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RINGS OF SKEW POLYNOMIALS IN ALGEBRAICAL APPROACH TO CONTROL THEORY

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The paper aims at building a mathematical device, generalizing the polynomial approach in the control theory from time invariant systems to time varying ones. For that purpose, the algebraical rings are equipped with some operations more: the shift, the difference or the derivation. Based on that, the skew polynomials are defined which are non-commutative but satisfy a commutation equation.

1. INTRODUCTION

The *polynomial approach* [3] proved to be a very powerful tool for analysis and synthesis of control systems. It considers linear time invariant sytems with input and output signals $u(t), y(t)$ satisfying the equation

$$A(q)\,y = B(q)\,u, \tag{1}$$

where $A(q), B(q)$ are polynomials in the derivative or the delay operator q:

$$A(q) = \sum_{k=0}^{n} A_k q^k, \tag{2}$$

the coefficients A_k being real numbers or matrices of them.

It appears that the most important properties of operators are algebraical: they form a *ring* with addition (parallel connection) and multiplication (series connection). The properties of divisibility, common divisors, coprimeness etc. play a significant role. The synthesis of control systems is performed by means of polynomial equations.

This paper aims at building a mathematical device, which makes possible to generalize this approach to *time varying systems*. It is made in a unified way both for continuous-time and for discrete-time systems, both for single-input-single-output

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(SISO) and for multiple-input-multiple-output (MIMO) ones. The coefficients A_k are no more constant but functions of time. Some more operations on coefficients are necessary, that is why the original structure of two rings (the ring $\mathcal{R}$ of coefficients, the ring $\mathcal{P}$ of polynomials over $\mathcal{R}$) must be modified. New operations of *shift* (delay or advance), *difference* or *derivation* are defined in $\mathcal{R}$. By means of them, the ring $\mathcal{P}$ of skew polynomials over $\mathcal{R}$ is constructed, generalizing that of polynomials from constant coefficients to varying ones. It is interesting to note that $\mathcal{P}$ is *non-commutative* even for SISO systems but a special *commutation equation* holds.

The paper deals not with the control theory itself but with mathematics needed for it. So it begins not with time signals and operators acting on them, but with abstract algebraical structures. It defines the skew polynomials not as those having certain time functions in coefficients but as those satisfying some axioms. This is the way how the rings have been defined in mathematics (summarized in Section 2), as well as the polynomials over them (in Section 3). Analogically to that, the rings with shift and difference are defined in Section 4 and their properties investigated. The skew polynomials over them are treated in Section 5. Some special cases: rings with shift only and rings with derivation are investigated in Section 6 as well as the skew polynomials over them. The proofs are omitted or only the main ideas of them are presented, as they are lengthy but straightforward. Afterwards, Section 7 contains the use of the mathematical device just developed in the system and control theory.

The theory of skew polynomials was originated by Øre [4]. The rings with derivation were introduced by Raudenbush [5] and are mentioned in various algebra books, e. g. [2], [1]. The unified approach for the continuous-time systems and the discrete-time ones is new here.

2. RINGS

In this section, a well-known material from mathematics is summarized.

Definition 1. A *ring* $\mathcal{R}$ (associative, with the identity element) is a set where for $a, b \in \mathcal{R}$, the operations $a+b \in \mathcal{R}$, $-a \in \mathcal{R}$, $ab \in \mathcal{R}$ and the elements $0 \in \mathcal{R}$, $1 \in \mathcal{R}$ are defined satisfying the following axioms:

$$\begin{aligned} (a+b)+c &= a+(b+c), && (3)\\ a+b &= b+a, && (4)\\ a+0 &= a, && (5)\\ a+(-a) &= 0, && (6)\\ (ab)c &= a(bc), && (7)\\ a\cdot 1 &= 1\cdot a = a, && (8)\\ (a+b)c &= ac+bc, && (9)\\ a(b+c) &= ab+ac. && (10)\end{aligned}$$

If moreover

$$ab = ba \qquad (11)$$

holds, the ring is called *commutative.*

Example 1. Real numbers, $N \times N$ matrices of real numbers.

Definition 2. A mapping $()^{\phi} : \mathcal{R} \to \mathcal{R}'$ where $\mathcal{R}, \mathcal{R}'$ are rings is called the *morphism* if it satisfies

$$\begin{aligned} (a+b)^{\phi} &= a^{\phi} + b^{\phi}, && (12) \\ (ab)^{\phi} &= a^{\phi} b^{\phi}, && (13) \\ 1^{\phi} &= 1. && (14) \end{aligned}$$

When

$$(ab)^{\phi} = b^{\phi} a^{\phi} \qquad (15)$$

holds instead of (13), the mapping is called the *antimorphism.*

Properties of composed mappings:

$$\begin{aligned} \text{morphism} \cdot \text{morphism} &= \text{morphism}, && (16) \\ \text{morphism} \cdot \text{antimorphism} &= \text{antimorphism}, && (17) \\ \text{antimorphism} \cdot \text{morphism} &= \text{antimorphism}, && (18) \\ \text{antimorphism} \cdot \text{antimorphism} &= \text{morphism}. && (19) \end{aligned}$$

3. POLYNOMIALS

Like the previous sections, this one also summarizes a well-known mathematical material.

Definition 3. Let $\mathcal{R}$ be a ring. A ring $\mathcal{P}$ is called the *ring of polynomials over* $\mathcal{R}$ if it satisfies the following axioms:

- There is an injective morphism $()^{P} : \mathcal{R} \to \mathcal{P}$. A convention for simpler writing: image $\mathcal{R}^{P}$ will be denoted by $\mathcal{R}$, its elements λ^{P} by λ.
- $\mathcal{P}$ contains a *basis element* x which *commutes* with all $\lambda \in \mathcal{R}$: $\lambda x = x\lambda$.
- x is *not algebraic* over $\mathcal{R}$: for finite number of $\lambda_k \in \mathcal{R}$, at least one nonzero, it is
$$\sum_k \lambda_k x^k \neq 0 \qquad (20)$$
- x *generates* $\mathcal{P}$ over $\mathcal{R}$: for $a \in \mathcal{P}$, ex. finite number of *coefficients* $a_k \in \mathcal{R}$ such that
$$a = \sum_k a_k x^k \qquad (21)$$

Properties. (Supposing that some $\mathcal{P}$ over $\mathcal{R}$ exists.) The coefficients a_k in (21) are unique. With all $a \in \mathcal{P}$, the basis element commutes: $xa = ax$, and so does its mth power x^m. The element $\lambda \in \mathcal{R}$ (more precisely, $\lambda^P \in \mathcal{R}^P$) has coefficients

$$\lambda_k = \lambda\delta_k \ , \tag{22}$$

using the Dirac notation $\delta_k = \left\{ \begin{array}{lll} 1 & \dots & k=0 \\ 0 & \dots & k \neq 0 \end{array} \right\}$. Specially

$$0_k = 0, \ \ 1_k = \delta_k \ . \tag{23}$$

The basis element x has coefficients

$$x_k = \delta_{k-1} \ , \tag{24}$$

its power x^m has $(x^m)_k = \delta_{k-m}$. The *degree* of polynomial is defined: for $a \neq 0$, $\deg a =$ highest k such that $a_k \neq 0$; $\deg 0 = -\infty$.

The operations satisfy:

$$\begin{aligned} \deg(a+b) &\leq \max(\deg a, \deg b), && (25) \\ \deg(-a) &= \deg a, && (26) \\ \deg ab &\leq \deg a + \deg b, && (27) \end{aligned}$$

$$\begin{aligned} (a+b)_k &= \left\{ \begin{array}{lll} a_k & \dots & k \leq \deg a \\ 0 & \dots & \text{else} \end{array} \right\} + \left\{ \begin{array}{lll} b_k & \dots & k \leq \deg b \\ 0 & \dots & \text{else} \end{array} \right\} , && (28) \\ (-a)_k &= -a_k \ , && (29) \\ (ab)_k &= \sum_{l=\max(0,k-\deg a)}^{\min(k,\deg b)} a_{k-l}b_l = \sum_{l=\max(0,k-\deg b)}^{\min(k,\deg a)} a_l b_{k-l} \ . && (30) \end{aligned}$$

A convention for simpler writing: the coefficients a_k are augmented by zeros outside the interval $0 \leq k \leq \deg a$ and the operations written

$$\begin{aligned} (a+b)_k &= a_k + b_k \ , && (31) \\ (ab)_k &= \sum_l a_{k-l}b_l = \sum_l a_l b_{k-l} \ , && (32) \end{aligned}$$

the sum always having only finite number of nonzero terms.

Illustration. $\deg a = 3, \ \ \deg b = 2, \ \ \deg ab = 5$

$$\begin{array}{lclclclcl} (ab)_0 & = & a_0b_0 \\ (ab)_1 & = & a_0b_1 & + & a_1b_0 \\ (ab)_2 & = & a_0b_2 & + & a_1b_1 & + & a_2b_0 \\ (ab)_3 & = & & & a_1b_2 & + & a_2b_1 & + & a_3b_0 \\ (ab)_4 & = & & & & & a_2b_2 & + & a_3b_1 \\ (ab)_5 & = & & & & & & & a_3b_2 \end{array}$$

Properties. (Continuation.) If $\mathcal{R}$ is commutative, so is $\mathcal{P}$. For the absolute term, $(ab)_0 = a_0 b_0$. For the leading term, $(ab)_{\deg a + \deg b} = a_{\deg a} \cdot b_{\deg b}$. If $\mathcal{R}$ is an integrity domain, so is $\mathcal{P}$ and it holds $\deg ab = \deg a + \deg b$.

Theorem 1. For every ring $\mathcal{R}$, there exists a ring $\mathcal{P}$ of polynomials over $\mathcal{R}$.

Proof. By construction. Let $\mathcal{P}$ be a set of two-sided sequences $a(k) \in \mathcal{R}$, $k = \dots -1, 0, 1, \dots$ such that $a(k) \neq 0$ only for $k \geq 0$, only for finite number of k's. Define the operations and the elements 0,1:

$$\begin{aligned}
(a+b)(k) &= a(k) + b(k), && (33)\\
0(k) &= 0, && (34)\\
(-a)(k) &= -a(k), && (35)\\
(ab)(k) &= \sum_{l=-\infty}^{\infty} a(k-l)\, b(l), && (36)\\
1(k) &= \delta_k\ , && (37)
\end{aligned}$$

following the required properties (31), (32), (29), (23). It can be proved that the operations are defined properly, i.e. that the sum in the multiplication formula has only finite number of nonzero terms, and that the results of all operations are of the required form, i.e. they are nonzero only for finite number of k's. It can be also proved that axioms of ring (3) – (10) are satisfied by these operations. Define the mapping $()^P : \mathcal{R} \to \mathcal{P}$ by $(\lambda^P)(k) = \lambda \delta_k$, following (22); it can be proved that it is injective morphism. Define the basis element $x \in \mathcal{P}$ by $x(k) = \delta_{k-1}$, following (24). Its mth power x^m is $x^m(k) = \delta_{k-m}$. It can be proved that x is not algebraic over $\mathcal{R}$ and that it generates $\mathcal{P}$ over $\mathcal{R}$, the needed coefficients a_k in (21) being $a_k = a(k)$.□

Theorem 2. (*The extension of morphism/antimorphism.*) Let $\mathcal{R}, \mathcal{R}'$ be rings, $\mathcal{P}, \mathcal{P}'$ rings of polynomials over them, $()^P : \mathcal{R} \to \mathcal{P}$, $()^{P'} : \mathcal{R}' \to \mathcal{P}'$ injective morphisms, $()_k$ coefficients in bases x, x'. For any morphism/antimorphism $()^{\phi} : \mathcal{R} \to \mathcal{R}'$, there exists an *extension* morphism/antimorphism $()^{\psi} : \mathcal{P} \to \mathcal{P}'$, defined by $(a^{\psi})_k = (a_k)^{\phi}$, i.e. coefficients of mapped polynomial are equal to mapped coefficients. For $\lambda \in \mathcal{R}$, it is $(\lambda^P)^{\psi} = (\lambda^{\phi})^{P'}$, i.e. the diagram

$$\begin{tikzcd}
\mathcal{R} \arrow[r, "\phi"] \arrow[d, "P"'] & \mathcal{R}' \arrow[d, "P'"] \\
\mathcal{P} \arrow[r, "\psi"] & \mathcal{P}'
\end{tikzcd} \qquad (38)$$

commutes. Furthermore, if ϕ is bijective, so is ψ.

Theorem 3. Given $\mathcal{R}$, the ring of polynomials over it exists uniquely up to isomorphism.

Proof. Let $\mathcal{P}, \mathcal{P}'$ be two rings of polynomials over $\mathcal{R}$. The identity mapping $()^{\iota} : \mathcal{R} \to \mathcal{R}$ is isomorphism and so is its extension $()^{\psi} : \mathcal{P} \to \mathcal{P}'$. □

A question arises naturally whether the basis element is unique or whether some other element of $\mathcal{P}$ has such a property. It appears that the latter is the case. So we have a group of basis transformations in $\mathcal{P}$, similarly to that in linear spaces.

Theorem 4. In $\mathcal{R}, \mathcal{P}$ with x, let $T_0, T_1 \in \mathcal{R}$ commute with all elements of $\mathcal{R}$, let T_1^{-1} exist. Then $x' = T_0 + T_1 x$ is also a basis element. All such transformations of basis form a group. The powers of x get transformed

$$x'^k = \sum_{l=0}^{k} \binom{k}{l} T_0^{k-l} T_1^l x^l \, . \tag{39}$$

In the new basis, the polynomial a has new coefficients a'_k. The coefficients get transformed contravariantly to the powers of x:

$$a_k = \sum_{l=k}^{\deg a} \binom{l}{k} T_0^{l-k} T_1^k a'_l \, . \tag{40}$$

Illustration.

$$\begin{bmatrix} 1 \\ x' \\ x'^2 \\ x'^3 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ T_0 & T_1 & & \\ T_0^2 & 2T_0T_1 & T_1^2 & \\ T_0^3 & 3T_0^2T_1 & 3T_0T_1^2 & T_1^3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix} ,$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & T_0 & T_0^2 & T_0^3 \\ & T_1 & 2T_0T_1 & 3T_0^2T_1 \\ & & T_1^2 & 3T_0T_1^2 \\ & & & T_1^3 \end{bmatrix} \cdot \begin{bmatrix} a'_0 \\ a'_1 \\ a'_2 \\ a'_3 \end{bmatrix} .$$

The transformation matrix for coefficients is inverse and transposed to that for powers of x.

4. RINGS WITH SHIFT, DIFFERENCE OR DERIVATION

In this section, a generalization of rings is defined, which is capable of describing the varying coefficients, for discrete-time systems. It is made by adding two new operations: the *shift* $()^{\varsigma}$ and the *difference* $()^{\nabla}$. The motivation is in the delay and the delayed difference of time functions $f(t)$, $t = \ldots -2T, -T, 0, T, 2T \ldots$:

$$f^{\varsigma}(t) = f(t-T), \tag{41}$$

$$f^{\nabla}(t) = \frac{f(t) - f(t-T)}{T}, \tag{42}$$

T being a sampling period. In (41), (42), the difference can be expressed by the shift or vice versa. However, a more elegant formulation can be obtained by ignoring this dependance and by defining the both operations axiomatically.

Definition 4. A ring $\mathcal{R}$ is called the *ring with shift and difference* if it is equipped by two operations more: for $a \in \mathcal{R}$, the shift $a^{\zeta} \in \mathcal{R}$ and the difference $a^{\nabla} \in \mathcal{R}$, satisfying:

$$\begin{aligned} (a+b)^{\zeta} &= a^{\zeta}+b^{\zeta}, && (43)\\ (ab)^{\zeta} &= a^{\zeta}b^{\zeta}, && (44)\\ ()^{\zeta} &\text{ is } \text{bijective}, && (45)\\ (a+b)^{\nabla} &= a^{\nabla}+b^{\nabla}, && (46)\\ (ab)^{\nabla} &= a^{\nabla}b+a^{\zeta}b^{\nabla}=a^{\nabla}b^{\zeta}+ab^{\nabla}, && (47)\\ (a^{\zeta})^{\nabla} &= (a^{\nabla})^{\zeta}. && (48) \end{aligned}$$

Example 2. Two-sided sequences $a(t)$ of real numbers, $t = \ldots -2T, -T, 0, T, 2T \ldots$ with pointwise addition and multiplication and with (41), (42).

Example 3. Similarly, finite sequences $a(t)$, $t = 0, T \ldots (N-1)T$, but in (41), (42) with $t-T$ replaced by $t-T$ mod NT. Equivalently: periodic sequences with the period NT.

Properties. $0^{\zeta}=0$, $1^{\zeta}=1$, $n^{\zeta}=n$ for natural n, $(-a)^{\zeta}=-a^{\zeta}$, $(a^{-1})^{\zeta}=(a^{\zeta})^{-1}$ if a^{-1} exists, the m-times iterated shift a^{ζ^p} has also properties of shift. $0^{\nabla}=0$, $1^{\nabla}=0$, $n^{\nabla}=0$ for natural n, $(-a)^{\nabla}=-a^{\nabla}$, $(a^{-1})^{\nabla}=-(a^{\zeta})^{-1}a^{\nabla}a^{-1}=-a^{-1}a^{\nabla}(a^{\zeta})^{-1}$ if a^{-1} exists. It holds $a^{\nabla}(b-b^{\zeta})=(a-a^{\zeta})b^{\nabla}$. The higher shifts commute with the higher differences: $(a^{\nabla^m})^{\zeta^n}=(a^{\zeta^n})^{\nabla^m}$. The higher differences satisfy:

$$(a+b)^{\nabla^m}=a^{\nabla^m}+b^{\nabla^m}, \tag{49}$$

$$(ab)^{\nabla^m}=\sum_{n=0}^{m}\begin{pmatrix} m\\ n\end{pmatrix}a^{\nabla^{m-n}\zeta^n}b^{\nabla^n}=\sum_{n=0}^{m}\begin{pmatrix} m\\ n\end{pmatrix}a^{\nabla^{m-n}}b^{\nabla^n\zeta^{m-n}}. \tag{50}$$

(proof by induction). The inverse of the *delay shift* $()^{\zeta}$ is the *advance shift* $()^{z}$, it satisfies $(a^{z})^{\zeta}=a$, $(a^{\zeta})^{z}=a$. The *delayed difference* $()^{\nabla}$ gives raise to the *advanced difference* $()^{\triangle}$ by $a^{\triangle}=a^{z\nabla}$. The couple of operators $()^{\triangle}, ()^{z}$ has the same properties as the couple $()^{\nabla}, ()^{\zeta}$. This is the *duality* of the theory: every theorem remains valid when exchanging the delayed operators with the advanced ones.

Definition 5. A mapping $()^{\phi}: \mathcal{R} \to \mathcal{R}'$ where $\mathcal{R}, \mathcal{R}'$ are rings with shift and difference, is called the *morphism* if it satisfies (12)–(14) and

$$\begin{aligned} (a^{\zeta})^{\phi} &= (a^{\phi})^{\zeta}, && (51)\\ (a^{\nabla})^{\phi} &= (a^{\phi})^{\nabla}. && (52) \end{aligned}$$

When (15) holds instead of (13) and

$$\begin{aligned} (a^{\zeta})^{\phi} &= (a^{\phi})^{z}, && (53)\\ (a^{\nabla})^{\phi} &= -(a^{\phi})^{\triangle} && (54) \end{aligned}$$

instead of (51), (52), the mapping is called the *antimorphism*. Note that it changes the sign of difference and exchanges the delayed operators with the advanced ones.

Theorem 5. Let T_0, $T_1 \in \mathcal{R}$ commute with all elements of $\mathcal{R}$, let $T_0^\zeta = T_0$, $T_1^\zeta = T_1$, $T_0^\nabla = 0$, $T_1^\nabla = 0$. Then $()^{\nabla'}$, defined by $a^{\nabla'} = T_0(a - a^\zeta) + T_1 a^\nabla$ is also a difference in $\mathcal{R}$, compatible with the shift $()^\zeta$.

5. SKEW POLYNOMIALS

In this section, the skew polynomials are constructed over a ring with shift and difference. The way is analogous to constructing the polynomials over a ring, described in Section 3.

Definition 6. Let $\mathcal{R}$ be a ring with shift and difference. A ring $\mathcal{P}$ is called the *ring of skew polynomials over* $\mathcal{R}$, if it satisfies the following axioms:

- There is an injective morphism $()^P : \mathcal{R} \to \mathcal{P}$. A convention for simpler writing like that in Definition 3.
- $\mathcal{P}$ contains a *basis element* x which satisfies a *commutation equation*
$$x\lambda = \lambda^\nabla + \lambda^\zeta x \tag{55}$$
with all $\lambda \in \mathcal{R}$
- $x^\zeta = x$
- $x^\nabla = 0$
- x is *not right algebraic* over $\mathcal{R}$: for finite number of $\lambda_k \in \mathcal{R}$, at least one nonzero, it is
$$\sum_k \lambda_k x^k \neq 0 \tag{56}$$
- x *right generates* $\mathcal{P}$ over $\mathcal{R}$: for $a \in \mathcal{P}$, ex. finite number of *left coefficients* $a_k \in \mathcal{R}$ such that
$$a = \sum_k a_k x^k. \tag{57}$$

Properties. (Supposing that some $\mathcal{P}$ over $\mathcal{R}$ exists.) The left coefficients (57) are unique. The commutation equation can be also written in the form
$$\lambda x = -\lambda^\Delta + x\lambda^z. \tag{58}$$

With all $a \in \mathcal{P}$, the basis element also satisfies the commutation equations
$$\begin{aligned} xa &= a^\nabla + a^\zeta x, & (59)\\ ax &= -a^\Delta + xa^z. & (60)\end{aligned}$$

Its mth power x^m satisfies the higher comutation equations
$$\begin{aligned} x^m a &= \sum_{n=0}^{m} \binom{m}{n} a^{\nabla^{m-n}\zeta^n} x^n, & (61)\\ ax^m &= \sum_{n=0}^{m} \binom{m}{n} (-1)^{m-n} x^n a^{\Delta^{m-n} z^n}. & (62)\end{aligned}$$

(proof by induction).

Illustration.

$$\begin{array}{rclllllll} a &=& a \\ xa &=& a^{\nabla} &+& a^{\zeta}x \\ x^2a &=& a^{\nabla^2} &+& 2a^{\nabla\zeta}x &+& a^{\zeta^2}x^2 \\ x^3a &=& a^{\nabla^3} &+& 3a^{\nabla^2\zeta}x &+& 3a^{\nabla\zeta^2}x^2 &+& a^{\zeta^3}x^3 \end{array} \tag{63}$$

$$\begin{array}{rcllllllll} a &=& & a \\ ax &=& - & a^{\triangle} &+& xa^{z} \\ ax^2 &=& & a^{\triangle^2} &-& 2xa^{\triangle z} &+& x^2a^{z^2} \\ ax^3 &=& - & a^{\triangle^3} &+& 3xa^{\triangle^2 z} &-& 3x^2a^{\triangle z^2} &+& x^3a^{z^3} \end{array} \tag{64}$$

Properties. (Continuation.) The basis element is *neither left algebraic over* $\mathcal{R}$: for finite number of $\mu_k \in \mathcal{R}$, at least one nonzero, it is $\sum_k x^k\mu^k \neq 0$. It also *left generates* $\mathcal{P}$ over $\mathcal{R}$: for $a \in \mathcal{P}$, ex. finite number of $[a]_k \in \mathcal{R}$ (*right coefficients*) such that $a = \sum_k x^k[a]_k$. Notation: the left coefficients $()_k$, the right ones $[]_k$. Element $\lambda \in \mathcal{R}$ has coefficients $(\lambda)_k = [\lambda]_k = \lambda\delta_k$. Specially $(0)_k = [0]_k = 0$, $(1)_k = [1]_k = \delta_k$. The basis element x has coefficients $(x)_k = [x]_k = \delta_{k-1}$, its power x^m has $(x^m)_k = [x^m]_k = \delta_{k-m}$. The *degree* of a skew polynomial is defined: for $a \neq 0$, $\deg a =$ highest k such that $(a)_k \neq 0$ (equivalently, $[a]_k \neq 0$); $\deg 0 = -\infty$. *Conversions* between the left and right coefficients are:

$$[a]_k = \sum_{l=k}^{\deg a} \binom{l}{k} (-1)^{l-k}((a)_l)^{\triangle^{l-k}z^k}, \tag{65}$$

$$(a)_k = \sum_{l=k}^{\deg a} \binom{l}{k} ([a]_l)^{\nabla^{l-k}\zeta^k}. \tag{66}$$

For the leading coefficients:

$$[a]_{\deg a} = ((a)_{\deg a})^{z^{\deg a}}, \tag{67}$$

$$(a)_{\deg a} = ([a]_{\deg a})^{\zeta^{\deg a}}. \tag{68}$$

Illustration. $\deg a = 3$

$$\begin{array}{rclllllll} [a]_0 &=& (a)_0 &-& ((a)_1)^{\triangle} &+& ((a)_2)^{\triangle^2} &-& ((a)_3)^{\triangle^3} \\ [a]_1 &=& & & ((a)_1)^{z} &-& 2((a)_2)^{\triangle z} &+& 3((a)_3)^{\triangle^2 z} \\ [a]_2 &=& & & & & ((a)_2)^{z^2} &-& 3((a)_3)^{\triangle z^2} \\ [a]_3 &=& & & & & & & ((a)_3)^{z^3} \end{array} \tag{69}$$

$$\begin{array}{rclllllll} (a)_0 &=& [a]_0 &+& ([a]_1)^{\nabla} &+& ([a]_2)^{\nabla^2} &+& ([a]_3)^{\nabla^3} \\ (a)_1 &=& & & ([a]_1)^{\zeta} &+& 2([a]_2)^{\nabla\zeta} &+& 3([a]_3)^{\nabla^2\zeta} \\ (a)_2 &=& & & & & ([a]_2)^{\zeta^2} &+& 3([a]_3)^{\nabla\zeta^2} \\ (a)_3 &=& & & & & & & ([a]_3)^{\zeta^3} \end{array} \tag{70}$$

Properties. (Continuation.) The operations satisfy

$$\begin{aligned}
\deg(a+b) &\le \max(\deg a, \deg b), \tag{71}\\
\deg(-a) &= \deg a, \tag{72}\\
\deg ab &\le \deg a + \deg b, \tag{73}\\
\deg a^{\zeta} &= \deg a, \tag{74}\\
\deg a^{z} &= \deg a, \tag{75}\\
\deg a^{\nabla} &\le \deg a, \tag{76}\\
\deg a^{\triangle} &\le \deg a, \tag{77}
\end{aligned}$$

$$\begin{aligned}
(a+b)_k &= \left\{ \begin{array}{lll} (a)_k & \dots & k \le \deg a \\ 0 & \dots & \text{else} \end{array} \right\} + \left\{ \begin{array}{lll} (b)_k & \dots & k \le \deg b \\ 0 & \dots & \text{else} \end{array} \right\}, \tag{78}\\
(-a)_k &= -(a)_k\ , \tag{79}\\
(ab)_k &= \sum_{m=\max(0,k-\deg b)}^{\min(k,\deg a)} \sum_{l=m}^{\deg a} \binom{l}{m} (a)_l ((b)_{k-m})^{\nabla^{l-m}\zeta^m}, \tag{80}\\
(\lambda a)_k &= \lambda (a)_k\ , \tag{81}\\
(a^{\zeta})_k &= ((a)_k)^{\zeta}, \tag{82}\\
(a^{z})_k &= ((a)_k)^{z}, \tag{83}\\
(a^{\nabla})_k &= ((a)_k)^{\nabla}, \tag{84}\\
(a^{\triangle})_k &= ((a)_k)^{\triangle}\ , \tag{85}
\end{aligned}$$

$$\begin{aligned}
[a+b]_k &= \left\{ \begin{array}{lll} [a]_k & \dots & k \le \deg a \\ 0 & \dots & \text{else} \end{array} \right\} + \left\{ \begin{array}{lll} [b]_k & \dots & k \le \deg b \\ 0 & \dots & \text{else} \end{array} \right\}, \tag{86}\\
[-a]_k &= -[a]_k\ , \tag{87}\\
[ab]_k &= \sum_{m=\max(0,k-\deg a)}^{\min(k,\deg b)} \sum_{l=m}^{\deg b} \binom{l}{m} (-1)^{l-m} ([a])_{k-m})^{\triangle^{l-m}z^m} [b]_l\ , \tag{88}\\
[a\lambda]_k &= [a]_k \lambda\ , \tag{89}\\
[a^{\zeta}]_k &= ([a]_k)^{\zeta}, \tag{90}\\
[a^{z}]_k &= ([a]_k)^{z}, \tag{91}\\
[a^{\nabla}]_k &= ([a]_k)^{\nabla}, \tag{92}\\
[a^{\triangle}]_k &= ([a]_k)^{\triangle}. \tag{93}
\end{aligned}$$

A convention for simpler notation, like that for polynomials:

$$\begin{aligned}
(a+b)_k &= (a)_k + (b)_k\ , \tag{94}\\
(ab)_k &= \sum_m \sum_l \binom{l}{m} (a)_l ((b)_{k-m})^{\nabla^{l-m}\zeta^m}, \tag{95}\\
[a+b]_k &= [a]_k + [b]_k\ , \tag{96}
\end{aligned}$$

$$[ab]_k = \sum_m \sum_l \binom{l}{m} (-1)^{l-m}([a]_{k-m})^{\Delta^{l-m} z^m}[b]_l\,, \tag{97}$$

the sums always having only finite number of nonzero terms.

Illustration. $\deg a = 3,\quad \deg b = 2,\quad \deg ab = 5$

$$\begin{aligned}
(ab)_0 &= (a)_0(b)_0 + (a)_1(b)_0^{\nabla} + (a)_2(b)_0^{\nabla^2} + (a)_3(b)_0^{\nabla^3}\\[1ex]
(ab)_1 &= (a)_0(b)_1 + (a)_1(b)_1^{\nabla} + (a)_2(b)_1^{\nabla^2} + (a)_3(b)_1^{\nabla^3}\\
&\quad + (a)_1(b)_0^{\zeta} + 2(a)_2(b)_0^{\nabla\zeta} + 3(a)_3(b)_0^{\nabla^2\zeta}\\[1ex]
(ab)_2 &= (a)_0(b)_2 + (a)_1(b)_2^{\nabla} + (a)_2(b)_2^{\nabla^2} + (a)_3(b)_2^{\nabla^3}\\
&\quad + (a)_1(b)_1^{\zeta} + 2(a)_2(b)_1^{\nabla\zeta} + 3(a)_3(b)_1^{\nabla^2\zeta}\\
&\quad + (a)_2(b)_0^{\zeta^2} + 3(a)_3(b)_0^{\nabla\zeta^2}\\[1ex]
(ab)_3 &= (a)_1(b)_2^{\zeta} + 2(a)_2(b)_2^{\nabla\zeta} + 3(a)_3(b)_2^{\nabla^2\zeta}\\
&\quad + (a)_2(b)_1^{\zeta^2} + 3(a)_3(b)_1^{\nabla\zeta^2}\\
&\quad + (a)_3(b)_0^{\zeta^3}\\[1ex]
(ab)_4 &= (a)_2(b)_2^{\zeta^2} + 3(a)_3(b)_2^{\nabla\zeta^2}\\
&\quad + (a)_3(b)_1^{\zeta^3}\\[1ex]
(ab)_5 &= (a)_3(b)_2^{\zeta^3}
\end{aligned}$$

$$\begin{aligned}
[ab]_0 &= [a]_0[b]_0 - [a]_0^{\Delta}[b]_1 + [a]_0^{\Delta^2}[b]_2\\[1ex]
[ab]_1 &= [a]_1[b]_0 - [a]_1^{\Delta}[b]_1 + [a]_1^{\Delta^2}[b]_2\\
&\quad + [a]_0^{z}[b]_1 - 2[a]_0^{\Delta z}[b]_2\\[1ex]
[ab]_2 &= [a]_2[b]_0 - [a]_2^{\Delta}[b]_1 + [a]_2^{\Delta^2}[b]_2\\
&\quad + [a]_1^{z}[b]_1 - 2[a]_1^{\Delta z}[b]_2\\
&\quad + [a]_0^{z^2}[b]_2\\[1ex]
[ab]_3 &= [a]_3[b]_0 - [a]_3^{\Delta}[b]_1 + [a]_3^{\Delta^2}[b]_2\\
&\quad + [a]_2^{z}[b]_1 - 2[a]_2^{\Delta z}[b]_2\\
&\quad + [a]_1^{z^2}[b]_2\\[1ex]
[ab]_4 &= [a]_3^{z}[b]_1 - 2[a]_3^{\Delta z}[b]_2\\
&\quad + [a]_2^{z^2}[b]_2\\[1ex]
[ab]_5 &= [a]_3^{z^2}[b]_2
\end{aligned}$$

Properties. (Continuation.) Ring $\mathcal{P}$ is not generally commutative, even if $\mathcal{R}$ is. For the leading term:

$$(ab)_{\deg a+\deg b} = (a)_{\deg a}((b)_{\deg b})^{\varsigma^{\deg a}}, \tag{98}$$

$$[ab]_{\deg a+\deg b} = ([a]_{\deg a})^{z^{\deg b}}[b]_{\deg b}. \tag{99}$$

If $\mathcal{R}$ is an integrity domain, so is $\mathcal{P}$ and it holds $\deg ab = \deg a + \deg b$. Note however that in Examples 2,3, $\mathcal{R}$ is not an integrity domain, neither is $\mathcal{P}$.

Theorem 6. For every ring $\mathcal{R}$ with shift and difference, there exists a ring $\mathcal{P}$ of skew polynomials over it.

Proof. By construction, like in proof of Theorem 1, but with modifications:

$$(ab)(k) = \sum_m \sum_l \binom{l}{m} a(l)\,(b(k-m))^{\nabla^{l-m}\varsigma^m}, \tag{100}$$

$$a^{\varsigma}(k) = (a(k))^{\varsigma}, \tag{101}$$

$$a^{\nabla}(k) = (a(k))^{\nabla}. \tag{102}$$

When proving the axioms of ring, (43)–(48) must be also proved. For mapping $()^P : \mathcal{R} \to \mathcal{P}$, (51),(52) must be also proved. □

Theorem 7. (*The extension of morphism.*) Let $\mathcal{R}, \mathcal{R}'$ be rings with shift and difference, $\mathcal{P}, \mathcal{P}'$ rings of skew polynomials over them, $()^P : \mathcal{R} \to \mathcal{P}$, $()^{P'} : \mathcal{R}' \to \mathcal{P}'$ injective morphisms, $()_k$ left, $[]_k$ right coefficients in bases x, x'. For any morphism $()^{\phi} : \mathcal{R} \to \mathcal{R}'$, there exists an *extension morphism* $()^{\psi} : \mathcal{P} \to \mathcal{P}'$, defined by $(a^{\psi})_k = ((a)_k)^{\phi}$, i.e. the left coefficients of the mapped skew polynomial are equal to the mapped left coefficients. It is $(\lambda^P)^{\psi} = (\lambda^{\phi})^{P'}$, i.e. the diagram (38) commutes. An equivalent definition of $()^{\psi}$ is $[a^{\psi}]_k = ([a]_k)^{\phi}$, i.e. the right coefficients of the mapped skew polynomial are equal to the mapped right coefficients. Furthermore, if $()^{\phi}$ is bijective, so is $()^{\psi}$.

Theorem 8. (*The extension of antimorphism.*) With the assumptions of Theorem 7, for any antimorphism $()^{\phi} : \mathcal{R} \to \mathcal{R}'$, there exists an *extension antimorphism* $()^{\psi} : \mathcal{P} \to \mathcal{P}'$, defined by $[a^{\psi}]_k = ((a)_k)^{\phi}$, i.e. the right coefficients of the mapped skew polynomial are equal to the mapped left coefficients. It is $(\lambda^P)^{\psi} = (\lambda^{\phi})^{P'}$, i.e. the diagram (38) commutes. An equivalent definition of $()^{\psi}$ is $(a^{\psi})_k = ([a]_k)^{\phi}$, i.e. the left coefficients of the mapped skew polynomial are equal to the mapped right coefficients. Furthermore, if $()^{\phi}$ is bijective, so is $()^{\psi}$.

Theorem 9. Given $\mathcal{R}$, the ring of skew polynomials over it exists uniquely up to isomorphism.

Proof. Like that of Theorem 3.

It appears that the ring $\mathcal{P}$ of skew polynomials over $\mathcal{R}$ has the basis transformation property, similar to that of the ring of polynomials (see Theorem 4). However, in these transformations, the operation of difference $()^{\nabla}$ in $\mathcal{R}$ gets also transformed.

Theorem 10. In $\mathcal{R},\mathcal{P}$ with x, let $T_0, T_1 \in \mathcal{R}$ commute with all elements of $\mathcal{R}$, let T_1^{-1} exist, let $T_0^\zeta = T_0$, $T_1^\zeta = T_1$, $T_0^\nabla = 0$, $T_1^\nabla = 0$. Then $x' = T_0 + T_1 x$ is also a basis element, when a new difference $()^{\nabla'}$ is used in $\mathcal{R}$, defined by $\lambda^{\nabla'} = T_0(\lambda - \lambda^\zeta) + T_1\lambda^\nabla$ (see Theorem 5). All such transformations of basis form a group. The powers of x get transformed by (39), the left and right coefficients of a skew polynomial by (40).

6. SPECIAL CASES

The first special case is a ring $\mathcal{R}$ with shift and difference, where $a^\nabla = 0$ for every a. The axioms of difference (46), (47), (48) are satisfied but the difference plays no role in this case. The ring $\mathcal{R}$ is *with shift only*, defined by the axioms (43), (44), (45). The Examples 2, 3 with (41) are of this case. The morphism/antimorphism of such rings is defined by (12), (13), (14), (51) or by (12), (15), (14), (53).

The ring $\mathcal{P}$ is also with shift only, the commutation equations (59), (60) being

$$\begin{aligned} xa &= a^\zeta x, && (103)\\ ax &= xa^z, && (104) \end{aligned}$$

the higher commutation equations (61), (62)

$$\begin{aligned} x^m a &= a^{\zeta^m} x^m, && (105)\\ ax^m &= x^m a^{z^m}. && (106) \end{aligned}$$

The conversions between left and right coefficients (65), (66) are

$$\begin{aligned} [a]_k &= ((a)_k)^{z^k}, && (107)\\ (a)_k &= ([a]_k)^{\zeta^k}, && (108) \end{aligned}$$

the formulae for multiplication (95), (97)

$$\begin{aligned} (ab)_k &= \sum_l (a)_l((b)_{k-l})^{\zeta^l}, && (109)\\ [ab]_k &= \sum_l ([a]_{k-l})^{z^l}[b]_l. && (110) \end{aligned}$$

$$\begin{array}{lcccccccc}
(ab)_0 & = & (a)_0(b)_0 & & & & & & \\
(ab)_1 & = & (a)_0(b)_1 & + & (a)_1(b)_0^\zeta & & & & \\
(ab)_2 & = & (a)_0(b)_2 & + & (a)_1(b)_1^\zeta & + & (a)_2(b)_0^{\zeta^2} & & \\
(ab)_3 & = & & & (a)_1(b)_2^\zeta & + & (a)_2(b)_1^{\zeta^2} & + & (a)_3(b)_0^{\zeta^3} \\
(ab)_4 & = & & & & & (a)_2(b)_2^{\zeta^2} & + & (a)_3(b)_1^{\zeta^3} \\
(ab)_5 & = & & & & & & & (a)_3(b)_2^{\zeta^3}
\end{array}$$

$$\begin{array}{lcccccccc}
[ab]_0 & = & [a]_0[b]_0 & & & & & & \\
[ab]_1 & = & [a]_0^z[b]_1 & + & [a]_1[b]_0 & & & & \\
[ab]_2 & = & [a]_0^{z^2}[b]_2 & + & [a]_1^z[b]_1 & + & [a]_2[b]_0 & & \\
[ab]_3 & = & & & [a]_1^{z^2}[b]_2 & + & [a]_2^z[b]_1 & + & [a]_3[b]_0 \\
[ab]_4 & = & & & & & [a]_2^{z^2}[b]_2 & + & [a]_3^z[b]_1 \\
[ab]_5 & = & & & & & & & [a]_3^{z^2}[b]_2
\end{array}$$

The second special case is a ring $\mathcal{R}$ with shift and difference, where $a^{\varsigma} = a$ for every a. The axioms of shift (43), (44), (45), (48) are satisfied but the shift plays no role in this case. The ring $\mathcal{R}$ is with difference only, which is called *derivation* and denoted $()^{p}$. A ring with derivation is defined by the following axioms

$$(a+b)^{p} = a^{p} + b^{p}, \tag{111}$$

$$(ab)^{p} = a^{p}b + ab^{p}. \tag{112}$$

It is motivated by a derivative of time function $f(t)$, t real, $-\infty < t < \infty$:

$$f^{p}(t) = \frac{\mathrm{d}f(t)}{\mathrm{d}t}\,. \tag{113}$$

Example 4. Functions $f(t)$ of real variable $-\infty < t < \infty$, having derivatives of all order, with pointwise addition and multiplication and with (113).

Example 5. Similarly, functions $f(t)$ of $0 \le t \le L$ with constraints $\mathrm{d}^{m}f(0)/\mathrm{d}t^{m} = \mathrm{d}^{m}f(L)/\mathrm{d}t^{m}$ for all m. Equivalently: functions on a circular manifold, once more equivalently: periodic function with the period L.

The properties of higher derivations (49), (50) are

$$(a+b)^{p^{m}} = a^{p^{m}} + b^{p^{m}}, \tag{114}$$

$$(ab)^{p^{m}} = \sum_{n=0}^{m} \binom{m}{n} a^{p^{m-n}} b^{p^{n}}. \tag{115}$$

The morphism/antimorphism of rings with derivation is defined by (12), (13), (14) and

$$(a^{p})^{\phi} = (a^{\phi})^{p}, \tag{116}$$

or by (12), (15), (14) and

$$(a^{p})^{\phi} = -(a^{\phi})^{p}. \tag{117}$$

The ring $\mathcal{P}$ is also with derivation, the commutation equations (59), (60) being

$$xa = a^{p} + ax, \tag{118}$$

$$ax = -a^{p} + xa, \tag{119}$$

the higher ones (61), (62)

$$x^{m}a = \sum_{n=0}^{m} \binom{m}{n} a^{p^{m-n}} x^{n}, \tag{120}$$

$$ax^{m} = \sum_{n=0}^{m} \binom{m}{n} (-1)^{m-n} x^{n} a^{p^{m-n}}. \tag{121}$$

The conversions (65), (66) are

$$[a]_k = \sum_{l=k}^{\deg a} \begin{pmatrix} l \\ k \end{pmatrix} (-1)^{l-k}((a)_l)^{p^{l-k}}, \tag{122}$$

$$(a)_k = \sum_{l=k}^{\deg a} \begin{pmatrix} l \\ k \end{pmatrix} ([a]_l)^{p^{l-k}}, \tag{123}$$

the formulae for multiplication (95), (97)

$$(ab)_k = \sum_m \sum_l \begin{pmatrix} l \\ m \end{pmatrix} (a)_l((b)_{k-m})^{p^{l-m}}, \tag{124}$$

$$[ab]_k = \sum_m \sum_l \begin{pmatrix} l \\ m \end{pmatrix} (-1)^{l-m}([a]_{k-m})^{p^{l-m}}[b]_l. \tag{125}$$

$$\begin{array}{lclclclcl}
(ab)_0 & = & (a)_0(b)_0 & + & (a)_1(b)_0^p & + & (a)_2(b)_0^{p^2} & + & (a)_3(b)_0^{p^3} \\
(ab)_1 & = & (a)_0(b)_1 & + & (a)_1(b)_1^p & + & (a)_2(b)_1^{p^2} & + & (a)_3(b)_1^{p^3} \\
 & & & + & (a)_1(b)_0 & + & 2(a)_2(b)_0^p & + & 3(a)_3(b)_0^{p^2} \\
(ab)_2 & = & (a)_0(b)_2 & + & (a)_1(b)_2^p & + & (a)_2(b)_2^{p^2} & + & (a)_3(b)_2^{p^3} \\
 & & & + & (a)_1(b)_1 & + & 2(a)_2(b)_1^p & + & 3(a)_3(b)_1^{p^2} \\
 & & & & & + & (a)_2(b)_0 & + & 3(a)_3(b)_0^p \\
(ab)_3 & = & & & (a)_1(b)_2 & + & 2(a)_2(b)_2^p & + & 3(a)_3(b)_2^{p^2} \\
 & & & & & + & (a)_2(b)_1 & + & 3(a)_3(b)_1^p \\
 & & & & & & & + & (a)_3(b)_0 \\
(ab)_4 & = & & & & & (a)_2(b)_2 & + & 3(a)_3(b)_2^p \\
 & & & & & & & + & (a)_3(b)_1 \\
(ab)_5 & = & & & & & & & (a)_3(b)_2
\end{array}$$

$$\begin{array}{lclclcl}
[ab]_0 & = & [a]_0[b]_0 & - & [a]_0^p[b]_1 & + & [a]_0^{p^2}[b]_2 \\
[ab]_1 & = & [a]_1[b]_0 & - & [a]_1^p[b]_1 & + & [a]_1^{p^2}[b]_2 \\
 & & & + & [a]_0[b]_1 & - & 2[a]_0^p[b]_2 \\
[ab]_2 & = & [a]_2[b]_0 & - & [a]_2^p[b]_1 & + & [a]_2^{p^2}[b]_2 \\
 & & & + & [a]_1[b]_1 & - & 2[a]_1^p[b]_2 \\
 & & & & & + & [a]_0[b]_2 \\
[ab]_3 & = & [a]_3[b]_0 & - & [a]_3^p[b]_1 & + & [a]_3^{p^2}[b]_2 \\
 & & & + & [a]_2[b]_1 & - & 2[a]_2^p[b]_2 \\
 & & & & & + & [a]_1[b]_2 \\
[ab]_4 & = & & & [a]_3[b]_1 & - & 2[a]_3^p[b]_2 \\
 & & & & & + & [a]_2[b]_2 \\
[ab]_5 & = & & & & & [a]_3[b]_2
\end{array}$$

7. APPLICATION IN SYSTEM AND CONTROL THEORY

In the system and control theory, the rings play a role as rings of operators, acting on signals. The signals (time functions) form a linear space over a field (real numbers). In the discrete-time case, the signals are two-sided sequences $y(t)$, $t = \dots -2T, -T, 0, T, 2T \dots$. The basic operators are that of shift ζ:

$$\zeta y(t) = y(t-T) \tag{126}$$

and that of difference ∇:

$$\nabla y(t) = \frac{y(t) - y(t-T)}{T}. \tag{127}$$

The time invariant systems are described by the equation (1) where A, B are polynomials in ζ or in ∇, coefficients A_k, B_k being real numbers.

For time varying systems, the coefficients are functions of time $A_k(t), B_k(t)$, $t = \dots -2T, -T, 0, T, 2T \dots$. They form a (commutative) ring with shift $()^\zeta$ and difference $()^\nabla$, given by (41), (42). The operators A, B form the ring $\mathcal{P}$ of skew polynomials over $\mathcal{R}$, the role of the element x played by the operator ∇. For a coefficient $a(t)$, the equality of signals

$$\nabla[a(t)\, y(t)] \quad = \quad a^\nabla(t)\, y(t) + a^\zeta(t)\, \nabla y(t) \tag{128}$$

means the equality of operators

$$\nabla a(t) \quad = \quad a^\nabla(t) + a^\zeta(t)\, \nabla. \tag{129}$$

Now it is clear that ∇ does not commute with $a(t)$ but satisfies the commutation equation (59).

Alternatively, the operators A, B can be thought as skew polynomials of the element x', whose role is played by the operator ζ. From (126), (127), we have $\zeta = 1 - T\nabla$, i.e. in the language of Theorem 10, $x' = 1 - Tx$. So the corresponding new difference ∇' is zero, the ring is with shift only. For a coefficient $a(t)$, the equality of signals

$$\zeta[a(t)\, y(t)] = a^\zeta(t)\, \zeta y(t) \tag{130}$$

means the equality of operators

$$\zeta a(t) = a^\zeta(t)\, \zeta, \tag{131}$$

with accord to the commutation equation (103).

The transformations (39) between the powers of ζ and ∇ are

$$\zeta^k \quad = \quad \sum_{l=0}^{k} \left(\begin{array}{c} k \\ l \end{array} \right) (-T\nabla)^l, \tag{132}$$

$$\nabla^k \quad = \quad \frac{1}{T^k} \sum_{l=0}^{k} \left(\begin{array}{c} k \\ l \end{array} \right) (-\zeta)^l, \tag{133}$$

the transformations (40) between the left/right coefficients a_k in ∇ and a'_k in ζ are

$$a_k = (-T)^k \sum_{l=k}^{\deg a} \binom{l}{k} a'_l, \tag{134}$$

$$a'_k = (-1)^k \sum_{l=k}^{\deg a} \binom{l}{k} \frac{a_l}{T^l}. \tag{135}$$

Note that our general approach in Sections 4, 5 can cover a case of nonuniform sampling ($T \neq$ const) as well.

In the continuous-time case, the signals are functions of real variable $y(t)$, $-\infty < t < \infty$ of some class, say, piecewise continuously differentiable of all orders, where the Dirac impulses $\delta^{(m)}$ are also allowed in the discontinuity points. The basic operator is that of (distributive) derivative p:

$$py(t) = \frac{\mathrm{d}y(t)}{\mathrm{d}t}. \tag{136}$$

For time varying systems, the coefficients are functions of real variable $A(t), B(t)$, $-\infty < t < \infty$, say, continuously differentiable of all orders. They form a (commutative) ring with derivation $()^p$ given by (113). The operators A, B are skew polynomials in p. For a coefficient $a(t)$, the equality of signals

$$p[a(t)\, y(t)] = a^p(t)\, y(t) + a(t)\, p\, y(t) \tag{137}$$

yields the equality of operators

$$pa(t) = a^p(t) + a(t)\, p \tag{138}$$

explaining the commutation equation (118).

The transformations in Theorems 5, 10 having $T_0 = 0$ are $p' = T_1 p$, $a^{p'} = T_1 a^p$.They correspond to transformations of time: given a function $t(t')$, monotonic and differentiable of all orders, it is $\frac{\mathrm{d}}{\mathrm{d}t'} = \frac{\mathrm{d}t}{\mathrm{d}t'} \cdot \frac{\mathrm{d}}{\mathrm{d}t}$, i.e. $T_1 = \frac{\mathrm{d}t}{\mathrm{d}t'}$.

A comment is necessary in this place. The signals and operators are considered here in the two-sided infinite horizon $-\infty < t < \infty$. For a one-sided one $0 \leq t < \infty$ or for a finite one $0 \leq t \leq L$, further modifications of rings and polynomials are necessary.

For LQ control problems, the adjoint operator $a^\star$ plays an important role. It is defined $a^\star(\zeta) = a(\zeta^{-1})$ or $a^\star(p) = a(-p)$. The operation $()^\star$ is an antimorphism (15) in the ring of operators. This is why the antimorphisms have been treated in this paper.

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