1$, $c_2 = 0.625$, $c_3 = 0.5$, $c_4 = 0.390625$. Put $k = 1$. Then the numbers $r_1 = r_2 = r_3 = 1$ represent a solution of the system (5.3.1), (5.3.2), but they obviously do not satisfy (5.3.3)–(5.3.5).

The solutions from Lemmas 5.3.5, 5.3.6 have the same property.

Choose $r_i = r_i(k,p)$, $i = 1,2,\ldots,p$ such that the conditions (5.3.3)–(5.3.5) for $k = 1$, the conditions (5.3.9)–(5.3.11) for $k = 2,\ldots,p$ and the conditions (5.3.12)–(5.3.14) for $k = p+1$ are fulfilled. This means that

$$\min_{j\geq 0} P_j = \min_{j\geq 0} P_j(k,p) = P_0 = P_0(k,p) = c_k, \quad k = 1,\ldots,p+1$$

and the variables

$$M_k = M_k(k,p) = \min_{p+1\leq t\leq n-k} \frac{1}{X_t}\left(X_{t+k} + \sum_{i=1}^{p} r_i X_{t-i}\right), \quad k = 1,\ldots,p+1$$

are according to Theorem 4.1 and Corollary 4.2 strongly consistent estimates for the coefficients c_k, $k = 1,2,\ldots,p+1$.

In the remaining part of this section we determine the system of linear equations the solution of which will give the strongly consistent estimates of the parameters $a_1, b_1, \ldots, b_p$ of our ARMA(p,1) model.

Lemma 5.3.8. Let $M_1,\ldots,M_k$ be real numbers, $M_i \neq 0$, $i = 1,\ldots,k$, $k > 2$. Denote

$$\boldsymbol{A}_k = \begin{pmatrix} M_1 & 1 & 0 & 0 & \ldots & 0 & 0 \\ M_2 & M_1 & 1 & 0 & \ldots & 0 & 0 \\ M_3 & M_2 & M_1 & 1 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ M_{k-1} & M_{k-2} & M_{k-3} & M_{k-4} & \ldots & M_1 & 1 \\ M_k & M_{k-1} & M_{k-2} & M_{k-3} & \ldots & M_2 & M_1 \end{pmatrix}$$

and

$$\tilde{D}_k = |\boldsymbol{A}_k|, \tag{5.3.15}$$

where $|\ |$ is the symbol for determinant. Then

$$\tilde{D}_k = \sum_{j=1}^{k-2} (-1)^{j-1} M_j \tilde{D}_{k-j} + (-1)^{k-4} M_1 M_{k-1} + (-1)^{k-5} M_k.$$

Proof. The matrix $\boldsymbol{A}_k$ can be written as follows

$$\begin{pmatrix} \boldsymbol{A}_{k-1} & \boldsymbol{B}_{k-1} \\ \boldsymbol{C}_{k-1} & \boldsymbol{D}_{k-1} \end{pmatrix}.$$

Here $\boldsymbol{B}_{k-1} = (0, 0, \dots, 0, 1)'$ is $(k-1)$-dimensional column vector, $\boldsymbol{C}_{k-1} = (M_k, M_{k-1}, \dots, M_3, M_2)$ is $(k-1)$-dimensional row vector and $\boldsymbol{D}_{k-1} = M_1$. It can be showed that

$$\tilde{D}_k = M_1 |\boldsymbol{A}_{k-1} - \boldsymbol{B}_{k-1} M_1^{-1} \boldsymbol{C}_{k-1}|$$

and the matrix $\boldsymbol{A}_{k-1} - \boldsymbol{B}_{k-1} M_1^{-1} \boldsymbol{C}_{k-1}$ is equal to

$$\begin{pmatrix} M_1 & 1 & 0 & \dots & 0 & 0 \\ M_2 & M_1 & 1 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ M_{k-2} & M_{k-3} & M_{k-4} & \dots & M_1 & 1 \\ M_{k-1} - \frac{M_k}{M_1} & M_{k-2} - \frac{M_{k-1}}{M_1} & M_{k-3} - \frac{M_{k-2}}{M_1} & \dots & M_2 - \frac{M_3}{M_1} & M_1 - \frac{M_2}{M_1} \end{pmatrix}$$

Applying the well known properties of determinants we can write

$$\begin{aligned} \tilde{D}_k &= M_1 \tilde{D}_{k-1} - M_2 |\boldsymbol{A}_{k-2} - \boldsymbol{B}_{k-2} M_2^{-1} \boldsymbol{C}_{k-2}| \\ &= M_1 \tilde{D}_{k-1} - M_2 \tilde{D}_{k-2} + M_3 |\boldsymbol{A}_{k-3} - \boldsymbol{B}_{k-3} M_3^{-1} \boldsymbol{C}_{k-3}| \\ \dots &= \sum_{j=1}^{k-2} (-1)^{j-1} M_j \tilde{D}_{k-j} + (-1)^{k-4} M_1 M_{k-1} + (-1)^{k-5} M_k. \end{aligned}$$

□

Lemma 5.3.9. Let $c_1, \dots, c_k$, $2 < k \le p$ be the coefficients from Remark 3.1. Denote

$$D_k = \begin{vmatrix} c_1 & 1 & 0 & 0 & \dots & 0 & 0 \\ c_2 & c_1 & 1 & 0 & \dots & 0 & 0 \\ c_3 & c_2 & c_1 & 1 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ c_{k-1} & c_{k-2} & c_{k-3} & c_{k-4} & \dots & c_1 & 1 \\ c_k & c_{k-1} & c_{k-2} & c_{k-3} & \dots & c_2 & c_1 \end{vmatrix}. \tag{5.3.16}$$

Then

$$D_k = a_1 D_{k-1} + (-1)^{k-1} b_k, \quad k = 1, 2, \dots, p. \tag{5.3.17}$$

Proof. The assertion can be proved by complete induction. □

Lemma 5.3.10. Consider the system of equations

$$\begin{aligned} c_1 &= \hat{a}_1 + \hat{b}_1 \\ c_2 &= \hat{b}_1 c_1 + \hat{b}_2 \\ c_3 &= \hat{b}_1 c_2 + \hat{b}_2 c_1 + \hat{b}_3 \\ &\dots\dots\dots\dots\dots\dots \\ c_p &= \hat{b}_1 c_{p-1} + \hat{b}_2 c_{p-2} + \dots + \hat{b}_p \\ c_{p+1} &= \hat{b}_1 c_p + \hat{b}_2 c_{p-1} + \dots + \hat{b}_p c_1, \end{aligned} \tag{5.3.18}$$

which is to be solved with respect to $\hat{a}_1, \hat{b}_1, \dots, \hat{b}_p$. Then the following conditions are equivalent:

C.1 The system (5.3.18) has a unique solution.

C.2 The polynomials

$$\Theta(z) = z + a_1, \quad \Psi(z) = z^p - \sum_{i=0}^{p-1} b_{p-i} z^i$$

have no common roots.

Proof. The matrix of the system (5.3.18) is

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & c_1 \\ 0 & c_1 & 1 & 0 & 0 & \dots & 0 & 0 & c_2 \\ 0 & c_2 & c_1 & 1 & 0 & \dots & 0 & 0 & c_3 \\ 0 & c_3 & c_2 & c_1 & 1 & \dots & 0 & 0 & c_4 \\ . & . & . & . & . & \dots & . & . & . \\ 0 & c_{p-1} & c_{p-2} & c_{p-3} & c_{p-4} & \dots & c_1 & 1 & c_p \\ 0 & c_p & c_{p-1} & c_{p-2} & c_{p-3} & \dots & c_2 & c_1 & c_{p+1} \end{pmatrix}$$

The condition C.1 is equivalent to the condition

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & c_1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & c_2 & c_1 & 1 & 0 & \dots & 0 & 0 \\ 0 & c_3 & c_2 & c_1 & 1 & \dots & 0 & 0 \\ . & . & . & . & . & \dots & . & . \\ 0 & c_{p-1} & c_{p-2} & c_{p-3} & c_{p-4} & \dots & c_1 & 1 \\ 0 & c_p & c_{p-1} & c_{p-2} & c_{p-3} & \dots & c_2 & c_1 \end{vmatrix} \neq 0. \tag{5.3.19}$$

The determinant from (5.3.19) is equal to the determinant D_p from (5.3.16). It is not difficult to prove that (5.3.17) is equivalent to the equation

$$D_k = a_1^k + \sum_{i=0}^{k-1} (-1)^{k-1-i} b_{k-i} a_1^i.$$

If we define the polynomial $D_p(z)$ by

$$D_p(z) = z^p + \sum_{i=0}^{p-1} (-1)^{p-1-i} b_{p-i} z^i,$$

we get $D_p(a_1) = D_p$. It is obvious that a is a root of $D_p(z)$ if and only if $-a$ is a root of $\Psi(z)$. Therefore $D_p = 0$ if and only if the root $-a_1$ of the polynomial $\Theta(z)$ is also a root of $\Psi(z)$. Using (5.3.19) we obtain the assertion. □

Theorem 5.3.11. Let $1 \le k \le p+1$ be a positive integer and let $r_i = r_i(k,p) > 0$, $i = 1,\dots,p$ satisfy the conditions (5.3.3)–(5.3.5) for $k = 1$, (5.3.9)–(5.3.11) for $k = 2,\dots,p$ and (5.3.12)–(5.3.14) for $k = p+1$. Then with probability 1 there exists for sufficiently large n a unique solution $\hat{a}_{1n}, \hat{b}_{1n}, \dots, \hat{b}_{pn}$ of the system of equations

$$\begin{aligned} M_1 &= \hat{a}_1 + \hat{b}_1 \\ M_2 &= \hat{b}_1 M_1 + \hat{b}_2 \\ M_3 &= \hat{b}_1 M_2 + \hat{b}_2 M_1 + \hat{b}_3 \\ &\dots\dots\dots\dots \\ M_p &= \hat{b}_1 M_{p-1} + \hat{b}_2 M_{p-2} + \dots + \hat{b}_p \\ M_{p+1} &= \hat{b}_1 M_p + \hat{b}_2 M_{p-1} + \dots + \hat{b}_p M_1 . \end{aligned} \tag{5.3.20}$$

This solution satisfies

$$\begin{aligned} \hat{a}_{1n} &\to a_1 \text{ a.s.}, \\ \hat{b}_{jn} &\to b_j \text{ a.s.}, \ j = 1,\dots,p. \end{aligned}$$

Proof. The variables M_k are strongly consistent estimates for c_k, $k = 1,2,\dots$ $\dots, p+1$. It follows from Remark 3.1 that a solution of the system of equations (5.3.20) represents strongly consistent estimates $\hat{a}_{1n}, \hat{b}_{1n}, \dots, \hat{b}_{pn}$ of the parameters $a_1, b_1, \dots, b_p$. The system (5.3.20) has a unique solution if and only if the determinant (5.3.15) satisfies $\tilde{D}_p \neq 0$. It is clear that

$$\tilde{D}_p \to D_p \ \text{ a.s.} \quad \text{for } n \to \infty$$

and $D_p \neq 0$ according to Lemma 5.3.10. Thus with probability 1 we have for sufficiently large n $\tilde{D}_p \neq 0$ from which the assertion immediately follows. □

6. APPENDIX

Theorem 6.1. Let $\eta_1, \eta_2, \dots$ be positive independent identically distributed random variables the distribution of which is non-degenerated and $E\eta_j^2 < \infty$, $j = 1,2,\dots$ Let α_j, β_j, $j = 1,2,\dots$ be non-negative real numbers. Let there exist a j_0 such that $\beta_{j_0} > 0$. Suppose further that

$$\sum_{j=1}^{\infty} \alpha_j < \infty, \quad \sum_{j=1}^{\infty} \beta_j < \infty. \tag{6.1}$$

Denote

$$P_j = \frac{\alpha_j}{\beta_j} \quad \text{for } \beta_j \neq 0$$

and define

$$P_j = \infty \quad \text{for } \beta_j = 0 \text{ and for } \alpha_j = \beta_j = 0.$$

Let there exist an index $s \geq 1$ such that $\beta_s > 0$ and

$$\min_{1 \leq j < \infty} P_j = P_s = \frac{\alpha_s}{\beta_s} < \infty.$$

Define

$$R = \left(\sum_{j=1}^{\infty} \beta_j \eta_j \right)^{-1} \sum_{j=1}^{\infty} \alpha_j \eta_j .$$

Then

$$R = \frac{\alpha_s}{\beta_s} + \left(\sum_{j=1}^{\infty} \beta_j \eta_j \right)^{-1} \sum_{j=1}^{\infty} \lambda_j \eta_j , \tag{6.2}$$

where $\lambda_j \geq 0$, $j = 1, 2, \ldots$ and $\lambda_s = 0$.

Proof. It follows from (6.1) that

$$\sum_{j=1}^{\infty} \alpha_j^2 < \infty, \quad \sum_{j=1}^{\infty} \beta_j^2 < \infty.$$

Thus

$$\sum_{j=1}^{\infty} E(\alpha_j \eta_j) < \infty, \quad \sum_{j=1}^{\infty} \operatorname{var}(\alpha_j \eta_j) < \infty,$$

$$\sum_{j=1}^{\infty} E(\beta_j \eta_j) < \infty, \quad \sum_{j=1}^{\infty} \operatorname{var}(\beta_j \eta_j) < \infty$$

and according to Theorem 2 in [4], p. 423 we have

$$\sum_{j=1}^{\infty} \alpha_j \eta_j < \infty \text{ a.s. }, \quad \sum_{j=1}^{\infty} \beta_j \eta_j < \infty \text{ a.s.}$$

If $\alpha_s = 0$ we get $\lambda_j = \alpha_j$, $j = 1, 2, \ldots$ Consider $\alpha_s \neq 0$. Clearly,

$$\eta_s = \left(\sum_{j=1}^{\infty} \beta_j \eta_j - \sum_{\substack{j=1 \\ j \neq s}}^{\infty} \beta_j \eta_j \right) \frac{1}{\beta_s} .$$

Therefore we can write

$$\begin{aligned} R &= \left(\sum_{j=1}^{\infty} \beta_j \eta_j \right)^{-1} \left(\alpha_s \eta_s + \sum_{\substack{j=1 \\ j \neq s}}^{\infty} \alpha_j \eta_j \right) \\ &= \frac{\alpha_s}{\beta_s} + \left(\sum_{j=1}^{\infty} \beta_j \eta_j \right)^{-1} \sum_{\substack{j=1 \\ j \neq s}}^{\infty} \left(\alpha_j - \frac{\alpha_s \beta_j}{\beta_s} \right) \eta_j . \end{aligned}$$

Since

$$\frac{\alpha_s}{\beta_s} = \min_{1\le j<\infty} \frac{\alpha_j}{\beta_j},$$

we have

$$\alpha_j\beta_s - \alpha_s\beta_j \ge 0 \quad \text{for } j = 1,2,\dots$$

This implies

$$\lambda_j = \frac{\alpha_j\beta_s - \alpha_s\beta_j}{\beta_s} \ge 0 \qquad \text{for } j = 1,2,\dots,\ \lambda_s = 0. \qquad \square$$

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REFERENCES

[1] J. Anděl: Non–negative autoregressive processes. J. Time Ser. Anal. *10* (1989), 1–11.
[2] J. Anděl: Non–negative moving–average models. Asymptotic Statistics. In: Proceedings of the Fifth Prague Symposium. Contributions to Statistics, Physica Verlag, Heidelberg 1994, pp. 163–171.
[3] C. B. Bell and E. P. Smith: Inference for non–negative autoregressive schemes. Comm. Statist. Theory Methods *15* (1986), 2267–93.
[4] A. Rényi: Probability Theory. Akadémiai Kiadó, Budapest 1970.
[5] J. Štěpán: Probability Theory. Academia, Praha 1987 (in Czech).
[6] J. Zichová: Estimating parameters in non–negative ARMA processes. In: Transactions of the Ninth European Young Statisticians Meeting, Erasmus University, Rotterdam 1995, pp. 49–54.

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