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NON-COMMUTATIVE RINGS OF FRACTIONS IN ALGEBRAICAL APPROACH TO CONTROL THEORY

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The paper describes a mathematical construction of the ring of fractions, generalized for non-commutative rings. This is an underlying concept for the algebraical approach to the control theory.

1. INTRODUCTION

The *algebraical approach* proved to be a very powerful tool for analysis and synthesis of control systems. It considers systems with input and output signals u, y, satisfying the equation

$$y = Fu \tag{1}$$

or

$$Ay = Bu. \tag{2}$$

Here F, A, B are some operators, acting on signals. It appears that the most important properties of operators are algebraical: they form a *ring* with addition (parallel connection) and multiplication (series connection). The properties of divisibility, common divisors, coprimeness etc. play a significant role.

Moreover, a structure of *two rings* $\mathcal{I}$ ("integer") and $\mathcal{F}$ ("fractional") seems to be natural for control problems. The ring $\mathcal{F}$ is formed by fractions B/A for $B, A \in \mathcal{I}$. The operation of making fractions corresponds to making closed-loop systems $F/(1+F)$. Typically, $\mathcal{I}$ consists of polynomials (of some operator, e. g. derivative or delay), $\mathcal{F}$ of polynomial fractions. However, other selections of $\mathcal{I}$ are also possible, e. g. the ring of proper stable functions. This *fractional approach* proved to be very useful also for control synthesis, $\mathcal{F}$ representing all systems, $\mathcal{I}$ those with a required property (stability, dead-beat). The approach was originated by Pernebo [7] and is now culminating by Vidyasagar [10]. For some systems with distributed parameters, still other $\mathcal{I}, \mathcal{F}$ have been used [4].

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In all the literature, the rings $\mathcal{I}, \mathcal{F}$ are supposed to be *commutative*. The non-commutative case of matrix rings for multiple-input-multiple-output (MIMO) systems is usually treated separately. The reason for such a limitation lies probably in the fact that time-invariant single-input-single-output (SISO) operators are commutative (both lumped and distributed). This is not the case of time varying operators. Non-commutative are also operators for space-distributed input and output signals $u(\zeta, t), y(\zeta, t)$ with $0 \leq \zeta \leq L$. So, it seems to be useful for the system and control theory, to formulate the *non-commutative* case from its very fundaments.

The mathematical construction of the ring of fractions in the commutative case goes to beginnings of mathematics, see any elementary algebra book, e. g. in great details [9]. The noncommutative case is due to Øre [6]. For some applications, the rings are equipped with some more operations. The rings with *conjugation* have been studied in [1], the rings with *derivation* have beed introduced in [8] and mentioned in more books, e. g. [2], [3].

The purpose of this paper is mainly to attract the attention of control theorists to the Øre's mathematical fundaments. His results are also more elaborated here, to be applicable in the system and control theory: including zero divisors, allowing a special set $\mathcal{D}$ of demoninators, showing a *duality* between left and right fractions, studying morphisms/antimorphisms and including further operations in rings.

From the system and control theory point of view, the paper is preparatory, it contains a background mathematics rather than the control theory itself. It follows the usual mathematical way of defining new objects axiomatically and then of giving a construction satisfying the axioms. The proofs are omitted or the main ideas of them are presented only, as they are relatively straightforward.

The structure of the paper is as follows: The well-known commutative case is summarized in Section 2. Then Section 3 follows this way for the non-commutative case. Section 4 includes further operations in the rings and constructs the fractions for such cases. Finally, Section 5 presents an application in the system and control theory.

2. THE COMMUTATIVE CASE

In this paper, a ring is assumed associative, with the identity element. A mapping $()^\phi$ of rings is called the *morphism* if it preserves the operations

$$(a+b)^\phi = a^\phi + b^\phi, \tag{3}$$

$$(ab)^\phi = a^\phi b^\phi, \tag{4}$$

$$1^\phi = 1. \tag{5}$$

For non-commutative rings, $()^\phi$ is called the *antimorphism* if

$$(ab)^\phi = b^\phi a^\phi \tag{6}$$

holds instead of (4). A subset $\mathcal{D}$ of the ring is called *multiplicative* if $a, b \in \mathcal{D}$ implies $ab \in \mathcal{D}$, $1 \in \mathcal{D}$. The *left zero-divisor* is such a that ex. $b \neq 0$, $ab = 0$. The *right zero-divisor* is such a that ex. $b \neq 0$, $ba = 0$. The *zero-divisor* is either the left or the right one, otherwise: the *zero-nondivisor*.

Definition 1. Let $\mathcal{I}$ be a commutative ring, $\mathcal{D} \subset \mathcal{I}$ a multiplicative set of some zero-nondivisors. A commutative ring $\mathcal{F}$ is called the *ring of fractions over* $\mathcal{I}$, $\mathcal{D}$ if it satisfies the following axioms:

- There is an injective morphism $()^F : \mathcal{I} \to \mathcal{F}$. A convention for simpler writing: the image $\mathcal{I}^F$ will be denoted by $\mathcal{I}$, its elements a^F by a.
- Every $a \in \mathcal{D}$ is invertible in $\mathcal{F}$.
- For $c \in \mathcal{F}$, there exist a *numerator* $b \in \mathcal{I}$ and a *denominator* $a \in \mathcal{D}$ such that $c = b/a$.

The maximal $\mathcal{D}$ is the set of all zero-nondivisors in $\mathcal{I}$. For such a set, $\mathcal{F}$ is called the *ring of all fractions over* $\mathcal{I}$.

Properties. (Supposing that some $\mathcal{F}$ over $\mathcal{I}$, $\mathcal{D}$ exists.) For $c \in \mathcal{I}$ (more precisely, for $c^F \in \mathcal{I}^F$), the numerator and the denominator are

$$c = \frac{c}{1}. \tag{7}$$

Specially

$$0 = \frac{0}{1}, \quad 1 = \frac{1}{1}. \tag{8}$$

The numerator and the denominator are not unique but two possible couples, denoted by $\begin{bmatrix} b \\ a \end{bmatrix}, \begin{bmatrix} b' \\ a' \end{bmatrix}$, satisfy

$$\begin{bmatrix} b \\ a \end{bmatrix} \sim \begin{bmatrix} b' \\ a' \end{bmatrix} \quad \ldots \quad \text{ex. } k, k' \in \mathcal{D} \text{ such that } bk = b'k',\ ak = a'k' \tag{9}$$

i.e. the both fractions can be broadened to yield the same. It can be proved that this relation is an equivalence. Let us show the proof of transitivity only:

$$\begin{bmatrix} b \\ a \end{bmatrix} \sim \begin{bmatrix} b' \\ a' \end{bmatrix}, \begin{bmatrix} b' \\ a' \end{bmatrix} \sim \begin{bmatrix} b'' \\ a'' \end{bmatrix} \Rightarrow \begin{bmatrix} b \\ a \end{bmatrix} \sim \begin{bmatrix} b'' \\ a'' \end{bmatrix} \tag{10}$$

It means: ex. k, k', l', $l'' \in \mathcal{D}$ such that

$$bk = b'k',\ ak = a'k', \tag{11}$$

$$b'l' = b''l'',\ a'l' = a''l''. \tag{12}$$

Multiplying (11) by l', (12) by k' we have

$$bkl' = b''l''k',\ akl' = a''l''k' \tag{13}$$

with $kl' \in \mathcal{D}$, $l''k' \in \mathcal{D}$, which proves (10). The equivalence can be alternatively defined

$$\begin{bmatrix} b \\ a \end{bmatrix} \sim \begin{bmatrix} b' \\ a' \end{bmatrix} \quad \ldots \quad ab' = ba'. \tag{14}$$

The operations satisfy:

$$\frac{b}{a}+\frac{c}{d} = \frac{bd+ca}{ad}, \tag{15}$$

$$-\frac{b}{a} = \frac{-b}{a}, \tag{16}$$

$$\frac{b}{a}\cdot\frac{c}{d} = \frac{bc}{ad}. \tag{17}$$

Furthermore: Every b/a with $b \in \mathcal{D}$ is invertible in $\mathcal{F}$. Every b/a is a zero-divisor in $\mathcal{F}$ just if b is a zero-divisor in $\mathcal{I}$. So, in the ring of all fractions, all zero-nondivisors are invertible. If $\mathcal{I}$ is an integrity domain (it does not contain zero-divisors), so is $\mathcal{F}$. In such a case, the ring of all fractions over $\mathcal{I}$ is a field.

Theorem 1. For any commutative ring $\mathcal{I}$ and any multiplicative set $\mathcal{D} \subset \mathcal{I}$ of zero-nondivisors, a ring of fractions over $\mathcal{I}$, $\mathcal{D}$ exists.

Proof. Consider a set of couples $b \in \mathcal{I}$, $a \in \mathcal{D}$, denoted by $\begin{bmatrix} b \\ a \end{bmatrix}$. Define $\mathcal{F}$ as a set of equivalence classes (9). Define the operations and the elements 0,1:

$$\begin{bmatrix} b \\ a \end{bmatrix}+\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} bd+ca \\ ad \end{bmatrix}, \tag{18}$$

$$0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{19}$$

$$-\begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}, \tag{20}$$

$$\begin{bmatrix} b \\ a \end{bmatrix}\cdot\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} bc \\ ad \end{bmatrix}, \tag{21}$$

$$1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{22}$$

following the required properties (15), (16), (17), (8). It can be proved that the operations are defined properly, i. e. that the all resulting denominators belong to $\mathcal{D}$ and that the results do not change when b, a get replaced by equivalent b', a' or c, d by c', d'. It can be also proved that all axioms of a commutative ring are satisfied by these operations. Define the mapping $()^F : \mathcal{I} \to \mathcal{F}$ by $c^F = \begin{bmatrix} c \\ 1 \end{bmatrix}$, following (7); it can be proved that it is an injective morphism. For $a \in \mathcal{D}$, it is $\frac{1}{a} = \begin{bmatrix} 1 \\ a \end{bmatrix}$. Finally, for $c = \begin{bmatrix} b \\ a \end{bmatrix}$, define the numerator b, the denominator a. □

Theorem 2. (*The extension of morphism.*) Let $\mathcal{F}$ be a ring of fractions over $\mathcal{I}$, $\mathcal{D}$, let $\mathcal{F}'$ be a ring of fractions over $\mathcal{I}'$, $\mathcal{D}'$, let $()^F : \mathcal{I} \to \mathcal{F}$, $()^{F'} : \mathcal{I}' \to \mathcal{F}'$ be injective morphisms. For any morphism $()^\phi : \mathcal{I} \to \mathcal{I}'$ such that $\mathcal{D}^\phi \subset \mathcal{D}'$, there exists an *extension* morphism $()^\psi : \mathcal{F} \to \mathcal{F}'$ defined by

$$\left(\frac{b}{a}\right)^\psi = \frac{b^\phi}{a^\phi}, \tag{23}$$

i.e. the numerator and denominator of the mapped fraction are equal to the mapped numerator and denominator. For $a \in \mathcal{I}$, it is

$$(a^F)^\psi = (a^\phi)^{F'} \tag{24}$$

i.e. the diagram

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{\phi} & \mathcal{I}' \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F'} \\ \mathcal{F} & \xrightarrow{\psi} & \mathcal{F}' \end{array} \tag{25}$$

commutes. Furthermore, if $()^\phi$ is bijective and $\mathcal{D}^\phi = \mathcal{D}'$ then $()^\psi$ is also bijective.

P r o o f. It must be proved that $()^\psi$ is properly defined, i.e. that the result does not change when b, a get replaced by equivalent b', a'. The rest of the proof is straightforward. □

Theorem 3. For a given commutative $\mathcal{I}$ with $\mathcal{D}$, the ring of fractions over them exists uniquely up to isomorphism.

P r o o f. Let $\mathcal{F}$, $\mathcal{F}'$ be two rings of fractions over $\mathcal{I}$, $\mathcal{D}$. The identity mapping $()^\iota : \mathcal{I} \to \mathcal{I}$ is an isomorphism and so is its extension $()^\psi : \mathcal{F} \to \mathcal{F}'$. □

3. THE NON-COMMUTATIVE CASE

This section provides the procedure described in the previous setion, for a noncommutative ring $\mathcal{I}$. Now the construction is possible not for any multiplicative set $\mathcal{D}$ of some zero-nondivisors but something more is required.

Definition 2. Let $\mathcal{I}$ be a ring. A multiplicative set $\mathcal{D} \subset \mathcal{I}$ of some zero-nondivisors is called the *right Øre set* if it satisfies:

- For $b \in \mathcal{I}$, $a \in \mathcal{D}$, ex. $\hat{b} \in \mathcal{I}$, $\hat{a} \in \mathcal{D}$ such that $a\hat{b} = b\hat{a}$, i.e. $\hat{b}$, $\hat{a}$ are right multipliers of a, b to a common multiple.
- For $b \in \mathcal{D}$, it is $\hat{b} \in \mathcal{D}$.

Properties. For given b, a, the solution $\hat{b}$, $\hat{a}$ is not unique but two possible couples, denoted by $\begin{bmatrix} \hat{b} \\ \hat{a} \end{bmatrix}$, $\begin{bmatrix} \hat{b}' \\ \hat{a}' \end{bmatrix}$, satisfy $\begin{bmatrix} \hat{b} \\ \hat{a} \end{bmatrix} \sim \begin{bmatrix} \hat{b}' \\ \hat{a}' \end{bmatrix}$ in the sense of (9), i.e. the both couples can be right broadened to yield the same. It can be proved that this relation is an equivalence even in the noncommutative case; it is called the *right equivalence* now. Let us show the proof of transitivity only. (10) means (11), (12). Given k', l', ex. $\hat{k}'$, $\hat{l}'$ such that

$$k'\hat{l}' = l'\hat{k}' \tag{26}$$

Multiplying (11) by $\hat{l}'$, (12) by $\hat{k}'$ and using (26) we have

$$bk\hat{l}' = b'k'\hat{l}' = b'l'\hat{k}' = b''l''\hat{k}', \tag{27}$$

$$ak\hat{l}' = a'k'\hat{l}' = a'l'\hat{k}' = a''l''\hat{k}' \tag{28}$$

with $k\hat{l}' \in \mathcal{D}$, $l''\hat{k}' \in \mathcal{D}$ which proves (10).

Definition 3. Let $\mathcal{I}$ be a ring, $\mathcal{D} \subset \mathcal{I}$ a right Øre set. A ring $\mathcal{F}$ is called the *ring of right fractions over* $\mathcal{I}$, $\mathcal{D}$ if it satisfies the following axioms:

- There is an injective morphism $()^F : \mathcal{I} \to \mathcal{F}$; a convention like that in Definition 1.
- Every $a \in \mathcal{D}$ is invertible in $\mathcal{F}$.
- For $c \in \mathcal{F}$, there exist a *right numerator* $b \in \mathcal{I}$ and a *right denominator* $a \in \mathcal{D}$ such that $c = ba^{-1}$.

If the set of all zero-nondivisors in $\mathcal{I}$ is a right Øre set then $\mathcal{I}$ is called the *right Øre ring*. For such $\mathcal{D}$, the ring $\mathcal{F}$ is called the *ring of all right fractions over* $\mathcal{I}$.

Properties. (Supposing that some $\mathcal{F}$ over $\mathcal{I}$, $\mathcal{D}$ exists.) For $c \in \mathcal{I}$, it is

$$c = c \cdot 1^{-1}, \tag{29}$$

specially

$$0 = 0 \cdot 1^{-1}, \qquad 1 = 1 \cdot 1^{-1}. \tag{30}$$

The right numerator and denominator are not unique but two possible couples are right equivalent. The operations satisfy:

$$ba^{-1} + cd^{-1} = (b\hat{d} + c\hat{a})(a\hat{d})^{-1} \tag{31}$$

where $\hat{a}$, $\hat{d}$ satisfy $a\hat{d} = d\hat{a}$,

$$-ba^{-1} = (-b)\,a^{-1}, \tag{32}$$

$$ba^{-1} \cdot cd^{-1} = (b\hat{c})(d\hat{a})^{-1} \tag{33}$$

where $\hat{a}$, $\hat{c}$ satisfy $a\hat{c} = c\hat{a}$. Furthermore: ba^{-1} is a zero-divisor in $\mathcal{F}$ just if b is a zero-divisor in $\mathcal{I}$. So, in the ring of all right fractions, all zero-nondivisors are invertible. If $\mathcal{I}$ is a (non-commutative) integrity domain, so is $\mathcal{F}$. In such a case, the ring of all right fractions over $\mathcal{I}$ is a (non-commutative) field.

Theorem 4. For any ring $\mathcal{I}$ and any right Øre set $\mathcal{D} \subset \mathcal{I}$, a ring of right fractions over $\mathcal{I}$, $\mathcal{D}$ exists.

Proof. Like the commutative case, construct $\mathcal{F}$ as a set of equivalence classes (9). Define the operations

$$\begin{bmatrix} b \\ a \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} b\hat{d} + c\hat{a} \\ a\hat{d} \end{bmatrix} \tag{34}$$

where $\hat{a}, \hat{d}$ satisfy $a\hat{d} = d\hat{a}$,

$$0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{35}$$

$$-\begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}, \tag{36}$$

$$\begin{bmatrix} b \\ a \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} b\hat{c} \\ d\hat{a} \end{bmatrix} \tag{37}$$

where $\hat{a}, \hat{c}$ satisfy $a\hat{c} = c\hat{a}$,

$$1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{38}$$

following (31), (16), (33), (30). In proving that the operations are properly defined, we must prove one thing more than in the commutative case: that the results do not change when the solution $\hat{a}, \hat{d}$ get replaced by equivalent $\hat{a}', \hat{d}'$. All that, together with proving that all axioms of ring are satisfied, is very lengthy exercise but no serious difficulties are met. The mapping $()^F$ is like that in the commutative case; for $c = \begin{bmatrix} b \\ a \end{bmatrix}$, the right numerator is b, the right denominator a. □

All what has been said can be modified for left fractions instead of right ones. This is a *duality* between the left and the right. So, the *left Øre set* satisfies:

- For $b \in \mathcal{I}$, $a \in \mathcal{D}$, ex. $\tilde{b} \in \mathcal{I}$, $\tilde{a} \in \mathcal{D}$ such that $\tilde{b}a = \tilde{a}b$, i.e. $\tilde{b}, \tilde{a}$ are left multipliers of a, b to a common multiple.
- For $b \in \mathcal{D}$, it is $\tilde{b} \in \mathcal{D}$.

Two solutions $\begin{bmatrix} \tilde{b} & \tilde{a} \end{bmatrix}$ and $\begin{bmatrix} \tilde{b}' & \tilde{a}' \end{bmatrix}$ of $\tilde{b}a = \tilde{a}b$ are left equivalent in the sense

$$\begin{bmatrix} b & a \end{bmatrix} \sim \begin{bmatrix} b' & a' \end{bmatrix} \quad \ldots \text{ ex. } k, k' \text{ such that } kb = k'b',\ ka = k'a' \tag{39}$$

i.e. both couples can be left broadened to yield the same. The *ring of left fractions* is defined by requiring existence of a left numerator and denominator $c = a^{-1}b$. The operations satisfy:

$$a^{-1}b + d^{-1}c = (\tilde{d}a)^{-1}(\tilde{d}b + \tilde{a}c) \tag{40}$$

where $\tilde{a}, \tilde{d}$ satisfy $\tilde{d}a = \tilde{a}d$, and

$$a^{-1}b \cdot d^{-1}c = (\tilde{d}a)^{-1}(\tilde{b}c) \tag{41}$$

where $\tilde{b}, \tilde{d}$ satisfy $\tilde{d}b = \tilde{b}d$.

If $\mathcal{D}$ is both a left and a right Øre set, call it the *Øre set*. For it, the ring of right fractions is in the same time ring of left fractions, call it the *ring of fractions*. Then conversion formulae between the right numerator and denominator b, a and the left ones β, α are

$$\alpha b = \beta a, \tag{42}$$

solvable either for α, β or for b, a. In this case, the right equivalence can be interpreted:

$$\begin{bmatrix} b \\ a \end{bmatrix} \sim \begin{bmatrix} b' \\ a' \end{bmatrix} \quad \ldots \text{ ex. } \beta \in \mathcal{I},\ \alpha \in \mathcal{D} \text{ such that } \beta a = \alpha b,\ \beta a' = \alpha b' \tag{43}$$

i.e. there exists a left fraction, corresponding to the both right fractions. The left equivalence can be interpreted:

$$\begin{bmatrix} \beta & \alpha \end{bmatrix} \sim \begin{bmatrix} \beta' & \alpha' \end{bmatrix} \quad \ldots \text{ ex. } b \in \mathcal{I},\ a \in \mathcal{D} \text{ such that } \beta a = \alpha b,\ \beta' a = \alpha' b \tag{44}$$

i.e. there exists a right fraction, corresponding to the both left fractions.

If the set of all zero-nondivisors in $\mathcal{I}$ is an Øre set then $\mathcal{I}$ is called the *Øre ring*. For such $\mathcal{D}$, the ring $\mathcal{F}$ is called the *ring of all fractions over* $\mathcal{I}$.

Theorem 5. (*The extension of morphism.*) Let $\mathcal{F}$ be a ring of right fractions over $\mathcal{I}$, $\mathcal{D}$, let $\mathcal{F}'$ be a ring of right fractions over $\mathcal{I}'$, $\mathcal{D}'$, let $()^F : \mathcal{I} \to \mathcal{F}$, $()^{F'} : \mathcal{I}' \to \mathcal{F}'$ be injective morphisms. For every morphism $()^\phi : \mathcal{I} \to \mathcal{I}'$ such that $\mathcal{D}^\phi \subset \mathcal{D}'$, there exists an *extension* morphism $()^\psi : \mathcal{F} \to \mathcal{F}'$ defined

$$(ba^{-1})^\psi = b^\phi (a^\phi)^{-1} \tag{45}$$

i.e. the right numerator and denominator of the mapped fraction are equal to the mapped right numerator and denominator. The properties of $()^\psi$ are as described in Theorem 2.

It is easy to modify the theorem from the right fractions to the left ones. The definition is

$$(a^{-1}b)^\psi = (a^\phi)^{-1} b^\phi \tag{46}$$

i.e. the left numerator and denominator of the mapped fraction are equal to the mapped left numerator and denominator. In case of (both-sided) fractions, the both definitions are equivalent.

Theorem 6. (*The extension of antimorphism.*) Let $\mathcal{F}$ be a ring of right fractions over $\mathcal{I}$, $\mathcal{D}$, let $\mathcal{F}'$ be a ring of left fractions over $\mathcal{I}'$, $\mathcal{D}'$, let $()^F : \mathcal{I} \to \mathcal{F}$, $()^{F'} : \mathcal{I}' \to \mathcal{F}'$ be injective morphisms. For every antimorphism $()^\phi : \mathcal{I} \to \mathcal{I}'$ such that $\mathcal{D}^\phi \subset \mathcal{D}'$, there exists an *extension* antimorphism $()^\psi : \mathcal{F} \to \mathcal{F}'$ defined by

$$(ba^{-1})^\psi = (a^\phi)^{-1} b^\phi \tag{47}$$

i.e. the left numerator and denominator of the mapped fraction are equal to the mapped right numerator and denominator. The properties of $()^{\psi}$ are as described in Theorem 2.

It is easy to modify the theorem from the right→left to the left→right. The definition is

$$(a^{-1}b)^{\psi} = b^{\phi}(a^{\phi})^{-1} \tag{48}$$

i.e. the right numerator and denominator of the mapped fraction are equal to the mapped left numerator and denominator. In case of (both-sided) fractions, the both definitions are equivalent.

Theorem 7. Given (non-commutative) $\mathcal{I}$, $\mathcal{D}$, the ring of right (left) fractions over them exists uniquely up to isomorphism.

4. FURTHER OPERATIONS

For some applications, the ring $\mathcal{I}$ can be equipped with further operations. The set $\mathcal{D}$ can be required to have some properties for them. It should be studied how the new operations can be extended from $\mathcal{I}$ to $\mathcal{F}$. The first case of new operations is that of *conjugation*, playing an important role for LQ-optimal control problems.

Definition 4. A ring $\mathcal{I}$ is called the *ring with conjugation* if the conjugation $()^{\star} : \mathcal{I} \longrightarrow \mathcal{I}$ is defined satisfying the following axioms:

$$(a+b)^{\star} = a^{\star} + b^{\star}, \tag{49}$$

$$(ab)^{\star} = b^{\star}a^{\star}, \tag{50}$$

$$1^{\star} = 1, \tag{51}$$

$$a^{\star\star} = a. \tag{52}$$

The axioms $(49), (50), (51)$ say that the conjugation is an antimorphism of the ring; (52) says it is bijective, the inverse being also $()^{\star}$. Such a mapping is called the *involutory anti-automorphism*. The morphism/antimorphism $()^{\phi}$ of rings with conjugation is defined to satisfy

$$(a^{\star})^{\phi} = (a^{\phi})^{\star} . \tag{53}$$

Theorem 8. (*The extension of conjugation, the commutative case.*) Let $\mathcal{I}$ be a commutative ring with conjugation, $\mathcal{D} \subset \mathcal{I}$ a multiplicative set of some zero-nondivisors, satisfying $\mathcal{D}^{\star} = \mathcal{D}$, let $\mathcal{F}$ be the ring of fractions over $\mathcal{I}$, $\mathcal{D}$. Then $\mathcal{F}$ is also a ring with conjugation defined

$$\left(\frac{b}{a}\right)^{\star} = \frac{b^{\star}}{a^{\star}}. \tag{54}$$

The injective morphism $()^F : \mathcal{I} \to \mathcal{F}$ is also a morphism of conjugation.

Proof. As the conjugation is a morphism of the commutative ring, the extension of it follows from Theorem 2. It is evident that $()^\star$ is also an involutory automorphism. The property of $()^F$ can be also shown. □

Theorem 9. (*The extension of conjugation, the non-commutative case.*) Let $\mathcal{I}$ be a ring with conjugation, $\mathcal{D} \subset \mathcal{I}$ an Øre set satisfying $\mathcal{D}^\star = \mathcal{D}$, let $\mathcal{F}$ be the ring of fractions over $\mathcal{I}$, $\mathcal{D}$. The $\mathcal{F}$ is also a ring with conjugation defined

$$(ba^{-1})^\star = (a^\star)^{-1} b^\star, \tag{55}$$

equivalently

$$(a^{-1}b)^\star = b^\star (a^\star)^{-1} \tag{56}$$

.

Proof. As the conjugation is an antimorphism of the ring, the extension of it follows from Theorem 6. □

The second case of new operations in $\mathcal{I}$ are the *shift* and the *difference* playing a role in a description of time varying systems, *discrete-time*.

Definition 5. A ring $\mathcal{I}$ is called the *ring with shift and difference* if the shift $()^\varsigma : \mathcal{I} \to \mathcal{I}$ and the difference $()^\nabla : \mathcal{I} \to \mathcal{I}$ are defined satisfying the following axioms:

$$\begin{aligned}
(a+b)^\varsigma &= a^\varsigma + b^\varsigma, && (57)\\
(ab)^\varsigma &= a^\varsigma b^\varsigma, && (58)\\
()^\varsigma &\ \text{is}\ \ \text{bijective}, && (59)\\
(a+b)^\nabla &= a^\nabla + b^\nabla, && (60)\\
(ab)^\nabla &= a^\nabla b + a^\varsigma b^\nabla \ = \ a^\nabla b^\varsigma + ab^\nabla, && (61)\\
(a^\varsigma)^\nabla &= (a^\nabla)^\varsigma. && (62)
\end{aligned}$$

In addition to the "delayed" shift and difference $()^\varsigma$, $()^\nabla$, the "advanced" shift and difference $()^z$, $()^\triangle$ are defined: $()^z$ is the inverse operation to $()^\varsigma$ and $a^\triangle = (a^\nabla)^z$. All formulae remain valid if the delayed operations are systematically exchanged with the advanced ones. This is a *duality* between the delayed and the advanced.

For the rings with shift and difference, the *morphism* $()^\phi$ is defined to satisfy

$$\begin{aligned}
(a^\varsigma)^\phi &= (a^\phi)^\varsigma, && (63)\\
(a^\nabla)^\phi &= (a^\phi)^\nabla, && (64)
\end{aligned}$$

the *antimorphism* to satisfy

$$\begin{aligned}
(a^\varsigma)^\phi &= (a^\phi)^z, && (65)\\
(a^\nabla)^\phi &= -(a^\phi)^\triangle && (66)
\end{aligned}$$

i.e. it changes the sign of the difference and exchanges the delayed operations with the advanced ones.

Theorem 10. (*The extension of shift and difference, the commutative case.*) Let $\mathcal{I}$ be a commutative ring with shift and difference, $\mathcal{D} \subset \mathcal{I}$ a multiplicative set of some zero-nondivisors satisfying $\mathcal{D}^\zeta = \mathcal{D}, \mathcal{D}^z = \mathcal{D}$, let $\mathcal{F}$ be the ring of fractions over $\mathcal{I}$, $\mathcal{D}$. Then $\mathcal{F}$ is also a ring with shift and difference, the operations being

$$\left(\frac{b}{a}\right)^\zeta = \frac{b^\zeta}{a^\zeta} \tag{67}$$

$$\left(\frac{b}{a}\right)^\nabla = \frac{b^\nabla a - b a^\nabla}{a a^\zeta} = \frac{b^\nabla a^\zeta - b^\zeta a^\nabla}{a a^\zeta}. \tag{68}$$

The injective morphism $()^F : \mathcal{I} \to \mathcal{F}$ is also a morphism of shift and difference.

Proof. As the shift is a morphism of the ring, the extension of it follows from Theorem 2. For the difference, it must be proved that the operation (68) is properly defined, i.e. the resulting denominator belongs to $\mathcal{D}$, the results not changing when b, a get replaced by equivalent b', a'. Then the axioms (60) – (62) are proved for the new operations and finally the property of $()^F$. □

Theorem 11. (*The extension of shift and difference, the non-commutative case.*) Let $\mathcal{I}$ be a ring with shift and difference, $\mathcal{D} \subset \mathcal{I}$ a right/left Øre set satisfying $\mathcal{D}^\zeta = \mathcal{D}$, $\mathcal{D}^z = \mathcal{D}$, let $\mathcal{F}$ be the ring of right/left fractions over $\mathcal{I}$, $\mathcal{D}$. Then $\mathcal{F}$ is also a ring with shift and difference, the operations being for the right case

$$(ba^{-1})^\zeta = b^\zeta (a^\zeta)^{-1} \tag{69}$$

and

$$(ba^{-1})^\nabla = (b^\nabla \hat{a} - b\widehat{a^\nabla})\,(a^\zeta \hat{a})^{-1} \tag{70}$$

where $\hat{a}, \widehat{a^\nabla}$ satisfy $a\widehat{a^\nabla} = a^\nabla \hat{a}$, or

$$(ba^{-1})^\nabla = (b^\nabla \widehat{a^\zeta} - b^\zeta \widehat{a^\nabla})(a\widehat{a^\zeta})^{-1} \tag{71}$$

where $\widehat{a^\zeta}$, $\widehat{a^\nabla}$ satisfy $a^\zeta \widehat{a^\nabla} = a^\nabla \widehat{a^\zeta}$. For the left case, they are

$$(a^{-1}b)^\zeta = (a^\zeta)^{-1} b^\zeta \tag{72}$$

and

$$(a^{-1}b)^\nabla = (\tilde{a} a^\zeta)^{-1} (\tilde{a} b^\nabla - \widetilde{a^\nabla} b) \tag{73}$$

where $\tilde{a}$, $\widetilde{a^\nabla}$ satisfy $\widetilde{a^\nabla} a = \tilde{a} a^\nabla$, or

$$(a^{-1}b)^\nabla = (\widetilde{a^\zeta} a)^{-1} (\widetilde{a^\zeta} b^\nabla - \widetilde{a^\nabla} b^\zeta) \tag{74}$$

where $\widetilde{a^\zeta}, \widetilde{a^\nabla}$ satisfy $\widetilde{a^\nabla} a^\zeta = \widetilde{a^\zeta} a^\nabla$.

The dual formulae are easily derived by replacing $()^\zeta$ by $()^z$ and $()^\nabla$ by $()^\Delta$.

The third case of new operations in $\mathcal{I}$ is that of *derivation*, playing a role in description of time-varying systems, *continuous-time*. It can be thought as a case of the shift and difference when the shift satisfies $a^\zeta = a$ for all a and plays no role.

Definition 6. A ring $\mathcal{I}$ is called the *ring with derivation* if the derivation $()^p : \mathcal{I} \to \mathcal{I}$ is defined satisfying the following axioms

$$\begin{aligned} (a+b)^p &= a^p + b^p &(75)\\ (ab)^p &= a^p b + ab^p. &(76) \end{aligned}$$

The morphism of rings with derivation is defined to satisfy

$$(a^p)^\phi = (a^\phi)^p, \tag{77}$$

the antimorphism

$$(a^p)^\phi = -(a^\phi)^p. \tag{78}$$

Theorem 12. (*The extension of derivation, the commutative case.*) Let $\mathcal{I}$ be a commutative ring with derivation, $\mathcal{D} \subset \mathcal{I}$ a multiplicative set of some zero-nondivisors, let $\mathcal{F}$ be a ring of fractions over $\mathcal{I}, \mathcal{D}$. Then $\mathcal{F}$ is also a ring with derivation

$$\left(\frac{b}{a}\right)^p = \frac{b^p a - ba^p}{a^2} \tag{79}$$

$()^F$ being also a morphism of derivation.

Theorem 13. (*The extension of derivation, the non-commutative case.*) Let $\mathcal{I}$ be a ring with derivation, $\mathcal{D} \subset \mathcal{I}$ a right/left Øre set, let $\mathcal{F}$ be the ring of right/left fractions over $\mathcal{I}, \mathcal{D}$. Then $\mathcal{F}$ is also a ring with derivation, for the right case

$$(ba^{-1})^p = (b^p \hat{a} - b\widehat{a^p})\,(a\hat{a})^{-1} \tag{80}$$

where $\hat{a}, \widehat{a^p}$ satisfy $a\widehat{a^p} = a^p \hat{a}$. For the left case

$$(a^{-1}b)^p = (\tilde{a}a)^{-1}\,(\tilde{a}b^p - \widetilde{a^p} b) \tag{81}$$

where $\tilde{a}, \widetilde{a^p}$ satisfy $\widetilde{a^p} a = \tilde{a}a^p$.

5. APPLICATION IN THE SYSTEM AND CONTROL THEORY

In the system and control theory, the rings play a role as rings of operators, acting on signals. The signals (time functions) form a linear space over a field (real numbers). In the discrete-time case, the signals are two-sided sequences $y(t)$, $t = \ldots - 2T, -T, 0, T, 2T \ldots$. The ring $\mathcal{I}$ is that of polynomials

$$a(\zeta) = \sum_k a_k \zeta^k \tag{82}$$

in the shift operator ζ:

$$\zeta y(t) = y(t - T). \tag{83}$$

The set $\mathcal{D}$ is that of polynomials with a_0 invertible, the ring $\mathcal{F}$ can represent impulse responses of causal systems, i.e. $c(\zeta) = b(\zeta)/a(\zeta)$ can be thought as a rational function of a complex variable ζ, not having a pole in the *causality point* $\zeta = 0$. Alternatively, the difference operator ∇:

$$\nabla y(t) = \frac{y(t) - y(t-T)}{T} \tag{84}$$

can be used instead of the shift. A similar construction runs in the continuous-time case with polynomials in the derivative operator. Other rings than those of polynomials can be also used as $\mathcal{I}$ in the system and control theory.

For SISO systems, the rings are commutative. However, for MIMO systems with signal vectors and operator matrices, we have a non-commutative case. Furthermore, for time-varying systems, the coefficients a_k are no more constants but functions of time: $a_k(t)$. This implies that they do not commute with the operator ζ:

$$a(t)\zeta \neq \zeta a(t) \tag{85}$$

as

$$a(t)\,\zeta y(t) = a(t)\,y(t-T) \tag{86}$$

is not equal to

$$\zeta a(t)\,y(t) = a(t-T)\,y(t-T). \tag{87}$$

Here we have a non-commutativity of an origin different from that for matrices: for the time-varying systems, even the SISO case is non-commutative. This is why the non-commutative theory was elaborated here for the system and control theory purposes. For more details on the polynomial-like approach in time-varying systems, see [5].

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