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A COMPUTATIONAL METHOD FOR REDUCED–ORDER OBSERVERS IN LINEAR SYSTEMS[1]

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A computationally stable method for reduced-order observers of linear systems is proposed. This method is based on orthogonal transformations and adopts Diophantine matrix polynomial equations.

1. INTRODUCTION

Reduced–order observer design has a long history spanning decades and involving various researchers in the control systems society. The first results on this problem were presented by Luenberger [1]. Thereafter, several papers have been presented examining the problems from different perspectives. One of these approaches is the Sylvester equation approach. This technique deals mainly with the computational aspects of the problem. In particular, it was Van Dooren who presented some fundamental concepts in this direction using block Hessenberg forms [8]. Later Tsui in [5] presented a parametrization using the aforementioned Sylvester equation.

In this paper an alternative algorithm is presented for designing reduced–order observers, which is based on a Diophantine equation. Specifically, the problem of reduced–order observer is studied as a full–order observer problem on a reduced–order subspace using a Diophantine equation. The algorithm uses the Hessenberg form of the pair (C, A). The computation of an output injection is achieved in a computationally efficient way.

2. PRELIMINARIES

Be given a linear time-invariant system (C, A, B) governed by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

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where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$. In addition $\operatorname{rank} C = p$ and the observability of the pair (C, A) is further assumed.

Let $F(s)$, $G(s)$ be $p \times p$, $p \times n$ polynomial matrices over $\mathbb{R}[s]$. Then $F(s)$, $G(s)$ are said to form a *(left) normal external description of* (C, A) if [3]

- $[G(s),\ F(s)]$ is a minimal polynomial basis of the left kernel of
$$\begin{bmatrix} sI - A \\ C \end{bmatrix};$$
- $F(s)$ is noincreasingly row-degree ordered and row reduced;
- $G(s)$ is a minimal polynomial basis of the left kernel of $(sI - A)\,\Pi$ where Π is a matrix representation of the maximal annihilator of C, i.e. $C\Pi = 0$.

Let $P(s)$ be a $p \times (k_1 + k_2 + \cdots + k_p)$ polynomial and row-reduced matrix with row degrees $k_1 - 1, k_2 - 1, \dots, k_p - 1$ such that
$$P(s) = \text{ block diag}\left\{[\,1, s, \dots, s^{k_i - 1}\,]\right\} K$$
where $K \in \mathbb{R}^{k\times k}$ is nonsingular, $k = \sum_{i=1}^{p} k_i$. Then $P(s)$ is said to be a *polynomial basis of a k-dimensional and R-linear vector space* [4].

Given an $n \times m$ polynomial matrix $P(s)$, $\operatorname{rank} P(s) = k := \min(n, m)$, we shall say that $P(s)$ is irreducible if $\operatorname{rank} P(z) = k$ for every complex z. For instance, the observability of the pair (C, A) is equivalent to the irreducibility of $\begin{bmatrix} sI_n - A \\ C \end{bmatrix}$.

The concept of a (right) normal external description of the controllable pair (A, B) was established in [3] and used in [9] for compututing a state feedback F assigning a given pole structure to the system
$$\dot{x} = (A + BF)\,x.$$

Here, we shall use the dual version of this algorithm. That is why we have introduced the concept of a (left) normal external description of the pair (C, A).

In [1] it was shown that the problem of designing a reduced-order observer is equivalent to the problem of designing a full-order observer on a reduced-order system of order $n - p$. To get a description of the reduced-order system in a computationaly stable way, we shall exploit the properties of the block-upper Hessenberg form of the matrix A [7, 8].

To this end, let $x = Ux'$ be a state-space similarity transformation, where U is orthogonal, such that [7, 8]
$$A' = U^T A U = \begin{bmatrix} \bar{A} & \hat{A} \\ \tilde{C} & \tilde{A} \end{bmatrix} := \left[\begin{array}{cccc|c} A_{11} & A_{12} & \cdots & A_{1,k-1} & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{1,k-1} & A_{2k} \\ \ddots & \ddots & & \vdots & \vdots \\ & \ddots & \ddots & \vdots & \vdots \\ (0) & & \ddots & A_{k-1,k-1} & A_{k-1,k} \\ \hline (0) & & & A_{k,k-1} & A_{kk} \end{array}\right]$$

and

$$C' = CU = \begin{bmatrix} 0 & \cdots & 0 & C_p \end{bmatrix}, \; B' = U^T B,$$

i.e. the matrix A is brought into the block-upper Hessenberg form [7] and $C_p \in \mathbb{R}^{p\times p}$ is nonsingular.

Let further $x' = T \begin{bmatrix} w \\ v \end{bmatrix}$, where

$$T = \begin{bmatrix} I_{n-p} & 0 \\ 0 & C_p^{-1} \end{bmatrix},$$

be another similarity transformation that brings the system (C', A', B') into the form

$$\begin{aligned} \begin{bmatrix} \dot{w} \\ \dot{v} \end{bmatrix} &= \begin{bmatrix} A_1 & A_2 \\ C_1 & A_3 \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & I_p \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} = v \end{aligned}$$

and denote

$$\bar{A} := \begin{bmatrix} A_1 & A_2 \\ C_1 & A_3 \end{bmatrix}, \; \bar{B} := \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$
$$\bar{C} := \begin{bmatrix} 0 & I_p \end{bmatrix}.$$

Luenberger's theory of observers [1] now implies that an observer for the system

$$\begin{aligned} \dot{w} &= A_1 w + A_2 y + B_1 u \\ y_w : &= C_1 w = \dot{y} - A_3 y - B_2 u \end{aligned}$$

can be constructed using the reduced–order subsystem (C_1, A_1) and is of the form

$$\dot{\xi} = (A_1 - LC_1)\,\xi + (A_2 - LA_3)\,y + (B_1 - LB_2)\,u + L\dot{y}. \tag{1}$$

The relationship (1) reveals that the most crucial point in designing the observer is to find an output injection L such that the observation process ξ will converge to the state w, i.e. the eigenstructure of the matrix $A_1 - LC_1$ must be set up properly.

3. THE STRUCTURE OF THE OBSERVER

In this section we exploit the structure of the system in order to build a reduced–order observer using a reduced–order and square Diophantine equation.

Let $G_1(s)$, $G_2(s)$ and $F(s)$ form a left normal external description of the system $(\bar{C}, \bar{A})$, i.e.

$$\begin{bmatrix} G_1(s) & G_2(s) & F(s) \end{bmatrix} \begin{bmatrix} sI_{n-p} - A_1 & -A_2 \\ -C_1 & sI_p - A_3 \\ 0 & -I_p \end{bmatrix} = 0, \tag{2}$$

and let $k_1 \geq k_2 \geq \cdots \geq k_p$ be the row indices of $[G_1(s)\ G_2(s)\ F(s)]$. Clearly, k_i, $i = 1, 2, \ldots, \rho$ are the observability indices of the pair $(\bar{C}, \bar{A})$ and hence those of (C, A) .

It is natural to explore the relationship between the observability indices of the pairs (C, A) and (C_1, A_1). The following lemma provides this relationship.

Lemma 1. Let(C, A) be an observable pair with observability indices and $k_1, k_2, \ldots$ $\ldots, k_p$. Then the pair (C_1, A_1) defined above is also observable and its observability indices are $k_1 - 1, k_2 - 1, \ldots, k_p - 1$.

Proof. As (C, A) is observable, $(\bar{C}, \bar{A})$ is also observable and has the same observability indices. This implies that

$$\begin{bmatrix} sI_{n-p} - A_1 & -A_2 \\ -C_1 & sI_p - A_3 \\ 0 & -I_p \end{bmatrix}$$

is an irreducible matrix. In view of the fact that the second column of the above matrix forms an ireducible submatrix, we have that

$$\begin{bmatrix} sI_{n-p} - A_1 \\ -C_1 \end{bmatrix}$$

is also irreducible and hence, (C_1, A_1) is observable.

Let $K(s) := [G_1(s)\ G_2(s)\ F(s)]$ form a left normal external description of $(\bar{C}, \bar{A})$ and let $k_1, k_2, \ldots, k_p$ be the row degrees of $K(s)$. Since $\bar{C}(sI_n - \bar{A})^{-1}$ is a strictly proper rational function, $F^{-1}(s)\,[G_1(s)\quad G_2(s)]$ is strictly proper, too. Then, $F(s)$ is row reduced with the row degrees $k_1, k_2, \ldots, k_p$ and $[G_1(s)\quad G_2(s)]$ is also row reduced with the row degrees $k_1 - 1, k_2 - 1, \ldots, k_p - 1$.

As $[G_1(s),\ G_2(s)]$ forms a left normal external description of (C_1, A_1), we can repeat the above considerations and the claim follows. □

The effect of an output injection L on the system $(\bar{C}, \bar{A}, \bar{B})$ can be described by

$$\begin{bmatrix} G_1(s) & G_2(s) & F(s) \end{bmatrix} M M^{-1} \begin{bmatrix} sI_{n-p} - A_1 & -A_2 \\ -C_1 & sI_p - A_3 \\ 0 & -I_p \end{bmatrix} = 0$$

where

$$M = \begin{bmatrix} I_{n-p} & L & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix}.$$

It follows from the above equation that

$$\begin{bmatrix} G_1(s) & G_2(s) + G_1(s)\,L \end{bmatrix} \begin{bmatrix} sI_{n-p} - A_1 + LC_1 \\ -C_1 \end{bmatrix} = 0 \tag{3}$$

and hence, the output injection gain L can be obtained from a constant solution pair X, Y (i.e. X nad Y have entries in $\mathbb{R}$) with X nonsingular [2] of the equation

$$G_2(s)\,X + G_1(s)\,Y = D(s) \tag{4}$$

where $D(s)$ is an $(n-p)\times(n-p)$ polynomial and row-reduced matrix having the same row indices as $G_2(s)$ and reflecting the desired pole structure of the observer (1). Indeed, it follows from (3) that $D(s)$ reflects the pole structure of the observer. Thus, if we find a constant solution pair X, Y with X nonsingular to equation (4), we get the desired output injection L on putting $L = YX^{-1}$.

It should be noted that there exists an efficient and numerically stable way to compute the matrices $G_1(s)$, $G_2(s)$ and $D(s)$; see [9] for more details. In [2] it has been shown that there exists a constant solution to (4), where X is nonsingular, if and only if $D(s)$ is a row reduced matrix with the same row indices as $G_2(s)$.

In order to calculate a constant solution X, Y of the equation (4), we can proceed as follows.

Let $c_i = k_i - 1, \quad i = 1, 2, \ldots, p$ and let

$$S(s) := \text{ block diag } \left\{[1, s, \ldots, s^{c_i-1}]\right\}.$$

Then the equation (4) can be written in the form

$$\begin{aligned} &\text{diag }[s^{c_1}, \ldots, s^{c_{n-p}}]\,G_{2lr}X + S(s)\,G_{2l}Y = \\ &= \text{diag }[s^{c_1}, \ldots, s^{c_{n-p}}]\,D_{lr} + S(s)\,D_l \end{aligned}$$

where G_{2lr} and D_{lr} are the leading coefficient matrices of $G_2(s)$ and $D(s)$, respectively, and G_{2l} and D_l are constant matrices (i.e. having elements in $\mathbb{R}$). In fact, we are to solve the following two systems of linear equations

$$\begin{aligned} G_{2lr}X &= D_{lr} \\ G_{2l}Y &= D_l, \end{aligned}$$

which can be done for instance using some of the well-known methods exploiting orthogonal transformations to achieve high numerical stability. The output injection L is then given by

$$L = YX^{-1}.$$

To sum up, the proposed algorithm for finding L can be described as follows:

1. Using orthogonal transformations, bring the matrix $\begin{bmatrix} A \\ C \end{bmatrix}$ into the block-upper Hessenberg form and put $C_1 = C_p\bar{C}$, $A_1 = \bar{A}$.
2. Find a left normal external description of $(C_1,\ A_1)$.
3. Construct a row-reduced matrix $D(s)$ having the desired invariant factors and the same row degrees as $G_2(s)$.

4. Find a constant solution pair X, Y with X nonsingular of the equation (4).

5. Put $L = YX^{-1}$.

It is to be noted that the solution given by the above algorithm is by no means unique [2]. The particular form of L depends mainly on the particular form of $D(s)$.

4. CONCLUSIONS

In this paper we presented a computationally efficient method for reduced-order observers in linear systems. The proposed technique is based on the Hessenberg form of the pair (C, A) and a square reduced-order Diophantine equation (4), which enables us to modify the dynamics of observer, i.e. the zero structure of $sI_{n-p} - A_1 + LC_1$, in the limits given by the Rosenbrock theorem; see [4] for details. The method was tested using MATLAB.

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