KYBERNETIKA — VOLUME 32 (1996), NUMBER 1, PAGES 17–42

DYNAMIC DISTURBANCE DECOUPLING FOR NONLINEAR DISCRETE-TIME SYSTEMS

THOMAS FLIEGNER AND HENK NIJMEIJER

In this paper we study the dynamic disturbance decoupling problem for nonlinear discrete-time systems that are considered in a neighbourhood of a given reference trajectory. Furthermore the connection between the solvability of this problem and the solvability of the corresponding problem for the time-varying linear discrete-time system obtained by linearizing the original system along the given reference trajectory is investigated. For this purpose, a geometric disturbance decoupling theory for time-varying linear discrete-time systems is developed.

1 INTRODUCTION

The problem of decoupling disturbances from the outputs of a given dynamical control system by means of *static* (DDP) or *dynamic state feedback* (DDDP) has received a lot of attention.

Complete solutions have been obtained for time-invariant linear systems, both in continuous and discrete time (see [1], [16], [26], [27]). Remarkably, the seemingly more restrictive problem of disturbance decoupling by static state feedback is, for this class of systems, solvable if and only if the disturbance decoupling problem can be solved by means of a dynamic state feedback (cf. [2]). In the nonlinear continuous-time setting, the derivation of solvability conditions proved to be harder. Nevertheless, local results have been obtained for special classes of nonlinear systems, provided certain regularity conditions are met (cf. [7], [8], [15]). In contrast with linear systems, however, the application of dynamic state feedback enlarged the class of disturbance decouplable nonlinear systems ([8], [10], [12], [18], [23]).

It is worth mentioning that the methods which are used to solve the problem fundamentally differ according to whether static or dynamic feedback is to be applied. Disturbance decoupling by static state feedback relies on the concepts of *controlled invariant* (or [A,B]-*invariant*) *subspaces* and *controlled invariant distributions*, respectively, depending on whether linear or nonlinear systems are considered. Controlled invariant subspaces have been introduced in [1] and [26] and their nonlinear counterparts in [15] and [7]. The resulting solvability conditions are basically of geometric nature. In dynamic disturbance decoupling, a crucial role is played by the so-called *Singh algorithm* and the resulting *Singh compensator*. The algorithm was introduced by Singh in connection with calculating left inverses for nonlinear systems (see e. g. [25]) and is a generalization of the algorithms in [24] and [6]. It turned out that this algorithm together with a *linear algebraic framework* developed in [3] gave a lot of insight into the system's structure and helped to solve DDDP for a class of nonlinear systems. The derived solvability conditions in terms of Singh's algorithm have then been translated into geometric language (cf. [8], [23]).

A similar but time-delayed development took place for *discrete-time systems*. It is known that in the linear setting there is no essential difference in solving DDP for time-invariant continuous- and discrete-time systems (cf. the corresponding chapters in [27] and [16]). Further steps in the development of a discrete-time theory are along the lines described above; generalization of the concepts of *invariance* and *controlled invariance* to the nonlinear discrete-time setting (see [4], [20]), deducing local solvability conditions for disturbance decoupling by means of static state feedback in terms of controlled invariant distributions (cf. [4], [22]), extension of Singh's algorithm to discrete-time systems in e. g. [18] and solving the dynamic disturbance decoupling problem locally about an equilibrium point of the system (see [18] and [17]). The last step turned out to be necessary since, similar to the continuous-time nonlinear case, the class of disturbance decouplable discrete-time nonlinear systems can be enlarged by allowing for dynamic state feedback.

Until now little is known about disturbance decoupling for time-varying systems. A geometric theory for a class of time-varying linear continuous-time systems can be found in [13], [14], but no discrete-time counterpart was derived.

The present paper aims to investigate the relation between the solvability of the dynamic disturbance decoupling problem for a nonlinear discrete-time system that is given in the neighbourhood of a reference trajectory and the solvability of the same problem for the linear time-varying system obtained by linearizing the original nonlinear system along a given reference trajectory. This approach is motivated by the understandable wish to study the solvability of nonlinear control problems by means of the linearization about a working point or trajectory as to simplify considerations. An example of this in discrete time may be found in [19].

Connected with the main objective of the paper, we first develop a geometric disturbance decoupling theory for time-varying linear discrete-time systems that essentially parallels the more familiar time-invariant case. In a second step we extend results obtained in [17], [18] concerning dynamic disturbance decoupling for nonlinear discrete-time systems that are given in the neighbourhood of an equilibrium point of the system to the case of systems that are given in a neighbourhood of a reference trajectory of the system. With this as preparation we are eventually able to discuss the mutual relation between the solvability of the dynamic disturbance decoupling problem for the nonlinear system and its linearization.

The organization of the paper closely follows these three steps. In Section 2 we develop a geometric disturbance decoupling theory for time-varying linear discrete-time systems. Section 3 is dedicated to the solution of the dynamic disturbance decoupling problem for nonlinear discrete-time systems that are given in the neighbourhood of a reference trajectory of the system. Solvability conditions are formulated in terms of Singh's algorithm. Finally, in Section 4 a connection is made between the proceeding sections by alternatively characterizing the solvability of the nonlinear dynamic disturbance decoupling problem by means of the solvability of the same problem for the linear time-varying system arising from linearizing the original nonlinear system along the given reference trajectory.

2. DISTURBANCE DECOUPLING FOR LINEAR TIME–VARYING SYSTEMS

2.1. Definitions and Formulation of the Problem

In this section we develop a geometric disturbance decoupling theory for time-varying linear discrete-time systems similar to the time-invariant case. As usual $\mathbb{N}$ denotes the set of natural numbers (including 0) and $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$. In the sequel we use bold face letters to denote sequences of objects defined for all $\mathbb{N}$. Thus for instance $\boldsymbol{q} = \{q(k)\}_{k\geq 0}$ denotes the sequence of disturbances, $\boldsymbol{V} = \{\mathcal{V}(k)\}_{k\geq 0}$ refers to a sequence of subspaces of the state space $\mathcal{X}$ and so on. Subsequences beginning with a $k_0 > 0$ are indexed correspondingly ($\boldsymbol{q}_{k\geq k_0}$, $\boldsymbol{V}_{k\geq k_0}, \ldots$).

Consider the following time-varying linear discrete-time system:

$$\Sigma_0 : \begin{cases} x(k+1) &= A(k)\,x(k) + E(k)\,q(k) \\ y(k) &= C(k)\,x(k) \end{cases}$$

where $\{q(k)\}$, $k \in \mathbb{N}$ represents a sequence of unknown disturbances. Furthermore, for all k, $x(k) \in \mathcal{X} = \mathbb{R}^n$, $q(k) \in \mathbb{R}^r$ and $y(k) \in \mathbb{R}^p$. $A(k)$, $E(k)$ and $C(k)$ are matrices of appropriate sizes. For a given initial condition $x_0 = x(k_0) \in \mathcal{X}$, $k_0 \in \mathbb{N}$, and disturbances $\boldsymbol{q}_{k\geq k_0}$, the output of the system is given by:

$$y(k) = C(k)\,x(k) = C(k)\,\Phi_{k,k_0}x_0 + C(k)\sum_{l=k_0}^{k-1}\Phi_{k,l+1}E(l)\,q(l), \quad k \geq k_0 \tag{1}$$

where the summation term on the right hand side is understood to be 0 for $(k-1) < k_0$ and $\Phi_{k,l}$ denotes the transition matrix which, for $0 \leq l \leq k$, is defined as follows:

$$\Phi_{k,l} = A(k-1)\,A(k-2)\ldots A(l), \quad \Phi_{l,l} = I\,. \tag{2}$$

Definition 2.1. The system Σ_0 is said to be disturbance decoupled if for every $k_0 \in \mathbb{N}$ and arbitrary sequence of disturbances $\boldsymbol{q}_{k\geq k_0}$, the output $\boldsymbol{y}_{k\geq k_0}$ is only depending on the initial state $x_0 = x(k_0)$.

Comparing with (1), this is obviously equivalent to

$$C(k)\sum_{l=k_0}^{k-1}\Phi_{k,l+1}E(l)\,q(l) = 0, \quad k > k_0 \tag{3}$$

for arbitrary $q_{k \geq k_0}$. Condition (3) can be given in a more geometric way as follows:

$$\operatorname{im} E(k-1) + \Phi_{k,k-1} \operatorname{im} E(k-2) + \ldots + \Phi_{k,k_0+1} \operatorname{im} E(k_0) \subseteq \ker C(k), \quad k > k_0 . \quad (4)$$

Unfortunately, a system of the form Σ_0 usually does not enjoy this property. One can then try to make it disturbance decoupled by either applying static or dynamic state feedback (also referred to as *dynamic compensation*). This requires the possibility of changing the dynamics of the system, usually modelled by adding a linear control term. Consider, therefore, the linear control system

$$\Sigma : \left\{ \begin{array}{rcl} x(k+1) & = & A(k)\,x(k) + B(k)\,u(k) + E(k)\,q(k) \\ y(k) & = & C(k)\,x(k) \end{array} \right.$$

with the notation as above and additionally $u(k) \in \mathcal{U} = \mathbb{R}^m$ and compatible matrices $B(k)$. We are now in the position to formulate the two disturbance decoupling problems which we are concerned with in the sequel.

Definition 2.2. The Static Disturbance Decoupling Problem (DDP) consists in finding a static feedback law $u(k) = F(k)\,x(k)$ such that in the closed loop system

$$\Sigma_{BF} : \left\{ \begin{array}{rcl} x(k+1) & = & [A(k) + B(k)F(k)]\,x(k) + E(k)\,q(k) \\ y(k) & = & C(k)\,x(k) \end{array} \right.$$

the outputs are not influenced by the disturbances.

Remark 2.3. In linear static feedback control, one usually applies feedbacks of the form $u(k) = F(k)\,x(k) + G(k)\,v(k)$ considering $v(k)$ as new inputs that can be used for further control design. We omit this term because it is not relevant to our purposes.

Definition 2.4. The Dynamic Disturbance Decoupling Problem (DDDP) consists in finding a dynamic compensator

$$C : \left\{ \begin{array}{rcl} z(k+1) & = & P(k)z(k) + Q(k)\,x(k) \\ u(k) & = & S(k)z(k) + T(k)\,x(k) \end{array} \right.$$

with $z(k) \in \mathbb{R}^\nu$ and the matrices $P(k), Q(k), S(k)$, and $T(k)$ of appropriate sizes such that the compensated system

$$\Sigma \circ C : \begin{bmatrix} x(k+1) \\ z(k+1) \end{bmatrix} = \begin{bmatrix} A(k) + B(k)\,T(k) & B(k)\,S(k) \\ Q(k) & P(k) \end{bmatrix} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} + \begin{bmatrix} E(k) \\ 0 \end{bmatrix} q(k)$$

$$y(k) = [C(k) \;\; 0] \begin{bmatrix} x(k) \\ z(k) \end{bmatrix}$$

is disturbance decoupled.

2.2. The Static Disturbance Decoupling Problem (DDP)

In this subsection, we define the concepts ***A**-invariant* and *controlled-invariant* sequences $\boldsymbol{V}$, respectively, and solve the disturbance decoupling problem by static state feedback. Furthermore the connection between the solvability of DDP and DDDP is investigated.

Consider the system

$$x(k+1) = A(k)\,x(k)\,.$$

Definition 2.5. A sequence $\boldsymbol{V}$ of subspaces of the state space $\mathcal{X}$ is called $\boldsymbol{A}$-invariant if

$$A(k)\mathcal{V}(k) \subseteq \mathcal{V}(k+1) \quad \text{for all} \quad k \in \mathbb{N}\,.$$

Now consider the system

$$x(k+1) = A(k)\,x(k) + B(k)\,u(k)\,.$$

We define controlled (or $[\boldsymbol{A}, \boldsymbol{B}]$-)invariance in the following way.

Definition 2.6. A sequence $\boldsymbol{V}$ of subspaces of the state space $\mathcal{X}$ is said to be controlled-invariant if for every $k_0 \in \mathbb{N}$ and arbitrary $x_0 \in \mathcal{V}(k_0)$, there exists an input $u_0 \in \mathcal{U}$ such that $x_{u_0}(k_0+1) \in \mathcal{V}(k_0+1)$ where $x_{u_0}(k_0+1) := A(k_0)\,x_0 + B(k_0)\,u_0$.

Remark 2.7. We note that Definition 2.6 could be equivalently given by saying that for arbitrary $k_0 \in \mathbb{N}$ and $x_0 \in \mathcal{V}(k_0)$, there exists a sequence of inputs $\boldsymbol{u}_{k \geq k_0}$ such that the resulting states $x(k)$ satisfy $x(k) \in \mathcal{V}(k),\ k > k_0$.

Besides this open loop definition we have similar to the linear time-invariant case

Lemma 2.8. The following statements are equivalent:

(i) $\boldsymbol{V}$ is controlled-invariant.

(ii) $A(k)\mathcal{V}(k) \subseteq \mathcal{V}(k+1) + \operatorname{im} B(k)$ for all $k \in \mathbb{N}$.

(iii) There exists a family $\boldsymbol{F}$ of linear maps $F(k) : \mathcal{X} \to \mathcal{U}$ such that for all $k \in \mathbb{N}$ $[A(k) + B(k)\,F(k)]\,\mathcal{V}(k) \subseteq \mathcal{V}\ (k+1)$.

P r o o f. (i) $\rightarrow$ (ii): Let for an arbitrary $k \in \mathbb{N}$ an $x_0 \in \mathcal{V}(k)$ be given. By definition of controlled invariance there exists a $u_0 \in \mathcal{U}$ such that $x_{u_0}(k+1) \in \mathcal{V}(k+1)$. (ii) then follows immediately from $A(k)\,x_0 = x_{u_0}(k+1) - B(k)\,u_0$.
(ii) $\rightarrow$ (iii): Let $\{b_1(k), \ldots, b_{l_k}(k)\}$ be a basis of $\mathcal{V}(k)$. Extend it to a basis $\{b_1(k), \ldots \ldots, b_n(k)\}$ of $\mathcal{X}$. By (ii) there exist $b_i(k+1) \in \mathcal{V}(k+1),\ 1 \leq i \leq l_k$, and $\{u_1(k), \ldots, u_{l_k}(k)\}$ such that $A(k)\,b_i(k) = b_i(k+1) + B(k)\,u_i(k),\ 1 \leq i \leq l_k$. Now

define $F(k) : \mathcal{X} \to \mathcal{U}$ by $F(k)\, b_i(k) = -u_i(k),\ 1 \le i \le l_k$, and $F(k)\, b_i(k)$ arbitrary vectors in $\mathcal{U}$ for $i > l_k$.
(iii) $\to$ (i): Let $x_0 \in \mathcal{V}(k_0)$ be arbitrary. Define $u_0 = F(k_0)\, x_0$. Then

$$x_{u_0}(k_0+1) = A(k_0)\, x_0 + B(k_0)\, u_0 = [A(k_0) + B(k_0)F(k_0)]\, x_0 \in \mathcal{V}(k_0+1)$$

which shows the controlled invariance of $\boldsymbol{V}$ and concludes the proof. □

With the help of the concept of $\boldsymbol{A}$-invariance, we can characterize a disturbance decoupled system in the following way.

Lemma 2.9. The system $x(k+1) = A(k)\, x(k) + E(k)\, q(k), y(k) = C(k)\, x(k)$ is disturbance decoupled if and only if there exists an $\boldsymbol{A}$-invariant sequence $\boldsymbol{V}$ such that

$$\operatorname{im} E(k-1) \subseteq \mathcal{V}(k) \subseteq \ker C(k) \quad \text{for all} \quad k \in \mathbb{N}_+ .$$

Proof. (only if) Let the system be disturbance decoupled. Then, using (4) with $k_0 = 0$, it holds for all $k \in \mathbb{N}_+$

$$\operatorname{im} E(k-1) + \Phi_{k,k-1} \operatorname{im} E(k-2) + \ldots + \Phi_{k,1} \operatorname{im} E(0) \subseteq \ker C(k) .$$

Now define

$$\mathcal{V}(k) = \operatorname{im} E(k-1) + \Phi_{k,k-1} \operatorname{im} E(k-2) + \ldots + \Phi_{k,1} \operatorname{im} E(0) .$$

Obviously

$$\operatorname{im} E(k-1) \subseteq \mathcal{V}(k) \subseteq \ker C(k) .$$

It remains to show that $\boldsymbol{V}$ is $\boldsymbol{A}$-invariant. To show this, consider

$$\begin{aligned} A(k)\mathcal{V}(k) &= A(k) \operatorname{im} E(k-1) + A(k)\, \Phi_{k,k-1} \operatorname{im} E(k-2) + \ldots + A(k)\, \Phi_{k,1} \operatorname{im} E(0) \\ &= \Phi_{k+1,k} \operatorname{im} E(k-1) + \Phi_{k+1,k-1} \operatorname{im} E(k-2) + \ldots + \Phi_{k+1,1} \operatorname{im} E(0) \\ &\subseteq \operatorname{im} E(k) + \Phi_{k+1,k} \operatorname{im} E(k-1) + \ldots + \Phi_{k+1,1} \operatorname{im} E(0) \\ &= \mathcal{V}(k+1) . \end{aligned}$$

(if) By induction we show that for an arbitrary $k_0 \in \mathbb{N}$

$$\sum_{l=k_0}^{k-1} \Phi_{k,l+1} \operatorname{im} E(l) \subseteq \mathcal{V}(k) \subseteq \ker C(k), \ k > k_0 .$$

For $k = k_0 + 1$ we have

$$\Phi_{k_0+1,k_0+1} \operatorname{im} E(k_0) = \operatorname{im} E(k_0) \subseteq \mathcal{V}(k_0+1) \subseteq \ker C(k_0+1) .$$

Let now $\sum_{l=k_0}^{k-1} \Phi_{k,l+1} \operatorname{im} E(l) \subseteq \mathcal{V}(k) \subseteq \ker C(k)$ for $k > k_0 + 1$. Then

$$\begin{aligned} \sum_{l=k_0}^{k} \quad \Phi_{k+1,l+1} \operatorname{im} E(l) &= \operatorname{im} E(k) + \sum_{l=k_0}^{k-1} \Phi_{k+1,l+1} \operatorname{im} E(l) \\ &= \quad \operatorname{im} E(k) + A(k) \sum_{l=k_0}^{k-1} \Phi_{k,l+1} \operatorname{im} E(l) . \end{aligned}$$

Since $\operatorname{im} E(k) \subseteq \mathcal{V}(k+1)$ by assumption and $\sum_{l=k_0}^{k-1} \Phi_{k,l+1} \operatorname{im} E(l) \subseteq \mathcal{V}(k)$, we get

$$\sum_{l=k_0}^{k} \Phi_{k+1,l+1} \operatorname{im} E(l) \subseteq \mathcal{V}(k+1) \subseteq \ker C(k+1) .$$

It follows by (4) that the system is disturbance decoupled. □

The question whether or not a system is disturbance decouplable by a static state feedback is completely answered by the following lemma.

Lemma 2.10. There exists a sequence $\boldsymbol{F}$ of maps $F(k) : \mathcal{X} \to \mathcal{U}$ such that system Σ_{BF} is disturbance decoupled if and only if there exists a controlled-invariant sequence $\boldsymbol{V}$ of subspaces of $\mathcal{X}$ such that

$$\operatorname{im} E(k-1) \subseteq \mathcal{V}(k) \subseteq \ker C(k), \quad k \in \mathbb{N}_+ .$$

Proof. The proof is similar to that of the previous lemma. □

Remark 2.11. In [13], [14] disturbance decoupling for time-varying linear continuous-time systems with piecewise analytic system matrices is considered. Although the results obtained there are basically similar to those of this article, things appear to be much more involved.

For time-invariant linear systems it is known that the class of systems for which DDP is solvable coincides with the class of systems for which DDDP is solvable (cf. [2]). The same holds true if one considers the discrete-time case. In the next lemma we show that the situation is the same for time-varying linear discrete-time systems as well.

Lemma 2.12. For time-varying linear discrete-time systems Σ, DDP is solvable if and only if DDDP is solvable.

Proof. (only if) Trivial.

(if) Let there exist a compensator C such that the compensated system $\Sigma \circ C$ is disturbance decoupled. Define

$$A_e(k)=\begin{bmatrix} A(k)+B(k)T(k) & B(k)S(k) \\ Q(k) & P(k) \end{bmatrix}, \quad E_e(k)=\begin{bmatrix} E(k) \\ 0 \end{bmatrix}, \quad C_e(k)=[C(k) \;\; 0].$$

By Lemma 2.9 there exists an $\boldsymbol{A}_e$-invariant sequcnce $\boldsymbol{V}_e$ of subspaces of the extended state space $\mathcal{X} \times \mathcal{Z}$ such that

$$\operatorname{im} E_e(k-1) \subseteq \mathcal{V}_e(k) \subseteq \ker C_e(k), \quad k = 1, 2, \ldots .$$

Consider the projection $\mathcal{V}(k)$ of $\mathcal{V}_e(k)$ along $\mathcal{Z}$, that is

$$\mathcal{V}(k) = \pi_1 \mathcal{V}_e(k) = \{x \in \mathcal{X} : \exists\, z_x \in \mathcal{Z} \text{ such that } (x, z) \in \mathcal{V}_e(k)\}.$$

First observe that by definition of $\mathcal{V}(k)$, we obviously have

$$\operatorname{im} E(k-1) \subseteq \mathcal{V}(k) \subseteq \ker C(k).$$

Since

$$A_e(k)\, \mathcal{V}_e(k) \subseteq \mathcal{V}_e(k+1),$$

we have

$$[A(k) + B(k)\, T(k)]\, x + B(k)\, S(k) z_x \in \mathcal{V}(k+1) \text{ for all } x \in \mathcal{V}(k).$$

This implies

$$(A + BT)\,(k)\, \mathcal{V}(k) \subseteq \mathcal{V}(k+1) + \operatorname{im} B(k).$$

The latter means that $\boldsymbol{V}$ is $[\boldsymbol{A}+\boldsymbol{BT}, \boldsymbol{B}]$-invariant. By Lemma 2.8 there exist maps $\tilde{F}(k)$ such that

$$(A + BT\tilde{F})\,(k)\, \mathcal{V}(k) \subseteq \mathcal{V}(k+1) \text{ for all } k = 0, 1, 2, \ldots .$$

Lemma 2.9 now says that the system $\Sigma_{BT\tilde{F}}$ is disturbance decoupled. The required static feedback which decouples Σ is given by $u(k) = T(k)\, \tilde{F}(k)\, x(k)$. □

2.3. The Existence of a Maximal Controlled Invariant Sequence

For time-invariant systems influenced by disturbances, the effectiveness of the approach relies on the existence of an algorithm (the so-called *Invariant Subspace Algorithm*) which enables one to compute the (always existing) maximal controlled-invariant subspace $\mathcal{V}^*$ contained in $\ker C$. The decision whether or not a given system is disturbance decouplable then comes down to check $\operatorname{im} E \subseteq \mathcal{V}^*$.

In this section we are going to show that there exists a maximal controlled-invariant sequence $\boldsymbol{V}^*$ of subspaces of the state space $\mathcal{X}$ contained in $\ker \boldsymbol{C}$ as well where inclusion is to be understood pointwise. Before we do this we give some definitions.

Definition 2.13. Let M be a non-empty set. A partial order on M is a binary relation, denoted by $\leq$, with the following properties:

1. For all $x \in M$, $x \leq x$. (reflexivity)
2. For all x, $y \in M$, $x \leq y$ and $y \leq x$ implies $x = y$. (antisymmetry)
3. For all x, y, $z \in M$, $x \leq y$ and $y \leq z$ implies $x \leq z$. (transitivity)

A nonempty set M on which there exists a partial order is called partially ordered.

Definition 2.14. If M is a partially ordered set and if $m \in M$ has the property that $m \leq x$ implies $m = x$, then m is called maximal element in M.

The definition of a partially ordered set does not require that all elements of this set are comparable. A partially ordered set in which every pair of elements is comparable is called *totally* or *linearly ordered.* Any totally ordered subset of a partially ordered set is called a *chain*.

Finally, an element u of a partially ordered set M is called an *upper bound* for a subset V of M if $v \leq u$ for all $v \in V$.

With the help of these concepts we can state Zcrn's lemma.

Throrem 2.15. Let M be a partially ordered set in which every chain has an upper bound. Then M has a maximal element.

Let $\ker \boldsymbol{C}$ denote the sequence $\{\ker C(0), \ker C(1), \ldots\}$. Introduce a partial order on the set of all sequences of subspaces of $\mathcal{X}$ by $\boldsymbol{V}^1 \leq \boldsymbol{V}^2 \Leftrightarrow \mathcal{V}^1(k) \subseteq \mathcal{V}^2(k)$ for all $k = 0, 1, 2, \ldots$. Finally, define $\Omega := \{\boldsymbol{V} : \boldsymbol{V} \leq \ker \boldsymbol{C},\ \boldsymbol{V} \text{ controlled-invariant}\}$.

Theorem 2.16. $\ker \boldsymbol{C}$ contains a maximal controlled-invariant sequence $\boldsymbol{V}^*$ of subspaces of $\mathcal{X}$.

Proof. Let $\boldsymbol{V}^1$, $\boldsymbol{V}^2$, $\boldsymbol{V}^3, \ldots$ be a chain in Ω. Let furthermore without loss of generality

$$\boldsymbol{V}^1 \leq \boldsymbol{V}^2 \leq \boldsymbol{V}^3 \leq \cdots .$$

It follows $\mathcal{V}^i(k) \subseteq \mathcal{V}^j(k)$ for all $k \geq 0$ and $i \leq j$. For all $k \geq 0$, the finite dimension of $\mathcal{X}$ assures the existence of a minimal number l_k such that

$$\mathcal{V}^1(k) \subset \mathcal{V}^2(k) \subset \cdots \subset \mathcal{V}^{l_k}(k) = \mathcal{V}^{l_k+1}(k) = \cdots .$$

Define $\boldsymbol{V} := \{\mathcal{V}^{l_0}(0), \mathcal{V}^{l_1}(1), \mathcal{V}^{l_2}(2), \ldots\}$. Obviously we have $\boldsymbol{V} \leq \ker \boldsymbol{C}$. Moreover from

$$A(k)\, \mathcal{V}^{l_k}(k) \subseteq \mathcal{V}^{l_k}(k+1) + \operatorname{im} B(k)$$

follows

$$A(k)\, \mathcal{V}^{l_k}(k) \subseteq \mathcal{V}^{l_{k+1}}(k+1) + \operatorname{im} B(k)$$

because for $l_k \geq l_{k+1}$ we have

$$\mathcal{V}^{l_k}(k+1) = \mathcal{V}^{l_{k+1}}(k+1)$$

and for $l_k \leq l_{k+1}$,

$$\mathcal{V}^{l_k}(k+1) \subset \mathcal{V}^{l_{k+1}}(k+1).$$

So the controlled invariance of $\boldsymbol{V}$ follows. Since by construction $\boldsymbol{V}^i \leq \boldsymbol{V}$, $\boldsymbol{V}$ is maximal in the considered chain. Zorn's Lemma then guarantees the existence of a maximal element $\boldsymbol{V}^*$ in Ω, that is, for every $\boldsymbol{V} \in \Omega$ which is comparable with $\boldsymbol{V}^*$ follows $\boldsymbol{V} \leq \boldsymbol{V}^*$. Suppose now there is a $\boldsymbol{V} \in \Omega$ which is not comparable with $\boldsymbol{V}^*$. Define $\boldsymbol{W} := \boldsymbol{V} + \boldsymbol{V}^*$. $\boldsymbol{W}$ is again an element of Ω and $\boldsymbol{V}^* \leq \boldsymbol{W}$ which contradicts the maximal element property of $\boldsymbol{V}^*$. Consequently, $\boldsymbol{V}^*$ is maximum of Ω. □

2.4. A modified invariant subspace algorithm

In the foregoing subsection we have proved the existence of a maximal controlled-invariant sequence $\boldsymbol{V}^*$ of subspaces of the state space $\mathcal{X}$ contained in $\ker \boldsymbol{C}$. The same proof applies of course if $\ker \boldsymbol{C}$ is replaced by any sequence $\boldsymbol{K}$ of subspaces.

The question arises how one can actually obtain the maximal controlled-invariant sequence $\boldsymbol{V}^*(\boldsymbol{K})$ contained in $\boldsymbol{K}$. It will turn out that an algorithm similar to the *Invariant Subspace Algorithm* for time-invariant systems can be employed in order to compute $\boldsymbol{V}^*(\boldsymbol{K})$.

Recall that, given a linear time invariant system

$$x(k+1) = Ax(k) + Bu(k)$$

and a subspace $\mathcal{K} \subseteq \mathcal{X}$, the largest controlled-invariant subspace $\mathcal{V}^*(\mathcal{K})$ in $\mathcal{K}$ can be calculated via the following algorithm:

$$\mathcal{V}^0 := \mathcal{K} \qquad \mathcal{V}^{i+1} := \mathcal{K} \cap A^{-1}(\mathcal{V}^i + \operatorname{im} B).$$

This algorithm terminates after at most $l = \dim\ \mathcal{K}$ steps and $\mathcal{V}^l = \mathcal{V}^*(\mathcal{K})$.

For linear time-varying systems, this algorithm is slightly modified. Let $\boldsymbol{K}$ be given and denote by σ the forward time-shift operator. Perform the following algorithm:

$$\boldsymbol{V}^0 := \boldsymbol{K} \qquad \boldsymbol{V}^{i+1} := \boldsymbol{K} \cap \boldsymbol{A}^{-1}(\sigma \boldsymbol{V}^i + \operatorname{im} \boldsymbol{B})$$

where as usual all operations are to be understood pointwise. It is clear that this algorithm produces a family of subspaces contained in $\boldsymbol{K}$ satisfying

$$\boldsymbol{V}^0 \geq \boldsymbol{V}^1 \geq \cdots \geq \boldsymbol{V}^l = \boldsymbol{V}^{l+1} = \cdots.$$

Moreover we have

Theorem 2.17. The sequence of subspaces $\boldsymbol{V}^l$ obtained in the final step of the algorithm is controlled-invariant. Any other sequence $\boldsymbol{W}$ of controlled-invariant subspaces of $\boldsymbol{K}$ satisfies $\boldsymbol{W} \leq \boldsymbol{V}^l$ or, put another way, $\boldsymbol{V}^l = \boldsymbol{V}^*(\boldsymbol{K})$.

Proof. To prove the controlled invariance we have to show, for arbitrary $x_0 \in \mathcal{V}^l(k)$, the existence of a $u_0 \in \mathcal{U}$ such that $A(k)\,x_0 + B(k)\,u_0 \in \mathcal{V}^l(k+1)$. Let therefore an arbitrary $x_0 \in \mathcal{V}^l(k)$ be given. According to the algorithm we have $x_0 \in \mathcal{K}(k)$ and $x_0 \in A^{-1}(k)\,(\mathcal{V}^{l-1}(k+1) + \operatorname{im} B(k))$.

$$\Longrightarrow \quad A(k)\,x_0 \;+\; B(k)\,u_0 \;\in\; \mathcal{V}^{l-1}(k+1) \supseteq \mathcal{V}^l(k+1)$$

for some $u_0 \in \mathcal{U}$. Suppose now that for all such u_0

$$A(k)\,x_0 \;+\; B(k)\,u_0 \in \mathcal{V}^{l-1}(k+1) \setminus \mathcal{V}^l(k+1).$$

$$\Longrightarrow \quad A(k)\,x_0 \notin \mathcal{V}^l(k+1) + \operatorname{im} B(k) \Longrightarrow x_0 \notin \mathcal{V}^{l+1}(k)$$

which contradicts $\mathcal{V}^{l+1}(k) = \mathcal{V}^l(k)$ and, hence, proves the controlled invariance of $\boldsymbol{V}^l$.

Let now be given another controlled-invariant sequence $\boldsymbol{W}$ in $\boldsymbol{K}$ and consider an arbitrary $x_0 \in \mathcal{W}(k) \subseteq \mathcal{K}(k)$. We have to show that $x_0 \in \mathcal{V}^l(k)$ as well. Since $\boldsymbol{W}$ is supposed to be controlled-invariant, we can find a $u_0 \in \mathcal{U}$ such that

$$A(k)\,x_0 + B(k)\,u_0 \in \mathcal{W}(k+1) \subseteq \mathcal{K}(k+1) \quad \text{resp.} \quad x_0 \in A^{-1}(k)\,(\mathcal{K}(k+1) + \operatorname{im} B(k)).$$

It follows that x_0 is also an element of $\mathcal{V}^1(k)$ or equivalently $\mathcal{W}(k) \subseteq \mathcal{V}^1(k)$ for all $k = 0, 1, 2, \ldots$ because the arbitrary choice of k. This especially means that the element $A(k)\,x_0 + B(k)\,u_0 \in \mathcal{W}(k+1)$ also belongs to $\mathcal{V}^1(k+1)$ and consequently $x_0 \in \mathcal{V}^2(k)$. Repeating this argument finally yields $x_0 \in \mathcal{V}^l(k)$.

$$\Longrightarrow \mathcal{W}(k) \subseteq \mathcal{V}^l(k) \;\; \text{for all } k \geq 0 \Longrightarrow \boldsymbol{W} \leq \boldsymbol{V}^l = \boldsymbol{V}^*(\boldsymbol{K})$$

which proves the statement. □

Remark 2.18. It is evident that this method of computing the maximal controlled-invariant sequence of subspaces relies very much on the discrete-time assumption such that it is not clear how to generalize it to the continuous-time setting. In this case the maximal controlled-invariant family of subspaces can be computed algorithmically utilizing the duality between the concepts controlled invariance and conditioned invariance (cf. [13], [14]).

We conclude this section with an example.

Example 2.19. Consider the following mathematical system:

$$x(k+1) = A(k)\,x(k) + B(k)\,u(k) + E(k)\,q(k), \quad y(k) = C(k)\,x(k) \quad \text{where}$$

$$A(k) = \begin{bmatrix} \frac{(k+2)^2}{k+1} & 0 & 0 \\ 0 & -k^2 & -k^2 \\ 0 & -(k+2) & (k+1) \end{bmatrix}, \quad B(k) = \begin{bmatrix} 0 \\ (k+1) \\ 0 \end{bmatrix},$$

$$E(k) = \begin{bmatrix} (k+2)^2 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad C(k) = \begin{bmatrix} 1 & 0 & (k+1)^2 \end{bmatrix}.$$

Pointwise, $\ker C(k)$ is given by

$$\ker C(k) = \operatorname{span}_{\mathbb{R}}\{\ (0 \ \ 1 \ \ 0)^T\ ,\ ((k+1)^2 \ \ 0 \ \ -1)^T\ \}.$$

Performing the algorithm introduced above we define first

$$\mathcal{V}^0(k) = \ker C(k)$$

and compute

$$\mathcal{V}^1(k) = \ker C(k) \cap A^{-1}(k)\,(\mathcal{V}^0(k+1) + \operatorname{im} B(k))\,.$$

This may be done by investigating which images under $A(k)$ of elements of $\ker C(k)$ lie in $(\mathcal{V}^0(k+1) + \operatorname{im} B(k))$. Since the general form of an element $x \in \ker C(k)$ is

$$x = (a_1(k+1)^2, a_2, -a_1)^T$$

a_1, $a_2 \in \mathbb{R}$, and obviously

$$(\mathcal{V}^0(k+1) + \operatorname{im} B(k)) = \mathcal{V}^0(k+1)$$

this leads to the equations

$$\begin{bmatrix} a_1(k+2)^2(k+1) \\ (a_1 - a_2)\,k^2 \\ -a_1(k+1) - a_2(k+2) \end{bmatrix} = \begin{bmatrix} b_1(k+2)^2 \\ b_2 \\ -b_1 \end{bmatrix}$$

which imply $b_1 = a_1(k+1)$ and $-b_1 = -a_2(k+2) - a_1(k+1)$. The two last equations yield $a_2 = 0$ which on its turn implies

$$\mathcal{V}^1(k) = \operatorname{span}_{\mathbb{R}}\{((k+1)^2 \ \ 0 \ \ -1)\}\,.$$

We start the next step by computing

$$\begin{aligned} (\mathcal{V}^1(k+1) + \operatorname{im} B(k)) &= \operatorname{span}_{\mathbb{R}}\{((k+2)^2 \ \ 0 \ \ -1)^T, (0 \ \ (k+1) \ \ 0)^T\} \\ &= \operatorname{span}_{\mathbb{R}}\{((k+2)^2 \ \ 0 \ \ -1)^T, (0 \ \ 1 \ \ 0)^T\} \end{aligned}$$

from which we may immediately conclude

$$\mathcal{V}^2(k) = \mathcal{V}^1(k)$$

and so $\boldsymbol{V}^*(\ker \boldsymbol{C}) = \boldsymbol{V}^1$. One immediately verifies $\operatorname{im} E(k-1) \subseteq \mathcal{V}^*(k)$ and so the DDP is solvable. The computation of a decoupling feedback can now be done following the procedure in the proof of Lemma 2.8. A basis of $\mathcal{V}^*(k)$ is given by $q_1(k) = ((k+1)^2 \;\; 0 \;\; -1)^T$. This can be extended by the two vectors $q_2(k) = (0 \;\; 1 \;\; 1)^T$ and $q_3(k) = (0 \;\; 0 \;\; 1)^T$ to a basis of $\mathcal{X} = \mathbb{R}^3$. Next we have to find $u(k)$ such that

$$\begin{aligned} A(k)\, q_1(k) + B(k)\, u(k) &= \begin{bmatrix} (k+2)^2\,(k+1) \\ k^2 \\ -(k+1) \end{bmatrix} + \begin{bmatrix} 0 \\ (k+1)\, u(k) \\ 0 \end{bmatrix} \\ &\subseteq \operatorname{span}_{\mathbb{R}} \begin{bmatrix} (k+2)^2 \\ 0 \\ -1 \end{bmatrix} \end{aligned}$$

which certainly holds for $u(k) = \frac{-k^2}{k+1}$. $F(k) = [a(k) \;\; b(k) \;\; c(k)]$ can be defined by setting $F(k)\, q_1(k) = \frac{-k^2}{k+1}$ which imposes the restriction

$$a(k)\,(k+1)^2 - c(k) = \frac{-k^2}{k+1} \quad \text{or} \quad c(k) = \frac{a(k)\,(k+1)^3 + k^2}{k+1}\,.$$

Choosing $a(k) = b(k) = 0$, we get $F(k) = [0 \;\; 0 \;\; \frac{k^2}{k+1}]$. The resulting matrices $[A(k) + B(k)\, F(k)]$ differ from $A(k)$ in element (2,3) which turns 0 in the closed loop matrix. One immediately verifies that

$$[A(k) + B(k)\, F(k)]\, q_1(k) \in \mathcal{V}^1(k+1)\,.$$

3. THE (REGULAR) DYNAMIC DISTURBANCE DECOUPLING PROBLEM FOR NONLINEAR DISCRETE-TIME SYSTEMS

3.1. Definitions and Problem Formulation

In this section we are concerned with the solution of the *(regular) dynamic disturbance decoupling problem (DDDP)* for nonlinear discrete-time systems which are given in a neighbourhood of a reference trajectory. Consider, therefore, the following system:

$$\Sigma : \begin{cases} x(k+1) &= f(x(k),\, u(k),\, q(k)), \quad x_0 = x(0) \\ y(k) &= h(x(k)) \end{cases}$$

where the states $x(k)$ belong to an open part $\mathcal{X}$ of $\mathbb{R}^n$, the inputs $u(k)$ are in some open part $\mathcal{U}$ of $\mathbb{R}^m$, the unmeasurable disturbances $q(k)$ take their values in some open $\mathcal{W} \subseteq \mathbb{R}^r$, and the outputs $y(k)$ belong to some open part $\mathcal{Y}$ of $\mathbb{R}^p$. f and h are supposed to be real analytic mappings. Let us assume furthermore that their exists a reference trajectory for Σ, that is, a set of time functions $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{u}}, \overline{\boldsymbol{q}}, \overline{\boldsymbol{y}}) \in \mathcal{X}^{\mathbb{N}} \times \mathcal{U}^{\mathbb{N}} \times \mathcal{W}^{\mathbb{N}} \times \mathcal{Y}^{\mathbb{N}}$ that satisfies the system equations (for the meaning of bold face letters see Section 2.1). The restriction of the system to a neighbourhood of a reference trajectory (including equilibrium points) is the usual way to deal with the inherent nonlocal character of discrete-time systems. By a proper choice of the initial state x_0 and input sequence $\boldsymbol{u}$ one can so keep the system's states and outputs (at least up to a finite time instant k_F) in the neighbourhood of known points in $\mathcal{X} \times \mathcal{Y}$ given by the sequences $\overline{\boldsymbol{x}}$ and $\overline{\boldsymbol{y}}$, provided the disturbances $\boldsymbol{q}$ stay close to $\overline{\boldsymbol{q}}$. This way, the application of local methods become possible. We now define the problem which will concern us in the sequel.

Definition 3.1. The Regular dynamic disturbance decoupling problem (DDDP) consists in finding a regular dynamic compensator

$$R: \begin{cases} z(k+1) &= \psi(z(k),\, x(k),\, v(k)), \quad z_0 = z(0) \\ u(k) &= \phi_k(z(k),\, x(k),\, v(k)) \end{cases} \tag{5}$$

with ν-dimensional states $z(k)$ and new inputs $v(k)$ of dimension m, defined locally around a set of time functions $(\overline{\boldsymbol{z}}, \overline{\boldsymbol{x}}, \overline{\boldsymbol{v}}, \overline{\boldsymbol{u}})$ that satisfy the compensator equations such that in the compensated system $\Sigma \circ R$ the disturbances do not influence the outputs for $0 \leq k \leq k_F$.

Here the term 'regular' refers to the invertibility of the relation between the inputs $v(k)$ and outputs $u(k)$ of the control system given by R (cf. [18]).

We remark that the restriction to 'finite-time decoupling' is not imposed by properties of the considered system but by the mathematical apparatus used; we will comment on this issue later on in the construction of the Singh compensator.

Instrumental in the solution of the formulated problem is the so-called *Singh compensator*. This compensator is constructed via *Singh's algorithm* which has been introduced in [25]. A modified version of this algorithm introduced in [3] is used nowadays in the treatment of numerous synthesis problems for continuous-time systems (cf. [8], [10], [11], [12]).

Essentially there are two versions of Singh's algorithm for discrete-time systems. One has been introduced in [5] and parallels conceptually very much the continuous-time algorithm. The advantage here is that by transition to the differentials of all involved signals, computations are linearized. Furthermore, results obtained via this algorithm are easier to interpret. On the other side, a globally defined system is required and termination of the algorithm may take $2n$ steps. We, therefore, decided to choose the version introduced in [18] which is recapitulated in the next section.

3.2. Singh's algorithm

In the sequel $i \in \mathcal{I}_{jl}$ for $j \leq l$ is a short hand notation for $j \leq i \leq l$. Moreover, we consider the system $\Sigma_{\overline{\boldsymbol{q}}}$ obtained from Σ by keeping the disturbances fixed to $\overline{\boldsymbol{q}}$. Perform Singh's algorithm about every point $(\overline{x}(k),\, \overline{u}(k),\, \overline{q}(k))$, $k = 0, 1, \ldots, k_F$. Observe that we apply the algorithm only in the neighbourhood of the finite time path $(\overline{x}(k), \overline{u}(k), \overline{q}(k))|_{0 \leq k \leq k_F}$ in connection with the problems already mentioned. 'For all k' is to be understood accordingly.

STEP 1

Calculate $y(k+1) = h[f(x(k), u(k), \overline{q}(k))]$ and define

$$\rho^1(k) := \text{rank } D_u(h \circ f)\,(x, u, \overline{q}(k)) \ .$$

Let us assume that there exist neighbourhoods $\mathcal{O}^1(k)$ of $(\overline{x}(k), \overline{u}(k))$ in which $\rho^1(k) =: \rho^1$ is constant for every k. Moreover, assume that the independent rows of $D_u(h \circ f)\,(x, u, \overline{q}(k))$ are the same for every k. Permute, if necessary, the components of the output in such a way that the first ρ^1 rows of the matrix $D_u(h \circ f)\,(x, u, \overline{q}(k))$ are linearly independent and decompose $h[f(x(k), u(k), \overline{q}(k))]$ and $y(k+1)$ accordingly.

$$y(k+1) = \begin{bmatrix} \tilde{y}^1(k+1) \\ \hat{y}^1(k+1) \end{bmatrix}, \quad h[f(x(k), u(k), \overline{q}(k))] = \begin{bmatrix} \tilde{a}^1(x(k), u(k), \overline{q}(k)) \\ \hat{a}^1(x(k), u(k), \overline{q}(k)) \end{bmatrix}$$

where $\tilde{y}^1(k+1)$ and $\tilde{a}^1(x(k),\, u(k),\, \overline{q}(k))$ consist of the first ρ^1 (independent) components of $y(k+1)$ and $h[f(x(k),\, u(k), \overline{q}(k))]$, respectively. Since the last $p - \rho^1$ rows of the matrix $D_u(h \circ f)\,(x, u, \overline{q}(k))$ are linearly dependent on the first ρ^1 rows, the corresponding components of h and y, respectively, viewed as functions of u and with parameters x (and $\overline{q}$), are functionally dependent on the first ρ^1 components. Hence, we can write

$$\begin{aligned} \tilde{y}^1(k+1) &= \tilde{a}^1(x(k), u(k), \overline{q}(k)) \\ \hat{y}^1(k+1) &= \hat{a}^1(x(k), u(k), \overline{q}(k)) = \psi^1(x(k), \overline{q}(k), \tilde{y}^1(k+1)) \,. \end{aligned}$$

Denote $\tilde{a}^1(x(k), u(k), \overline{q}(k))$ by $A^1(x(k), u(k), \overline{q}(k))$.
End of **STEP 1**

STEP l+1

Suppose that in Steps 1 through to l, $\tilde{y}^1(k+1), \tilde{y}^2(k+2), \ldots, \tilde{y}^l(k+l)$, $\hat{y}^l(k+l)$ have been defined in such a way that

$$\begin{aligned} \tilde{y}^1(k+1) &= \tilde{a}^1(x(k), u(k), \overline{q}(k)) \\ \tilde{y}^2(k+2) &= \tilde{a}^2(x(k), u(k), \overline{q}(k), \overline{q}(k+1), \tilde{y}^1(k+2)) \\ &\ \vdots \\ \tilde{y}^l(k+l) &= \tilde{a}^l(x(k), u(k), \{\overline{q}(k+i) : i \in \mathcal{I}_{0l-1}\}, \{\tilde{y}^i(k+j) : i \in \mathcal{I}_{1l-1}, j \in \mathcal{I}_{i+1l}\}) \\ \hat{y}^l(k+l) &= \psi^l(x(k), \{\overline{q}(k+i) : i \in \mathcal{I}_{0l-1}\}, \{\tilde{y}^i(k+j) : i \in \mathcal{I}_{1l}, j \in \mathcal{I}_{il}\}) \,. \end{aligned}$$

Suppose also that the matrix $D_u A^l = D_u[\tilde{a}^{1^T}, \ldots, \tilde{a}^{l^T}]^T$ has full row rank ρ^l in some neighbourhood $\mathcal{O}^l(k)$ of $(\overline{x}(k), \overline{u}(k), \{\overline{\tilde{y}}^i(k+j) : i \in \mathcal{I}_{1l-1}, j \in \mathcal{I}_{i+1l}\})$. Compute

$$\begin{aligned} \hat{y}^l(k+l+1) &= \psi^l(x(k+1), \{\overline{q}(k+i+1) : i \in \mathcal{I}_{0l-1}\}, \{\tilde{y}^i(k+j+1) : i \in \mathcal{I}_{1l},\ j \in \mathcal{I}_{il}\}) \\ &= \psi^l(f(x(k), u(k), \overline{q}(k)), \cdots) \\ &=: a^{l+1}(x(k), u(k), \{\overline{q}(k+i) : i \in \mathcal{I}_{0l}\}, \{\tilde{y}^i(k+j) : i \in \mathcal{I}_{1l},\ j \in \mathcal{I}_{i+1l+1}\}) \end{aligned}$$

and define

$$\rho^{l+1}(k) := \text{rank } D_u \begin{bmatrix} A^l(\cdot) \\ a^{l+1}(\cdot) \end{bmatrix}.$$

Let us assume that there exist neighbourhoods $\mathcal{O}^{l+1}(k)$ of $(\overline{x}(k), \overline{u}(k), \{\overline{\tilde{y}^i}(k+j) : i \in \mathcal{I}_{1l}, j \in \mathcal{I}_{i+1l+1}\})$ in which $\rho^{l+1}(k) =: \rho^{l+1}$ is constant for every k. Moreover assume that the independent rows of $D_u[A^{l^T}, a^{l+1^T}]^T$ are the same for all k. Permute, if necessary, the components of $\hat{y}^l(k+l+1)$ such that the first ρ^{l+1} rows of the matrix $D_u[A^{l^T}, a^{l+1^T}]^T$ are linearly independent. Decompose $\hat{y}^l(k+l+1)$ and a^{l+1} according to

$$\hat{y}^l(k+l+1) = \begin{bmatrix} \tilde{y}^{l+1}(k+l+1) \\ \hat{y}^{l+1}(k+l+1) \end{bmatrix}, \quad a^{l+1} = \begin{bmatrix} \tilde{a}^{l+1} \\ \hat{a}^{l+1} \end{bmatrix}.$$

Then we have

$$\begin{aligned} \tilde{y}^1(k+1) &= \tilde{a}^1(x(k), u(k), \overline{q}(k)) \\ \tilde{y}^2(k+2) &= \tilde{a}^2(x(k), u(k), \overline{q}(k), \overline{q}(k+1), \tilde{y}^1(k+2)) \\ &\vdots \\ \tilde{y}^{l+1}(k+l+1) &= \tilde{a}^{l+1}(x(k), u(k), \{\overline{q}(k+i) : i \in \mathcal{I}_{0l}\}, \{\tilde{y}^i(k+j) : i \in \mathcal{I}_{1l}, j \in \mathcal{I}_{i+1l+1}\}) \\ \hat{y}^{l+1}(k+l+1) &= \psi^{l+1}(x(k), \{\overline{q}(k+i) : i \in \mathcal{I}_{0l}\}, \{\tilde{y}^i(k+j) : i \in \mathcal{I}_{1l+1}, j \in \mathcal{I}_{il+1}\}). \end{aligned}$$

Denote $A^{l+1} := [A^{l^T}, \tilde{a}^{l+1^T}]^T$.

End of **STEP l+1**

The application of the algorithm is certainly not unique because one has in general a multiple choice in selecting functionally independent components. Moreover, it is not sure at all if the imposed assumptions are satisfied in each step of the algorithm. In order to proceed we have to define a notion of regularity associated with the given trajectory $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{u}}, \overline{\boldsymbol{q}}, \overline{\boldsymbol{y}})$.

Definition 3.2. The reference trajectory $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{u}}, \overline{\boldsymbol{q}}, \overline{\boldsymbol{y}})$ is said to be regular if there is an application of Singh's algorithm such that all assumptions made in performing the algorithm are satisfied. It is said to be strongly regular if this holds true for an arbitrary application of the algorithm.

In [17] it has been shown that around a regular reference trajectory $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{u}}, \overline{\boldsymbol{q}}, \overline{\boldsymbol{y}})$ the algorithm terminates in at most n steps, that is, if one were to continue the algorithm then $\rho^{n+j} = \rho^n$ for all integers $j \geq 0$. In the sequel let $\rho^* := \max\{\rho^l,\ l \geq 1\}$ and define α as the smallest $l \in \mathbb{N}$ such that $\rho^l = \rho^*$. Moreover we assume that we are working around a strongly regular reference trajectory.

It can be shown that around a strongly regular reference trajectory the integers $\rho^1, \ldots, \rho^*$, the so-called invertibility indices, do not depend on the particular permutation of the components of $\hat{y}^l(k+l+1)$. Thus, using this algorithm around a strongly regular trajectory, we obtain a uniquely defined sequence of integers $0 \leq \rho^1 \leq \ldots \leq \rho^* \leq \min(m, p)$.

3.3. Construction of the Singh compensator

Applying the algorithm around $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{u}}, \overline{\boldsymbol{q}}, \overline{\boldsymbol{y}})$ yields at the αth step:

$$\tilde{Y}^{\alpha}(k+\alpha) = A^{\alpha}(x(k), u(k), \{\overline{q}(k+i) : i \in \mathcal{I}_{0\alpha-1}\}, \{\tilde{y}^{i}(k+j) : i \in \mathcal{I}_{1\alpha-1}, j \in \mathcal{I}_{i+1\alpha}\}) \quad (6)$$

$$\hat{y}^{\alpha}(k+\alpha) = \hat{y}^{\alpha}(x(k), \{\overline{q}(k+i) : i \in \mathcal{I}_{0\alpha-1}\}, \{\tilde{y}^{i}(k+j) : i \in \mathcal{I}_{1\alpha}, j \in \mathcal{I}_{i\alpha}\}) \quad (7)$$

where $\tilde{Y}^{\alpha} = [\tilde{y}^{1^T}(k+1), \tilde{y}^{2^T}(k+2), \ldots, \tilde{y}^{\alpha^T}(k+\alpha)]^T$ and the matrix $D_u A^{\alpha}$ has full row rank ρ^* in neighbourhoods $\mathcal{O}^{\alpha}(k)$ of $(\overline{x}(k), \overline{u}(k), \{\overline{\tilde{y}^{i}}(k+j) : i \in \mathcal{I}_{1\alpha-1}, j \in \mathcal{I}_{i+1\alpha}\})$. For $i = 1, 2, \ldots, \rho^*$, let $k + \gamma_i$ be the lowest time instant and $k + \delta_i$ the highest time instant at which $\tilde{y}_i$ appears in (6). Then we can write (6) for $l = 1, \ldots, \alpha$ as:

$$\begin{bmatrix} \tilde{y}_{\rho^{l-1}+1}(k+l) \\ \vdots \\ \tilde{y}_{\rho^{l}}(k+l) \end{bmatrix}$$

$$= \tilde{a}^{l}(x(k), u(k), \{\overline{q}(k+i) : i \in \mathcal{I}_{0l-1}\}, \{\tilde{y}_i(k+j) : i \in \mathcal{I}_{1\rho^{l-1}}, j \in \mathcal{I}_{\gamma_i+1 min(l,\delta_i)}\}). \quad (8)$$

After a possible permutation of the inputs we may assume that the Jacobian matrix of (8) with respect to $u^1 := (u_1, \ldots, u_{\rho^*})^T$ has full row rank ρ^* about the points

$$p(k) := (\overline{x}(k), \overline{u}(k), \{\overline{q}(k+i) : i \in \mathcal{I}_{0\alpha}\}, \{\overline{\tilde{y}}_i(k+j) : i \in \mathcal{I}_{1\rho^*}, j \in \mathcal{I}_{\gamma_i+1\delta_i}\}).$$

Therefore, (8) can be uniquely solved for $u^1(k)$ about $p(k)$ by applying the Implicit Function Theorem. Defining $u^2 := (u_{\rho^*+1}, \ldots, u_m)$, we obtain from (8)

$$u^1(k) = \phi_k(x(k), \{\overline{q}(k+i) : i \in \mathcal{I}_{0\alpha-1}\}, \{\tilde{y}_i(k+j) : i \in \mathcal{I}_{1\rho^{\alpha-1}}, j \in \mathcal{I}_{\gamma_i+1\delta_i}\}, u^2(k))$$

which is such that (8) is satisfied identically when $u^1(k) = \phi_k(\cdot)$ is re-substituted into (8). Notice that, no matter how the initial state x_0 and inputs $u(\cdot)$ are chosen, the resulting trajectories will in general drift away from the reference trajectory. Therefore the solvability of (8) for u^1 can only be guaranteed up to a finite time k_F.

A Singh compensator is constructed in the following way. Let

$$z_i = (z_{i,1}, \ldots, z_{i,\delta_i-\gamma_i})^T, \; i \in \mathcal{I}_{1\rho^*}$$

be $(\delta_i - \gamma_i)$-dimensional vectors, v^2 a vector of dimension $(m - \rho^*)$ and consider the system S with inputs $v^1 = (v_1, \ldots, v_{\rho^*})^T$, v^2 and outputs (u^1, u^2)

$$\left.\begin{array}{rcl} z_{i,1}(k+1) & = & z_{i,2}(k) \\ & \vdots & \\ z_{i,\delta_i-\gamma_i-1}(k+1) & = & z_{i,\delta_i-\gamma_i}(k) \\ z_{i,\delta_i-\gamma_i}(k+1) & = & v_i(k) \end{array}\right., \qquad i = 1, \ldots, \rho^* \quad (9)$$

where the output equations are given by

$$u^1(k) = \phi_k(x(k), \{\overline{q}(k+i) : i \in \mathcal{I}_{1\alpha-1}\}, \{z_{i,j}(k) : j \in \mathcal{I}_{1\delta_i-\gamma_i}, v_i(k) : i \in \mathcal{I}_{1\rho^*}\}, v^2(k)) \quad (10)$$
$$u^2(k) = v^2(k).$$

Notice that by construction of the compensator, a reference trajectory is given by $(\overline{z}(k), \overline{x}(k), \overline{v}(k), \overline{u}(k))$ for $0 \le k \le k_F$ where

$$\overline{z} = (\overline{z}_{1,1}, \ldots, \overline{z}_{1,\delta_1-\gamma_1}, \ldots, \overline{z}_{\rho^*,\delta_{\rho^*}-\gamma_{\rho^*}})^T, \ \overline{z}_{i,j}(k) = \bar{\tilde{y}}_i(k+\gamma_i+j-1), \ \overline{v}_i(k) = \bar{\tilde{y}}_i(k+\delta_i)$$

and

$$\overline{v}^2(k) = \overline{u}^2(k), \quad i = 1, \ldots, \rho^*, \quad j = 1, \ldots, \delta_i - \gamma_i .$$

It may be shown that the so defined compensator is regular in a neighbourhood of the trajectory defined above (see [17]). Applying the compensator $(9, 10)$ with arbitrary initial state to $\Sigma_{\overline{q}}$ results locally about

$$(\overline{x}(k), \overline{u}(k), \{\overline{q}(k+i) : i \in \mathcal{I}_{1\alpha-1}\}, \{\bar{\tilde{y}}_i(k+j) : i \in \mathcal{I}_{1\rho^*}, j \in \mathcal{I}_{\gamma_i\delta_i}\})$$

in the following modification of the output equations:

$$\begin{array}{rcll} \tilde{y}_i(k+\gamma_i+j-1) & = & z_{i,j}(k), & j = 1, \ldots, \delta_i - \gamma_i \\ \tilde{y}_i(k+\delta_i) & = & v_i(k), & 0 \le k \le k_F \end{array} \quad i = 1, \ldots, \rho^* . \tag{11}$$

Moreover, inspection of Singh's algorithm reveals that for the compensated system the outputs $\tilde{y}_i(0), \ldots, \tilde{y}_i(\gamma_i - 1)$, $i = 1, \ldots, \rho^*$, only depend on the initial conditions x_0, z_0, and $\overline{q}$.

Having studied Singh's algorithm and compensator in some detail, deriving conditions for the solvability of the announced disturbance decoupling problem turns out to be rather straightforward.

3.4. Solution of the regular dynamic disturbance decoupling problem

Performing Singh's algorithm gives in each step l a function ψ^l representing the functionally dependent part of $\hat{y}(k+l)$. Considering the effects of applying the constructed compensator to the output components $\tilde{y}_i$ (see Section 3.3), it comes as no surprise that the ψ^l's play a crucial role in solving the dynamic disturbance decoupling problem. Before stating the main result of this section, let us define $\psi^0(x) := h(x)$.

Theorem 3.3. Consider system Σ in a neighbourhood of a strongly regular reference trajectory $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{u}}, \overline{\boldsymbol{q}}, \overline{\boldsymbol{y}})$. Apply Singh's algorithm to Σ with $\boldsymbol{q} = \overline{\boldsymbol{q}}$. Then the regular dynamic disturbance decoupling problem is finite time solvable for Σ around the given reference trajectory if and only if

$$D_z\psi^l(z, \{\overline{q}(k+i+1), \tilde{y}^{i+1}(k+j+1) : i \in \mathcal{I}_{0l-1}, j \in \mathcal{I}_{il}\})\big|_{z=f(x,u,q)} D_q f(x,u,q) = 0 \tag{12}$$

for $0 \le l \le (n-1)$ and for all (x, u, q) in a neighbourhood of $(\overline{x}(k), \overline{u}(k), \overline{q}(k))$, $0 \le k \le k_F$. Moreover, DDDP can then be solved by means of the Singh compensator constructed in Section 3.3.

Proof. (if) Inspection of Singh's algorithm easily reveals that under assumption (12) the application of the algorithm to Σ ($\boldsymbol{q}$ not considered fixed) gives the same result as applying it to $\Sigma_{\overline{q}}$. Hence, applying compensator $(9, 10)$ decouples

$\tilde{y}_i(k)$, $i = 1, \ldots, \rho^*$, from the disturbances for a (possibly) finite time span (cf. (11)). Let $\alpha \le n$ be given. In case $\alpha \le (n-2)$, continuing Singh's algorithm through to $(n-1)$ does not further increase the number of functionally independent output components. For $\alpha \le l \le (n-1)$ it follows, recalling the equality $\psi^l(f(x(k), u(k), q(k)), \ldots) = \hat{y}^l(k+l+1)$, that

$$\psi^l(f(x(k), u(k), q(k)), \ldots) = \hat{y}^\alpha(k+l+1)\,.$$

Moreover, if (12) holds for $0 \le l \le (n-1)$, it will also hold for $l > (n-1)$. But that means

$$\hat{y}^1(k+1), \hat{y}^2(k+2), \ldots, \hat{y}^\alpha(k+\alpha), \hat{y}^\alpha(k+\alpha+1), \hat{y}^\alpha(k+\alpha+2), \ldots$$

are independent of the disturbances. Therefore (9, 10) solves the DDDP.

(only if) Let there exist an arbitrary regular compensator (5) which solves DDDP. Without giving details we remark that in case (12) is not satisfied, a contradiction with the assumed regularity of the compensator (5) occurs. □

4. NONLINEAR DDDP AND LINEARIZATION

In this section, we want to go into the question what can be said about the solvability of the nonlinear DDDP by means of addressing the same problem for the linearization of the original nonlinear system along a given reference trajectory. Basically, we show that under suitable 'regularity' conditions on the nonlinear system/reference trajectory the nonlinear DDDP is solvable locally around the reference trajectory if and only if the linear DDP is solvable for the corresponding linearized system. In other words, the solvability of the nonlinear problem can be decided by verifying the solvability of the associated linear problem which we have analyzed in Section 2. The analysis of the connection between the solvability of the nonlinear DDDP and the corresponding DDP of its linearization is done in terms of a careful study of the relation between Singh's algorithm for a nonlinear system and that for its linearization.

We again consider a nonlinear system Σ as defined in Section 3.1 around a strongly regular reference trajectory $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{u}}, \overline{\boldsymbol{q}}, \overline{\boldsymbol{y}})$ and furthermore its linearization Σ_{li} along this trajectory

$$\Sigma_{li} : \left\{ \begin{array}{rcl} x_{li}(k+1) & = & F(k)\, x_{li}(k) + G(k)\, u_{li}(k) + E(k)\, q_{li}(k) \\ y_{li}(k) & = & H(k)\, x_{li}(k) \end{array} \right. \tag{13}$$

where

$$\begin{array}{rcl} F(k) & = & D_x f(x,u,q)|_{\overline{x}(k),\overline{u}(k),\overline{q}(k)}\,, \\ G(k) & = & D_u f(x,u,q)|_{\overline{x}(k),\overline{u}(k),\overline{q}(k)}\,, \\ E(k) & = & D_q f(x,u,q)|_{\overline{x}(k),\overline{u}(k),\overline{q}(k)}\,, \qquad \text{and} \\ H(k) & = & D_x h(x)|_{\overline{x}(k)}\,. \end{array}$$

x_{li}, u_{li}, q_{li}, and y_{li} denote the first variations of x, u, q, and y, respectively.

4.1. Singh's algorithm and linearization

In this subsection, we prove some results concerning the connection between Singh algorithms for Σ and Σ_{li}. They can be seen as the discrete-time counterparts of results obtained in [9] for continuous-time systems.

The next lemma is very instrumental in the sequel for the analysis between the Singh algorithm for a nonlinear system and the same algorithm for its linearization. Its proof is given in the appendix.

Lemma 4.1. Consider the equations

$$\begin{aligned} \tilde{y} &= \tilde{a}(x,u,q) && (14)\\ \hat{y} &= \hat{a}(x,u,q) && (15) \end{aligned}$$

and let $(\overline{x},\overline{u},\overline{q},\overline{y})$ be such that they satisfy (14, 15) where $\overline{y} = (\bar{\tilde{y}}^T, \bar{\hat{y}}^T)^T$. Suppose that

$$\tilde{G} = D_u\tilde{a}(x,u,q)$$

has full row rank ρ in a neighbourhood of $(\overline{x},\overline{u},\overline{q})$ and that the rows of

$$\hat{G} = D_u\hat{a}(x,u,q)$$

are linearly dependent on the rows of $\tilde{G}$. Linearize (14, 15) about $(\overline{x},\overline{u},\overline{q})$ to obtain

$$\begin{aligned} \tilde{y}_{li} &= \tilde{F}x_{li} + \tilde{G}u_{li} + \tilde{E}q_{li}\\ \hat{y}_{li} &= \hat{F}x_{li} + \hat{G}u_{li} + \hat{E}q_{li}\,. \end{aligned}$$

Write, using the Implicit Function Theorem about $(\overline{x},\overline{u},\overline{q},\overline{y})$

$$\hat{y} = \psi(x,q,\tilde{y}) \tag{16}$$

and let $\tilde{G}^+$ be a right inverse of $\tilde{G}$. Then

$$\hat{y}_{li} = \hat{F}x_{li} + \hat{G}\tilde{G}^+[\tilde{y}_{li} - \tilde{F}x_{li} - \tilde{E}q_{li}] + \hat{E}q_{li} \tag{17}$$

and (17) can be obtained by linearizing (16).

With the help of this lemma one can show that there exists a close connection between the Singh algorithms for Σ and Σ_{li}.

Lemma 4.2. Consider a given system Σ in a neighbourhood of a strongly regular trajectory $(\overline{\boldsymbol{x}},\overline{\boldsymbol{u}},\overline{\boldsymbol{q}},\overline{\boldsymbol{y}})$. Apply Singh's algorithm to Σ yielding a permutation $\tilde{y}^1,\ldots,\tilde{y}^\alpha$ of the outputs such that for $1 \le l \le \alpha$

$$\begin{aligned} \tilde{y}^l(k+l) &= \tilde{a}^l(x(k),u(k),\{q(k+i) : i\in\mathcal{I}_{0l-1}\},\{\tilde{y}^i(k+j) : i\in\mathcal{I}_{1l-1}, j\in\mathcal{I}_{i+1l}\})\\ \hat{y}^l(k+l) &= \psi^l(x(k),\{q(k+i) : i\in\mathcal{I}_{0l-1}\},\{\tilde{y}^i(k+j) : i\in\mathcal{I}_{1l}, j\in\mathcal{I}_{il}\})\,. \end{aligned}$$

Then there exists an application of Singh's algorithm to Σ_{li} resulting in the same permutation of outputs and where the results in each step of Singh's algorithm applied to Σ_{li} can alternatively be obtained by linearizing the results in the corresponding step of Singh's algorithm applied to Σ around

$$(\overline{x}(k), \overline{u}(k), \{\overline{q}(k+i) : i \in \mathcal{I}_{0\alpha-1}\}, \{\overline{\tilde{y}}^i(k+j) : i \in \mathcal{I}_{1\alpha-1}, j \in \mathcal{I}_{i+1\alpha}\}) .$$

P r o o f. Considering the first step of Singh's algorithm for Σ yields

$$\tilde{y}^1(k+1) = \tilde{a}^1(x(k), u(k), q(k)) \tag{18}$$

$$\hat{y}^1(k+1) = \hat{a}^1(x(k), u(k), q(k)) = \psi^1(x(k), q(k), \tilde{y}^1(k+1)) . \tag{19}$$

For the linearization Σ_{li} one obtains

$$y_{li}(k+1) = H(k+1)F(k)\,x_{li}(k) + H(k+1)G(k)\,u_{li}(k) + H(k+1)E(k)\,q_{li}(k) .$$

Permuting the outputs of Σ_{li} in the same way as for Σ yields

$$\tilde{y}^1_{li}(k+1) = \tilde{H}(k+1)\,F(k)\,x_{li}(k) + \tilde{H}(k+1)G(k)\,u_{li}(k) + \tilde{H}(k+1)E(k)\,q_{li}(k) \tag{20}$$

$$\hat{y}^1_{li}(k+1) = \hat{H}(k+1)F(k)\,x_{li}(k) + \hat{H}(k+1)G(k)\,u_{li}(k) + \hat{H}(k+1)E(k)\,q_{li}(k) \tag{21}$$

where, by the strong regularity assumption on $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{u}}, \overline{\boldsymbol{q}}, \overline{\boldsymbol{y}})$, $\tilde{H}(k+1)G(k)$ has full row rank ρ^1.
Let $[\tilde{H}(k+1)G(k)]^+$ be a right inverse. Then (20, 21) gives

$$\begin{aligned} \hat{y}^1_{li}(k+1) &= \hat{H}(k+1)F(k)\,x_{li}(k) + \hat{H}(k+1)E(k)\,q_{li}(k) \\ &+ \hat{H}(k+1)G(k)[\tilde{H}(k+1)G(k)]^+ \\ &\times \{\tilde{y}^1_{li}(k+1) - \tilde{H}(k+1)F(k)\,x_{li}(k) - \tilde{H}(k+1)E(k)\,q_{li}(k)\} . \end{aligned} \tag{22}$$

Observe that for the pairs (18, 19) and (20, 21) we are exactly in the situation of Lemma 4.1. Therefore (22) can be obtained by linearizing ψ^1 around

$$(\overline{x}(k), \overline{u}(k), \overline{q}(k), \overline{\tilde{y}}^1(k+1)) .$$

In the second step of Singh's algorithm we have

$$\hat{y}^1(k+2) = \psi^1(f(x(k), u(k), q(k)), q(k+1), \tilde{y}^1(k+2)) \tag{23}$$

and $\hat{y}^1_{li}(k+2)$ is obtained from (22) by replacing k by $(k+1)$. One easily computes that $\hat{y}^1_{li}(k+2)$ can be obtained by linearizing (23) around $(\overline{x}(k), \overline{u}(k), \overline{q}(k), \overline{q}(k+1), \overline{\tilde{y}}^1(k+2))$. Utilizing again Lemma 4.1 one proves the statement for $l = 2$. Proceeding in the same way as above, it can be shown that the claim also holds for $l = 3, \ldots, \alpha$. □

From Lemma 4.2, we can derive the following result.

Theorem 4.3. Consider a system Σ defined in a neighbourhood of a strongly regular reference trajectory $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{u}}, \overline{\boldsymbol{q}}, \overline{\boldsymbol{y}})$ and let Σ_{li} be its linearization along this trajectory. Then

1. The linearization of a Singh compensator for Σ is a Singh compensator for Σ_{li}.
2. Conversely, each Singh compensator for Σ_{li} is a first order approximation of a Singh compensator for Σ.

4.2. Nonlinear DDDP and linearization

In this subsection we finally give the connection between the solvability of DDDP for a system Σ and the solvability of the corresponding problem for its linearization Σ_{li}.

Suppose DDDP is solvable for Σ. By Theorem 3.3 it follows that for all applications of Singh's algorithm to Σ, we have for $0 \le l \le (n-1)$

$$D_z\psi^l(z, \{\overline{q}(k+i+1),\, \tilde{y}^{i+1}(k+j+1) : i \in \mathcal{I}_{0l-1}, j \in \mathcal{I}_{il}\})\big|_{z=f(x,u,q)} D_q f(x,u,q) = 0.$$

We know by Lemma 4.2 that to each application of Singh's algorithm to Σ, there corresponds an application of Singh's algorithm to Σ_{li} which can be obtained by linearization. It is immediately clear that the conditions necessary for the solvability of DDDP with respect to Σ_{li} will also be satisfied.

Unfortunately, one cannot decide on the solvability of DDDP for Σ by means of the solvability of DDDP for Σ_{li} without additional assumptions. The difficulties that one faces are of the following nature. Let Σ_{li} be given and assume that DDDP is solvable for Σ_{li}. Again by Theorem 3.3 one has then

$$D_{q_{li}} y_{li}(k+1) = H(k+1)\, E(k) = D_q(h \circ f)\,(x,u,q)|_{\overline{x}(k),\overline{u}(k),\overline{q}(k)} = 0\,.$$

Hence, without further assumptions,

$$D_z\psi^0(z)\big|_{z=f(x,u,q)} D_q f(x,u,q) = D_q(h \circ f)\,(x,u,q) = 0$$

as one of the necessary assumptions for the solvability of DDDP for Σ can only be assured using Σ_{li} for the points of the reference trajectory for Σ. The same problem occurs step by step. We therefore have to impose additional conditions to overcome this problem. It is clear that these conditions can only be given in terms of the original system Σ. On the other hand, they should be such that their verification does not require performing Singh's algorithm for Σ. In order to obtain such conditions, we have to reconsider Singh's algorithm.

Applying Singh's algorithm to Σ_{li} results in a reordering of the output components $(y_{li})_i$. By Lemma 4.2 we know that there is an application of Singh's algorithm to Σ which results in the same reordering of the output components y_i of Σ and which is such that the results obtained with respect to Σ_{li} can be alternatively obtained by its linearization. This includes that for all $l = 0, \ldots, (n-1)$, $\hat{y}^l_{li}(k+l+1)$ can be obtained by linearizing $\psi^l(f(x,u,q), \ldots)$.

We proceed with associating a number μ_i to each output component y_i of Σ in the following way. Let $(x,u,q) \in \mathcal{X} \times \mathcal{U} \times \mathcal{W}$. Then compute for $i = 1, \ldots, p$ the derivative $D_q(h_i \circ f)\,(x,u,q)$. From the analyticity of the mappings f and h it follows that this expression is either non-zero in an open and dense subset $\mathcal{O}_i$ of $\mathcal{X} \times \mathcal{U} \times \mathcal{W}$ or vanishes at all points (x,u,q). Define $\mu_i = 1$ in the first case whereas in the latter case we continue by observing that $h_i(f(x,u,q))$ does not depend on q. We can, therefore, write $h_i(f(x,u,q)) = h_i^0(x,u)$. Now compute $D_z h_i^0(z,u_2)\big|_{z=f(x,u_1,q)} D_q f(x,u_1,q)$.

If this expression is non-zero on an open and dense subset $\mathcal{O}_i$ of $\mathcal{X} \times \mathcal{U}^2 \times \mathcal{W}$, we set $\mu_i = 2$, otherwise we continue with the function $h_i^1(x, u_1, u_2) = h_i^0(f(x, u_1, q), u_2)$. If none of the iterated functions $h_i^{k+1}(x, u_1, \ldots, u_{k+2}) = h_i^k(f(x, u_1, q), u_2, \ldots, u_{k+2})$ depends on q, define $\mu_i = \infty$.

This procedure, performed with respect to the inputs u, leads to the concept of *'relative degree'*. Analogously with the situation there (cf. [21]), one proves $\mu_i \leq n$ or $\mu_i = \infty$, $i = 1, \ldots, p$.

Consider the permutation of the outputs y_i of Σ which is induced by performing Singh's algorithm for Σ_{li}. Investigating Singh's algorithm, it becomes immediately clear that

$$D_q(\psi^0 \circ f)(x, u, q) \quad = \quad D_q(h \circ f)(x, u, q) \quad = \quad 0$$

holds if $\mu_i > 1$ for all $i = 1, \ldots, p$. Furthermore,

$$D_z \psi^l(z, \ldots)\big|_{z=f(x,u,q)} D_q f(x, u, q) = 0\,, \quad 1 \leq l \leq \alpha - 1,$$

is satisfied in case $\mu_i > l + 1$ for μ_i's corresponding to the components of $\hat{y}_i^l$ and finally $D_q \psi^l(f(x, u, q), \ldots) = 0$, $\alpha \leq l \leq (n-1)$, if $\mu_i = \infty$ for the components of $\hat{y}_i^l$ $(= \hat{y}_i^\alpha)$. This we are going to summarize in the following assumption.

A sumption (A). *The numbers μ_i, $i = 1, \ldots, p$, satisfy*

1. $\mu_i > 1$.
2. $\mu_i > l + 1$ *for μ_i's belonging to output components $\hat{y}_i^l$, $1 \leq l \leq (\alpha - 1)$.*
3. $\mu_i = \infty$ *for μ_i's corresponding to output components $\hat{y}_i^l$, $\alpha \leq l \leq (n-1)$.*

With the help of these preparations, the following result can be concluded.

Theorem 4.4. Consider system Σ together with its linearization Σ_{li} along $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{u}}, \overline{\boldsymbol{q}}, \overline{\boldsymbol{y}})$. Perform Singh's algorithm to Σ_{li} and let Assumption $(\boldsymbol{A})$ be satisfied. Then DDDP is solvable for Σ if and only if DDDP is solvable for Σ_{li}.

Remark 4.5. The strong regularity assumption for the reference trajectory can be relaxed to the weaker condition of regularity if one takes care of choosing an admissible application of Singh's algorithm.

By virtue of Lemma 2.12 we finally get

Theorem 4.6. Under the assumptions of Theorem 4.4, DDDP is solvable for Σ if and only if DDP is solvable for Σ_{li}.

5. APPENDIX

Proof of Lemma 4.1. First observe that the representations (16) and (17) are unique. By the full row rank assumption on $\tilde{G}$, ρ components of u (without loss of generality $u_1, \ldots, u_\rho =: u^1$) are uniquely determined as functions of $\tilde{y}, x, q$ and $(u_{\rho+1}, \ldots, u_m) =: u^2$ by the equation

$$\tilde{y} - \tilde{a}(x, u, q) = 0$$

in a neighbourhood of $(\overline{x}, \overline{u}, \overline{q}, \overline{\tilde{y}})$, that is,

$$u^1 = \phi(x, u^2, q, \tilde{y}) .$$

Equation (16) can then be obtained by setting

$$\hat{y} = \hat{a}(x, \phi(x, u^2, q, \tilde{y}), u^2, q) =: \psi(x, u^2, q, \tilde{y}) \tag{24}$$

where by the rank assumptions on $\tilde{G}$ and $\hat{G}$, ψ does not depend on u^2. Write

$$\tilde{G} = [D_{u^1}\tilde{a} \vdots D_{u^2}\tilde{a}] =: [\tilde{G}_1 \vdots \tilde{G}_2] .$$

Since we have assumed that $\tilde{G}_1$ is invertible, we can choose $\tilde{G}^+$ of the form

$$\tilde{G}^+ = \begin{bmatrix} \tilde{G}_1^{-1} \\ \cdots \\ 0 \end{bmatrix} .$$

Decompose $\hat{G} = [\hat{G}_1 \vdots \hat{G}_2]$ accordingly. Linearizing (16) yields (cf. (24))

$$\begin{aligned} \hat{y}_{li} &= [D_x\hat{a} + D_{u^1}\hat{a}D_x\phi]\, x_{li} + [D_q\hat{a} + D_{u^1}\hat{a}D_q\phi]\, q_{li} + [D_{u^1}\hat{a}D_{\tilde{y}}\phi]\, \tilde{y}_{li} \\ &= [\hat{F} + \hat{G}_1 D_x\phi]\, x_{li} + [\hat{E} + \hat{G}_1 D_q\phi] q_{li} + [\hat{G}_1 D_{\tilde{y}}\phi]\, \tilde{y}_{li} . \end{aligned}$$

From the identity

$$\tilde{y} - \tilde{a}(x, \phi(x, u^2, q, \tilde{y}), u^2, q) \equiv 0$$

we obtain $D_x\phi = -\tilde{G}_1^{-1}\tilde{F}$ and thus

$$[\hat{F} + \hat{G}_1 D_x\phi] = [\hat{F} - \hat{G}_1\tilde{G}_1^{-1}\tilde{F}] = [\hat{F} - \hat{G}\tilde{G}^+\tilde{F}] .$$

Analogously one shows

$$[\hat{E} + \hat{G}_1 D_q\phi] = [\hat{E} - \hat{G}\tilde{G}^+\tilde{E}] \quad \text{and} \quad [\hat{G}_1 D_{\tilde{y}}\phi] = \hat{G}\tilde{G}^+ .$$

From this the statement follows. □

(Received December 21, 1994.)

REFERENCES

[1] G. Basile and G. Marro: Controlled and conditioned invariant subspaces in linear system theory. J. Optim. Theory Appl. *3* (1969), 306–315.

[2] S. P. Bhattacharyya: Disturbance rejection in linear systems. Internat. J. Systems Sci. *5* (1974), 633–637.

[3] M. D. Di Benedetto, J. W. Grizzle and C. H. Moog: Rank invariants of nonlinear systems. SIAM J. Control Optim. *27* (1989), 658–672.

[4] J. W. Grizzle: Controlled invariance for discrete–time nonlinear systems with an application to the disturbance decoupling problem. IEEE Trans. Automat. Control *AC-30* (1985), 868–874.

[5] J. W. Grizzle: A linear algebraic framework for the analysis of discrete–time nonlinear systems. SIAM J. Control Optim. *31* (1993), 1026–1044.

[6] R. M. Hirschorn: Invertibility of multivariable nonlinear control systems. IEEE Trans. Automat. Control *AC-24* (1979), 1–19.

[7] R. M. Hirschorn: (A, B)–invariant distributions and disturbance decoupling of nonlinear systems. SIAM J. Control Optim. *19* (1981), 1–19.

[8] H. J. C. Huijberts: Dynamic Feedback in Nonlinear Synthesis Problems. Ph.D. Thesis, University of Twente, Enschede 1991.

[9] H. J. C. Huijberts and H. Nijmeijer: Strong dynamic input–output decoupling: From linearity to nonlinearity. In: Proceedings of the IFAC Symposium on Nonlinear Control Systems Design, Bordeaux 1992.

[10] H. J. C. Huijberts, H. Nijmeijer and L. L. M. van der Wegen: Dynamic disturbance decoupling for nonlinear systems. SIAM J. Control Optim. *30* (1992), 336–349.

[11] H. J. C. Huijberts, H. Nijmeijer and L. L. M. van der Wegen: Minimality of dynamic input–output decoupling for nonlinear systems. Systems Control Lett. *18* (1992), 435–443.

[12] H. J. C. Huijberts, H. Nijmeijer and L. L. M. van der Wegen: Dynamic disturbance decoupling for nonlinear systems: The nonsquare and noninvertible case. In: Analysis of Controlled Dynamical Systems (B. Bonnard, B. Bride, J. P. Gauthier and I. Kupka, eds.), Birkhäuser, Boston 1991, pp. 243–252.

[13] A. Ilchmann. Contributions to Time–Varying Linear Control Systems. Verlag an der Lottbek, Ammersbek 1989.

[14] A. Ilchmann: Time–varying linear control systems: A geometric approach. IMA J. Math. Control Inform. *6* (1989), 411–440.

[15] A. Isidori, A. J. Krener, C. Gori–Giorgi and S. Monaco: Nonlinear decoupling via feedback: a differential geometric approach. IEEE Trans. Automat. Control *AC-26* (1981), 331–345.

[16] T. Kaczorek: Linear Control Systems II. Research Studies Press Ltd., Taunton 1993.

[17] Ü. Kotta: Dynamic disturbance decoupling for discrete–time nonlinear systems: the nonsquare and non–invertible case. In: Proc. Estonian Acad. Sci. Phys. Math. *41* (1991), 14–22.

[18] Ü. Kotta and H. Nijmeijer: Dynamic Disturbance Decoupling for Discrete–Time Nonlinear Systems. Memorandum No. 913, University of Twente, Enschede 1990.

[19] W. C. A. Maas and H. Nijmeijer: Dynamic path controllability in economic models: from linearity to nonlinearity. J. Econom. Dynamics Control *18* (1994), 781–805.

[20] S. Monaco and D. Normand–Cyrot: Invariant distributions for discrete time nonlinear systems. Systems Control Lett. *5* (1984), 191–196.

[21] H. Nijmeijer: Local (dynamic) input–output decoupling of discrete time nonlinear systems. IMA J. Math. Control Inform. *4* (1987), 237–250.

[22] H. Nijmeijer and A. van der Schaft: Nonlinear Dynamical Control Systems. Springer Verlag, New York 1990.

[23] W. Respondek: Disturbance decoupling via dynamic feedback. In: Analysis of Controlled Dynamical Systems (B. Bonnard, B. Bride, J. P. Gauthier and I. Kupka, eds.), Birkhäuser, Boston 1991, pp. 347–357.

[24] L. M. Silverman: Inversion of multivariable linear systems. IEEE Trans. Automat. Contr. *AC-14* (1969), 270–276.

[25] S. N. Singh: Generalised decoupled-control synthesis for nonlinear systems. IEE Proceedings *128* (1981), 157–161.

[26] W. M. Wonham and A. S. Morse: Decoupling and pole assignment in linear multivariable systems: A geometric approach. SIAM J. Control Optim. *8* (1970), 1–18.

[27] W. M. Wonham: Linear Multivariable Control. Third edition. Springer Verlag, New York 1985.

Dr. Thomas Fliegner and Prof. Dr. Henk Nijmeijer, Department of Applied Mathematics, University of Twente, P. O. Box 217, 7500 AE Enschede. The Netherlands.