KYBERNETIKA — VOLUME 31 (1995), NUMBER 6, PAGES 649–655

ON OBTAINING THE TIME–INVARIANT ASSOCIATED SYSTEM OF A PERIODIC SYSTEM THROUGH SYSTEM EQUIVALENCE[1]

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In this paper it is shown that the standard Rosenbrock's strict system equivalence technique can be used as a way for obtaining the associated system at a given initial time of a linear periodic discrete-time system S, starting from its "stacked form" at the same initial time. Therefore, by well-known results about the Rosenbrock's strict system equivalence, the stacked transfer matrix, the characteristic multipliers, the invariant zeros, the input decoupling zeros and the output decoupling zeros of system S at a given time and the corresponding ordered sets of structural indices, which were originally introduced on the basis of the associated system, can be equivalently characterized through the stacked form of S.

1. INTRODUCTION AND PROBLEM DEFINITION

The large amount of contributions on the analysis and control of linear periodic discrete-time systems, described by the following equations

$$x(k+1) = A(k)\,x(k) + B(k)\,u(k), \tag{1a}$$

$$y(k) = C(k)\,x(k) + D(k)\,u(k), \tag{1b}$$

where $k \in \mathbf{Z}$, $x(k) \in \mathbf{R}^n$, $u(k) \in \mathbf{R}^p$, $y(k) \in \mathbf{R}^q$, and $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$ are real periodic matrices of period ω (briefly, ω-periodic) [1]–[20], [23], is motivated by the variety of processes that can be modelled by linear periodic difference equations [1, 2, 19]. Some of those contributions (e. g., [10, 13, 19]) were based on time-invariant descriptions of system (1), and, specifically, on the following one:

$$x^{k_0}(h+1) = E_{k_0} x^{k_0}(h) + J_{k_0} u_{k_0}(h), \quad x^{k_0}(0) = x(k_0), \tag{2a}$$

$$y_{k_0}(h) = L_{k_0} x^{k_0}(h) + P_{k_0} u_{k_0}(h), \tag{2b}$$

where:

[1] This work has been supported by funds of Ministero dell'Università e della Ricerca Scientifica e Tecnologica.

$$E_{k_0} := \Phi(k_0 + \omega, k_0), \qquad J_{k_0} := [\, J_{k_0,0} \quad J_{k_0,1} \quad \cdots \quad J_{k_0,\omega-1}\,], \tag{3a}$$

$$L_{k_0} := \begin{bmatrix} L_{k_0,0} \\ L_{k_0,1} \\ \vdots \\ L_{k_0,\omega-1} \end{bmatrix}, \; P_{k_0} := \begin{bmatrix} P_{k_0,0,0} & P_{k_0,0,1} & \cdots & P_{k_0,0,\omega-1} \\ P_{k_0,1,0} & P_{k_0,1,1} & \cdots & P_{k_0,1,\omega-1} \\ \cdots & \cdots & \cdots & \cdots \\ P_{k_0,\omega-1,0} & P_{k_0,\omega-1,1} & \cdots & P_{k_0,\omega-1,\omega-1} \end{bmatrix}, \tag{3b}$$

with

$$\Phi(i,j) := A(i-1)\cdots A(j+1)\,A(j), \qquad \forall\, i,\, j \in \mathbf{Z},\; i > j, \tag{4a}$$

$$\Phi(j,j) := I_n, \qquad \forall\, j \in \mathbf{Z}; \tag{4b}$$

$$J_{k_0,j} := \Phi(k_0 + \omega, k_0 + j + 1)\, B(k_0 + j), \quad j = 0, \ldots, \omega - 1; \tag{4c}$$

$$L_{k_0,j} := C(k_0 + j)\, \Phi(k_0 + j, k_0), \qquad j = 0, \ldots, \omega - 1; \tag{4d}$$

$$P_{k_0,i,j} := 0, \qquad j = 1, \ldots, \omega - 1,\; i < j, \tag{4e}$$

$$P_{k_0,j,j} := D(k_0 + j), \qquad j = 0, \ldots, \omega - 1, \tag{4f}$$

$$P_{k_0,i,j} := C(k_0 + i)\, \Phi(k_0 + i, k_0 + j + 1)\, B(k_0 + j), \qquad j = 0, \ldots, \omega - 2,\; i = j + 1, \ldots, \omega - 1, \tag{4g}$$

and $u_{k_0}(h) := [u^T(k_0 + h\omega) \;\; u^T(k_0 + h\omega + 1) \;\; \cdots \;\; u^T(k_0 + h\omega + \omega - 1)]^T$, $x^{k_0}(h) := x(k_0 + h\omega)$, $y_{k_0}(h) := [y^T(k_0 + h\omega) \;\; y^T(k_0 + h\omega + 1) \;\; \ldots \;\; y^T(k_0 + h\omega + \omega - 1)]^T$.

The time-invariant equations (2) were called the *associated system at (the initial) time k_0 of system* (1). More recently, a different time-invariant description of system (1), which is not in state-space form and is called the *ω-stacked form at (the initial) time k_0 of system* (1), and consisting of the following pair of equations

$$R_n(\Delta)\, x_{k_0}(h) = \mathcal{A}_{k_0}\, x_{k_0}(h) + \mathcal{B}_{k_0}\, u_{k_0}(h), \tag{5a}$$

$$y_{k_0}(h) = \mathcal{C}_{k_0}\, x_{k_0}(h) + \mathcal{D}_{k_0}\, u_{k_0}(h), \tag{5b}$$

where $u_{k_0}(h)$ and $y_{k_0}(h)$ have the same meaning as before, $x_{k_0}(h) := [x^T(k_0 + h\omega) \;\; x^T(k_0 + h\omega + 1) \;\; \cdots \;\; x^T(k_0 + h\omega + \omega - 1)]^T$,

$$R_n(\Delta) := \begin{bmatrix} 0 & I_{(\omega-1)n} \\ \Delta I_n & 0 \end{bmatrix}, \tag{6}$$

with Δ being the one-step forward shift operator in the h variable, and

$$\mathcal{A}_{k_0} := \mathrm{diag}\{A(k_0), A(k_0 + 1), \ldots, A(k_0 + \omega - 1)\}, \tag{7a}$$

$$\mathcal{B}_{k_0} := \mathrm{diag}\{B(k_0), B(k_0 + 1), \cdots, B(k_0 + \omega - 1)\}, \tag{7b}$$

$$\mathcal{C}_{k_0} := \mathrm{diag}\{C(k_0), C(k_0 + 1), \ldots, C(k_0 + \omega - 1)\}, \tag{7c}$$

$$\mathcal{D}_{k_0} := \mathrm{diag}\{D(k_0), D(k_0 + 1), \ldots, D(k_0 + \omega - 1)\}, \tag{7d}$$

was introduced and used for studying the properties of system (1) [12, 16] through the following *ω-stacked system matrix at (the initial) time* k_0 *of system* (1):

$$S_{k_0}(\Delta) := \begin{bmatrix} \mathcal{A}_{k_0} - R_n(\Delta) & \mathcal{B}_{k_0} \\ \mathcal{C}_{k_0} & \mathcal{D}_{k_0} \end{bmatrix}. \tag{8}$$

Vectors $x_{k_0}(h)$, $u_{k_0}(h)$ and $y_{k_0}(h)$ are called the *ω-stacked forms at (the initial) time* k_0 of vectors $x(k)$, $u(k)$ and $y(k)$, respectively.

In [16] a procedure for finding a time-invariant description similar to (2), was given for a pair of periodic difference equations more general than (1), making use of an ω-stacked form of such equations that generalizes (5). Such a procedure, when applied to (1), consists of finding polynomial matrices $M(\Delta)$, $N(\Delta)$, $X(\Delta)$, and $Y(\Delta)$, with $M(\Delta)$ and $N(\Delta)$ being square and unimodular, such that

$$\begin{bmatrix} M(\Delta) & 0 \\ Y(\Delta) & I_{q\omega} \end{bmatrix} S_{k_0}(\Delta) \begin{bmatrix} N(\Delta) & X(\Delta) \\ 0 & I_{p\omega} \end{bmatrix} = \left[\begin{array}{cc|c} -I_{n(\omega-1)} & 0 & 0 \\ 0 & \overline{E}_{k_0} - \Delta I_n & \overline{J}_{k_0} \\ \hline 0 & \overline{L}_{k_0} & \overline{P}_{k_0} \end{array}\right] =: \overline{S}_{k_0}(\Delta), \tag{9}$$

where $\overline{E}_{k_0}, \overline{J}_{k_0}, \overline{L}_{k_0}$ and $\overline{P}_{k_0}$ are constant matrices of proper dimensions. Relation (9) is a strict system equivalence relation as formally defined in [22], and is a special case of the system equivalence relation defined in [16], as well as of the large system equivalence relation introduced in [15]. The purpose of this paper is to show that the quadruplet $(\overline{E}_{k_0}, \overline{J}_{k_0}, \overline{L}_{k_0}, \overline{P}_{k_0})$ thus obtained coincides with the quadruplet $(E_{k_0}, J_{k_0}, L_{k_0}, P_{k_0})$ in (2) within a nonsingular coordinate transformation in the state-space of (2), so that the application of the standard Rosenbrock's strict system equivalence technique [22], that is expressed by (9), can be seen as a different, but equivalent, way of obtaining the associated system (2) at time k_0 of system (1)—via the system equivalence defined in [16], instead of (3), (4).

2. MAIN RESULT

First of all, it is noted that the state $x^{k_0}(h)$ of the associated system (2) must not be confused with the ω-stacked form $x_{k_0}(h)$ at time k_0 of $x(k)$ appearing in (5). In particular, $x_{k_0}(h)$ can be computed from $x^{k_0}(h)$ and $u_{k_0}(h)$ by means of the following relation:

$$x_{k_0}(h) = N_a x^{k_0}(h) + X_a u_{k_0}(h), \tag{10}$$

where

$$N_a := \begin{bmatrix} I_n \\ \Phi(k_0+1, k_0) \\ \vdots \\ \Phi(k_0+\omega-1, k_0) \end{bmatrix},$$

$$X_a := \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \Psi(k_0+1, k_0) & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Psi(k_0+\omega-1, k_0) & \Psi(k_0+\omega-1, k_0+1) & \cdots & \Psi(k_0+\omega-1, k_0+\omega-2) & 0 \end{bmatrix},$$

with

$$\Psi(i,j) := \Phi(i,j+1)B(j), \quad j = k_0, ..., k_0+\omega-2, \; i = j+1, ..., \omega-1.$$

With reference to (9), defining

$$\overline{x}_{k_0}(h) := N^{-1}(\Delta)x_{k_0}(h) - N^{-1}(\Delta)X(\Delta)u_{k_0}(h) \tag{11}$$

and partitioning the vector $\overline{x}_{k_0}(h)$ as follows:

$$\overline{x}_{k_0}(h) = \begin{bmatrix} \overline{x}^1_{k_0}(h) \\ \overline{x}^2_{k_0}(h) \end{bmatrix}, \tag{12}$$

with $\overline{x}^1_{k_0}(h) \in \mathbf{R}^{n(\omega-1)}$ and $\overline{x}^2_{k_0}(h) \in \mathbf{R}^n$, the following time-invariant linear system is associated with (9):

$$\overline{x}^2_{k_0}(h+1) = \overline{E}_{k_0}\overline{x}^2_{k_0}(h) + \overline{J}_{k_0}u_{k_0}(h), \tag{13a}$$

$$y_{k_0}(h) = \overline{L}_{k_0}\overline{x}^2_{k_0}(h) + \overline{P}_{k_0}u_{k_0}(h). \tag{13b}$$

Now, it is possible to state and prove the following theorem, which shows that the time-invariant systems (2) and (13) are related by a state-space nonsingular coordinate transformation.

Theorem 1. The time-invariant systems (2) and (13) are system similar, i.e. there exists a square nonsingular matrix H such that:

$$\overline{E}_{k_0} = HE_{k_0}H^{-1}, \qquad \overline{J}_{k_0} = HJ_{k_0}, \tag{14a}$$

$$\overline{L}_{k_0} = L_{k_0}H^{-1}, \qquad \overline{P}_{k_0} = P_{k_0}. \tag{14b}$$

Proof. The solutions of

$$\overline{S}_{k_0}(\Delta)\begin{bmatrix} \overline{x}_{k_0}(h) \\ u_{k_0}(h) \end{bmatrix} = \begin{bmatrix} 0 \\ y_{k_0}(h) \end{bmatrix} \tag{15}$$

and those of

$$S_{k_0}(\Delta)\begin{bmatrix} x_{k_0}(h) \\ u_{k_0}(h) \end{bmatrix} = \begin{bmatrix} 0 \\ y_{k_0}(h) \end{bmatrix} \tag{16}$$

are exactly the same in the ω-stacked output, and are biuniquely related in the vectors $x_{k_0}(h)$ and $\overline{x}_{k_0}(h)$ by means of the functional relations (11) and the following one:

$$x_{k_0}(h) = N(\Delta)\,\overline{x}_{k_0}(h) + X(\Delta)\,u_{k_0}(h). \tag{17}$$

By substituting equation (10) into (11) and (17), it is obtained:

$$N_a x^{k_0}(h) + X_a u_{k_0}(h) = N(\Delta)\,\overline{x}_{k_0}(h) + X(\Delta)\,u_{k_0}(h), \tag{18a}$$

$$\overline{x}_{k_0}(h) = N^{-1}(\Delta)\,(N_a x^{k_0}(h) + X_a u_{k_0}(h)) - N^{-1}(\Delta)\,X(\Delta)\,u_{k_0}(h). \tag{18b}$$

From (9), (11) and (12), it is obtained that $\overline{x}^1_{k_0}(h) = 0,\ \forall\, h \in \mathbf{Z}^+$. Whence, by partitioning matrices $N(\Delta)$ and $N^{-1}(\Delta)$ according to (12), i.e.

$$\begin{aligned} N(\Delta) &= [N_1(\Delta) \quad N_2(\Delta)], \\ N^{-1}(\Delta) &= \begin{bmatrix} \hat{N}_1(\Delta) \\ \hat{N}_2(\Delta) \end{bmatrix}, \end{aligned}$$

equations (18) become

$$N_a x^{k_0}(h) + X_a u_{k_0}(h) = N_2(\Delta)\, \overline{x}^2_{k_0}(h) + X(\Delta)\, u_{k_0}(h), \tag{19a}$$

$$\overline{x}^2_{k_0}(h) = \hat{N}_2(\Delta)\, (N_a x^{k_0}(h) + X_a u_{k_0}(h)) - \hat{N}_2(\Delta)\, X(\Delta)\, u_{k_0}(h). \tag{19b}$$

Left multiplying (19a) by the pseudo-inverse $N_a^{\#}$ of N_a, $N_a^{\#} := \left(N_a^T N_a\right)^{-1} N_a^T$, which exists since $(N_a^T N_a)$ is nonsingular by its definition, it is found that:

$$x^{k_0}(h) = N_a^{\#} N_2(\Delta) \overline{x}^2_{k_0}(h) + N_a^{\#}(X(\Delta) - X_a) u_{k_0}(h). \tag{20}$$

Equations (19b), (20) can be rewritten in compact form as follows:

$$x^{k_0}(h) = \overline{N}(\Delta)\, \overline{x}^2_{k_0}(h) + \overline{X}(\Delta)\, u_{k_0}(h), \tag{21a}$$

$$\overline{x}^2_{k_0}(h) = \overline{\overline{N}}(\Delta)\, x^{k_0}(h) + \overline{\overline{X}}(\Delta)\, u_{k_0}(h), \tag{21b}$$

where

$$\overline{N}(\Delta) := N_a^{\#} N_2(\Delta), \qquad \overline{X}(\Delta) := N_a^{\#}(X(\Delta) - X_a), \tag{22a}$$

$$\overline{\overline{N}}(\Delta) := \hat{N}_2(\Delta) N_a, \qquad \overline{\overline{X}}(\Delta) := \hat{N}_2(\Delta)(X_a - X(\Delta)). \tag{22b}$$

Since, for a given input function in stacked form $u_{k_0}(\cdot)$, the solutions of (2) and those of (13) are biuniquely related in the vectors $x^{k_0}(\cdot)$ and $\overline{x}^2_{k_0}(\cdot)$ by (21), and are exactly the same in the output $y_{k_0}(\cdot)$, then the theorem follows from Theorem 3 in [21]. □

Notice that, by Theorem 1 and by well-known results about strict system equivalence [22], the ω-stacked transfer matrix, the characteristic multipliers, the invariant zeros, the input decoupling zeros and the output decoupling zeros of system (1) at time k_0 and the corresponding ordered sets of structural indices, introduced in [10] and [12] through the associated system (2), can be equivalently characterized through the ω-stacked form (5) of (1) (as already proved in [12]).

Now, in order to further clarify the procedure here introduced for obtaining the associated system of system (1), the computation of system (13) will be carried out in full details for the simplest case $\omega = 2$, for which the associated system (2) at the initial time $k_0 = 0$ of system (1) is characterized by the following matrices:

$$E_0 = A(1)A(0), \qquad J_0 = [A(1)B(0) \quad B(1)] \tag{23a}$$

$$L_0 = \begin{bmatrix} C(0) \\ C(1)A(0) \end{bmatrix}, \qquad P_0 = \begin{bmatrix} D(0) & 0 \\ C(1)B(0) & D(1) \end{bmatrix}. \tag{23b}$$

The 2-stacked system matrix at the initial time $k_0 = 0$ of system (1) is

$$S_0(\Delta) = \begin{bmatrix} A(0) & -I_n & B(0) & 0 \\ -\Delta I_n & A(1) & 0 & B(1) \\ C(0) & 0 & D(0) & 0 \\ 0 & C(1) & 0 & D(1) \end{bmatrix}. \tag{24}$$

Compute the left-hand side of relation (9) with the following polynomial matrices $M(\Delta), N(\Delta), X(\Delta)$ and $Y(\Delta)$, with $M(\Delta)$ and $N(\Delta)$ being square and unimodular:

$$M(\Delta) = \begin{bmatrix} I_n & 0 \\ A(1) & I_n \end{bmatrix}, \qquad Y(\Delta) = \begin{bmatrix} 0 & 0 \\ C(1) & 0 \end{bmatrix}, \tag{25a}$$

$$N(\Delta) = \begin{bmatrix} 0 & I_n \\ I_n & A(0) \end{bmatrix}, \qquad X(\Delta) = \begin{bmatrix} 0 & 0 \\ B(0) & 0 \end{bmatrix}. \tag{25b}$$

It is easy to see that (9) (with $k_0 = 0$) holds with the matrices $\overline{E}_0$, $\overline{J}_0$, $\overline{L}_0$ and $\overline{P}_0$ that coincide with the matrices E_0, J_0, L_0 and P_0, and Theorem 1 trivially holds with $H = I_n$.

3. CONCLUSION

It has been shown that the system equivalence technique introduced in [16] for linear discrete-time recurrent equations with ω-periodic coefficients described through their ω-stacked form, can be used as a different, but equivalent, way for obtaining the associated system at a given initial time of a linear ω-periodic discrete-time system S, starting from its ω-stacked form at the same initial time. In this case such a technique reduces to the standard Rosenbrock's strict system equivalence technique [22]. This, by well-known results about the Rosenbrock's strict system equivalence, implies that the ω-stacked transfer matrix, the characteristic multipliers, the invariant zeros, the input decoupling zeros and the output decoupling zeros of system S at a given time and the corresponding ordered sets of structural indices, as introduced in [10], [12], can be characterized through the ω-stacked form of S, rather than its associated system, which has a structure more involved than the one of the ω-stacked form of S.

(Received January 20, 1995.)

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