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OBSERVABILITY OF SATURATED SYSTEMS WITH AN OFFSET

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In the paper, observability of systems with output saturation is investigated under the condition that the system has an offset at the output.

1. INTRODUCTION AND PROBLEM FORMULATION

In [3], the problem of observability of systems with output saturation was considered. (See also [2] and [4].) In this paper, we want to consider the situation where the system has an offset at the output. Specifically, we assume that the system equations are of the form

$$\dot{x} = Ax + Bu, \quad y = cx + y_0, \quad z = \mathrm{sat}(y). \tag{1}$$

Here the *sat-function* is defined by

$$\mathrm{sat}(y) := \begin{cases} -1 & \text{if} \quad y < -1, \\ y & \text{if} \quad -1 \le y \le 1, \\ 1 & \text{if} \quad y > 1. \end{cases}$$

It is easily seen that the results will be very similar to the results given in [3] if $|y_0| < 1$. The problem is different, however, if $|y_0| \ge 1$. In this paper, we restrict ourselves to single-output systems. For such systems, rather complete results can be given. The multi-output case is more involved. For this, we refer to [2], [3] (for the case without an offset).

Let us now give a more specific description of the problems we are going to investigate. We consider the system $\Sigma = (c, A, B)$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $c \in \mathbb{R}^{1\times n}$. We will assume, without loss of generality, that B is injective (left invertible). We assume that the input is constrained to take values in a given set $U \subseteq \mathbb{R}^m$. We are interested in observability properties of the system determined by the system $\Sigma_s := (\Sigma, U, y_0)$ given by (1). Let us introduce some terminology and notation. The *state response* $x_u(t, x_0)$ is the state variable due to the input function u and initial state x_0. Hence

$$x_u(t, x_0) = e^{tA}x_0 + \int_0^t e^{(t-\tau)A}Bu(\tau)\,\mathrm{d}\tau.$$

Furthermore, the *nonsaturated output* is $y_u(t,x_0) := cx_u(t,x_0) + y_0$ and the (saturated) *output* is given by $z_u(t,x_0) := \operatorname{sat}(y_u(t,x_0))$. Then Σ_s is called *observable* if any two distinct states are distinguishable, i. e., if for any two distinct states x_1 and x_2 we have

$$\exists_{T>0}\, \exists_{u:[0,T]\to U}\ z_u(T,x_1) \neq z_u(T,x_2).$$

Obviously, for the saturated system to be observable, the pair (c,A) must be observable. This will be a standing assumption in this paper. We will treat consecutively the following cases:

- *The unconstrained-input case:* There is no restriction on the input, i. e., $U = \mathbb{R}^m$.
- *The zero-input case:* There is no input, i. e., $U = \{0\}$.
- *The small-input case:* U is bounded and contains the origin of $\mathbb{R}^m$ in its interior.

2. RESULTS

In this section, we will only give the results. The proofs will be given in the ensuing sections. Recall that B is assumed to be injective and hence nonzero.

Theorem 2.1. (The unconstrained-input case) If $U = \mathbb{R}^m$ and (c,A) is observable, then Σ_s is observable.

For the formulation of the zero-input case, we introduce some notation. For an $n \times n$ matrix A, we denote by $\sigma(A)$ the *spectrum*, i. e., the set of eigenvalues of A. Furthermore, we use the notation $\mathbb{R}^+ := \{\lambda \in \mathbb{R} : \lambda \geq 0\}$, $\mathbb{C}^- := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$ and $\overline{S}$ for the closure of a set S.

Theorem 2.2. (The zero-input case) Let (c,A) be observable, and assume $U = \{0\}$. Then we have

1. If $|y_0| < 1$, then Σ_s is observable iff $\sigma(A) \cap \mathbb{R}^+ = \emptyset$.
2. If $|y_0| = 1$, then Σ_s is observable iff $\sigma(A) \cap \mathbb{R} = \emptyset$.
3. If $|y_0| > 1$, then Σ_s is observable iff $\sigma(A) \cap (\mathbb{R} \cup \overline{\mathbb{C}^-}) = \emptyset$.

The case 1 is essentially equivalent to the case $y_0 = 0$ given in [2] and [3].

For the small-input case, we make the following assumption:

Assumption 2.3.

1. (c,A) is observable and (A,B) is stabilizable,
2. The constraint set $U \subseteq \mathbb{R}^m$ is bounded and $0 \in \operatorname{int} U$.

In order to formulate the main result, we need some concepts for matrices. The *spectral abscis* of A is defined as $\Lambda(A) := \max\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$. The matrix A is called *stable* if $\Lambda(A) < 0$. An eigenvalue λ_0 of A is called *dominant* if $\operatorname{Re}\lambda_0 = \Lambda(A)$ and the multiplicity of λ_0 is maximal among all eigenvalues λ with $\operatorname{Re}\lambda = \Lambda(A)$. The set of dominant eigenvalues of A is called *the dominant spectrum* of A and denoted $\sigma^*(A)$. Furthermore, we introduce the *U-norm* of an m-dimensional row vector ω by

$$\rho_U(\omega) := \sup_{u \in U} \omega\, u.$$

If Assumption 2.3, 2 is satisfied, $\rho_U(\omega)$ is bounded, positive for $\omega \neq 0$ and subadditive (i.e. $\rho_U(\omega_1 + \omega_2) \leq \rho_U(\omega_1) + \rho_U(\omega_2)$). We will use the notation $\mathbb{R}_{++} := \{\lambda \in \mathbb{R} : \lambda > 0\}$. The result is given by the following theorem:

Theorem 2.4. Let Assumption 2.3 hold. Then Σ_s is observable iff

1. $\sigma^*(A) \cap \mathbb{R}_{++} = \emptyset$,
2. $\int_0^\infty \rho_U(ce^{tA}B)\,\mathrm{d}t > |y_0| - 1$.

Condition 2 is always satisfied when A is unstable or $|y_0| \leq 1$.

Remark 2.5. If A is stable, we can give a simplified *sufficient* condition for 2, using the subadditivity of ρ_U. We have

$$\int_0^\infty \rho_U(ce^{tA}B)\,\mathrm{d}t \geq \rho_U\left(\int_0^\infty ce^{tA}B\,\mathrm{d}t\right) = \rho_U(cA^{-1}B).$$

Therefore, $\rho_U(cA^{-1}B) > |y_0| - 1$ implies 2.

We may also ask for a necessary and sufficient condition for the observability of Σ_s for all sets U satisfying Assumption 2.3, 2, for a given system Σ and given y_0. If $|y_0| \leq 1$, then Condition 1 of Theorem 2.4 is such a condition. However, if $|y_0| > 1$, we need the extra condition that A be unstable. This is an immediate consequence of Theorem 2.4.

3. THE UNCONSTRAINED-INPUT CASE

Our results will crucially be based on the following auxiliary result (which is valid for general nonempty U). In order to formulate this condition, we introduce the following terminology: We say that an initial state x_0 is *completely saturated* if for every control $u : [0, \infty) \to U$ and every $t > 0$, we have the inequality $|y_u(t, x_0)| \geq 1$.

Lemma 3.1. Let (c, A) be observable. Then Σ_s is observable iff there exists at most one completely saturated initial state x_0.

Proof. *'if'*: First assume that x_0 is not completely saturated, say $|y_u(T, x_0)| < 1$ for some T, u. Then there exists $\varepsilon > 0$ such that $|y_u(t, x_0)| < 1$ for $T < t < T+\varepsilon$. For these values of t, we have $y_u(t,x_0) = z_u(t,x_0)$. Since in an observable linear system, the state can be identified in an arbitrarily short interval, it follows that the initial state x_0 is uniquely determined by the values of $z_u(t,x_0)$ $(T < t < T+\varepsilon)$. If x_0 is completely saturated, then x_0 is the only state with this property, and consequently uniquely identifiable by this fact.

'only if': Assume that there are two distinct completely saturated states x_0, x_1. If $y_u(0,x_i) \geq 1$ holds for both $i = 1,2$, then $y_u(t,x_i) \geq 1$ and hence $z_u(t,x_i) = 1$ for all $t \geq 0$ and u, because y depends continuously on t. In that case, x_0 and x_1 are indistinguishable, contradicting the observability of Σ_s. A similar contradiction is obtained if $y_u(0,x_i) \leq -1$ holds for both $i = 1,2$. Therefore, we may assume that $y_u(0,x_1) \geq 1$ and $y_u(0,x_0) \leq -1$. Take any $t^* > 0$ and define $x^* := x_u(t^*,x_1)$. Then we have $y_u(t,x^*) \geq 1$ and hence $z_u(t,x^*) = 1$ for all $t \geq 0$ and u. Hence any state on the trajectory starting at x_1 will give rise to the same constant output $z = 1$, irrespective of u. If the system is observable, all these states must be equal. Hence, $t \mapsto x_u(t,x_1)$ is constant, i.e., x_1. Therefore, $Ax_1 + Bu_0 = 0$ for all $t \geq 0$ and $u_0 \in U$. Since B is injective, it follows that U consists of one element, say, $U = \{u_0\}$. A similar reasoning holds for the solution starting at x_0 with, of course, the same u_0. Define $x_2 := 2x_1 - x_0$ and $\xi(t) := x_u(t,x_2) - x_1$, where u is the constant input u_0. Then $\dot{\xi} = A\xi$, $\xi(0) = x_1 - x_0$. Because $A(x_1 - x_0) = 0$, it follows that $\xi(t) = x_1 - x_0$. Consequently,

$$y_u(t,x_2) - y_u(0,x_1) = c\xi(t) = c(x_1 - x_0) \geq 2,$$

so that $|y_u(t,x_2)| \geq 3$ for all $t \geq 0$ and u. It follows that $z_u(t,x_2) = 1$ for all $t \geq 0$ and u. Observability now implies that $x_2 = x_1$, which is a contradiction. □

Remark 3.2. It follows from the proof that, if in an observable system a completely saturated initial state x_0 exists, U contains exactly one element, say u_0. In addition, x_0 and u_0 have to satisfy the equation $Ax_0 + Bu_0 = 0$. Therefore, the (unique) trajectory starting in x_0 has to be constant. Finally, the inequality $|cx_0 + y_0| \geq 1$ has to be satisfied. These conditions together are easily seen to be sufficient. The following system shows an example of this:

$$A := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B := \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c := [1,0], \quad y_0 := 1, \quad U := \{0\}.$$

Here, $x_0 := 0$ is the unique completely saturated initial state.

Because of this lemma, we see that the observability problem is equivalent to a particular controllability problem: We have to find out what initial states can be controlled to the region $\{x \in \mathbb{R}^n : |cx + y_0| < 1\}$. We will apply this result for the specific choices of U mentioned in the previous section. This section will be concluded with the unconstrained-input case. Here, we choose a number $T > 0$ and a vector $p \in \mathbb{R}^m$ such that

$$q := \int_0^T ce^{\tau A} Bp \, d\tau \neq 0.$$

Because (c, A) is observable and $B \neq 0$, this is always possible. Then, if u is identically equal to αp, we have

$$y_u(T, x_0) = ce^{TA}x_0 + \alpha q + y_0.$$

By a proper choice of α, we can always achieve that $|y_u(T, x_0)| < 1$. Hence, Σ_s is observable. This proves Theorem 2.1.

4. THE ZERO-INPUT CASE

In this section, we assume that $U = \{0\} \subseteq \mathbb{R}^m$. Therefore, the state x satisfies the differential equation $\dot{x} = Ax$, and the output y is given by $y(t, x_0) = ce^{tA}x_0 + y_0$. In the next proof, we will use some results about Bohl functions. These can be found in Section 6.

Proof of Theorem 2.2. We use the notation $\tilde{y}(t, x_0) := y(t, x_0) - y_0 = ce^{tA}x_0$ for the unsaturated output without offset.

Case 1: Let $|y_0| < 1$ and Σ_s be observable. Assume that $\lambda \in \sigma(A) \cap \mathbb{R}^+$ and that v is a corresponding eigenvector. Then $|\tilde{y}(t, v)| = |ce^{tA}v| = |e^{\lambda t}cv| \geq |cv| = |y(0, v)| \neq 0$, because (c, A) is observable. By taking $|v|$ sufficiently large, we can achieve that both $|y(t, v)| \geq 1$ and $|y(t, 2v)| \geq 1$ for all $t > 0$, so that both v and $2v$ are completely saturated. For the converse, note that for given x_0, the function $\tilde{y}(t) = c^{tA}x_0$ is a Bohl function. Let σ_0 denote the set of its exponents. Then $\sigma_0 \subseteq \sigma(A)$. There are two possibilities:

- $\sigma_0 \subseteq \mathbb{C}^-$. Then $y(t) \to y_0$ $(t \to \infty)$ and hence $|y(t)| < 1$ for large t.
- $\exists_{\lambda \in \sigma_0} \mathrm{R}\, \lambda \geq 0$. Because $\sigma(A) \cap \mathbb{R}^+ = \emptyset$, it follows that $\tilde{y}$ is oscillating (see Section 6). Now Theorem 6.3 implies that $\tilde{y}$ has infinitely many zeros, which again ensures that $|y(t)| < 1$ for some $t > 0$.

Case 2: Assume that $|y_0| = 1$. If $\lambda \in \sigma(A) \cap \mathbb{R}$, and $v \in \mathbb{R}^n$ is a corresponding eigenvector then $x(t) := e^{\lambda t}\alpha v$ is the state with initial value αv. Correspondingly, $\tilde{y}(t) = \alpha e^{\lambda t}cv$. Again, $cv \neq 0$, because of observability. The function $\tilde{y}$ has a constant sign, which, by a suitable choice of α, can be chosen to be the same as the sign of y_0. Then we have $|y(t)| \geq 1$ for all $t > 0$. By making various choices for α, we obtain more than one completely saturated state. Hence the condition is necessary. Conversely, the condition implies that $\tilde{y}$ is oscillating, so that we must have $|y(t)| < 1$ infinitely often.

Case 3: Now let $|y_0| > 1$. If $\lambda \in \sigma(A) \cap \overline{\mathbb{C}^-}$, the differential equation has a nontrivial bounded solution $x(t)$ on $[0, \infty)$. By choosing the modulus of the initial state sufficiently small, we can make $\tilde{y}(t)$ so small that $|y(t)| \geq 1$ for all $t > 0$. If $\lambda \in \sigma(A) \cap \mathbb{R}$, we reason as in 2. Conversely, because of Theorem 6.3, we find that for any nontrivial x_0, the output $\tilde{y}$ is oscillating and unbounded (since $\bar{\Lambda}(\tilde{y}) > 0$). Consequently, $y(t)$ takes any value in $\mathbb{R}$. □

5. THE SMALL–INPUT CASE

In this section, we give a proof of Theorem 2.4.

Proof of Theorem 2.4. *"only if"*: First we note that, because U contains more than one element, there cannot be a completely saturated state if the system is observable. Assume that $\alpha := \Lambda(A) \in \sigma^*(A) \cap \mathbb{R}_{++}$. Let the multiplicity of α be k. According to Lemma 6.5, 2, there exists v be such that $\bar{\sigma}(ce^{tA}v) = \{\alpha\}$ and $\omega(ce^{tA}v) = \omega(e^{tA})$. Then there exists $N > 0$ such that $ce^{tA}v \geq Nt^{k-1}e^{\alpha t}$ for all $t > 0$. Hence, for arbitrary $\beta > 0$, we have $y_u(t, \beta v) = \beta y_0(t, v) + y_u(t, 0) \geq \beta N t^{k-1} e^{\alpha t} - Lt^{k-1}e^{\alpha t}$ for some constant L. By taking β sufficiently large, we can find an initial state (viz. $x_0 := \beta v$) such that $|y_u(t, x_0)| \geq 1$ for all $t \geq 0$ and all $u : [0, \infty) \to U$. Hence, Condition 1 is necessary.

Next, take $x_0 = 0$ and assume that $T > 0$ is given. Then we can compute the maximal and the minimal value of $y_u(T) := y_u(T, 0)$ that can be obtained by a suitable choice of the function u on the interval $[0, T]$, subject to the condition $u(t) \in \bar{U}$ $(0 \leq t \leq T)$. We denote these quantities by $y_{\max}(T)$ and $y_{\min}(T)$, respectively. The result is

$$\tilde{y}_{\max}(T) = -\tilde{y}_{\min}(T) = \int_0^T \rho_U(ce^{tA}B)\,dt,$$

where again, $\tilde{y}$ denotes $y - y_0$. If Condition 2 is not satisfied, we have $|y_u(T)| \geq 1$ for every $T > 0$ and every u. Hence, Condition 2 is necessary.

"if": Assume that the Conditions 1 and 2 is satisfied. Let the multiplicity of the eigenvalues in $\sigma^*(A)$ be k. Let x_0 be any initial state. We have got to find an input u such that $|y_u(t, x_0)| < 1$ for some $t > 0$. Again we write $\tilde{y}_u(t, x_0) := y_u(t, x_0) - y_0$. In addition, we use the notation $\tilde{y}(t) := y_0(t, x_0) - y_0 = ce^{tA}x_0$. There are the following possibilities:

1. $\Lambda(A) \leq 0$: Then there exists an admissible control function $u_0 : [0, \infty) \to U$ such that $x_{u_0}(t, x_0) \to 0$ $(t \to \infty)$ (see [5]). Because of Condition 2, there exist $T_1 > 0$, $\varepsilon > 0$ such that $\int_0^{T_1} \rho_U(ce^{tA}B)\,dt > |y_0| - 1 + 3\varepsilon$. Choose $T_0 > 0$ such that $\|ce^{T_1 A}x_{u_0}(T_0, x_0)\| < \varepsilon$. Next choose u_1 such that $\int_0^{T_1} ce^{tA}Bu_1(T_1 - t)dt > |y_0| - 1 + 2\varepsilon$. This is possible because $\int_0^{T_1} \rho_U(ce^{tA}B)\,dt$ is the supremum of $\int_0^{T_1} ce^{tA}Bu(T_1 - t)\,dt$. Then we choose the control u as follows:

$$u(t) := \begin{cases} u_0(t) & (0 \leq t \leq T_0), \\ u_1(t - T_1) & (T_1 < t \leq T_1 + T_0). \end{cases}$$

Then

$$\tilde{y}_u(T_0 + T_1, x_0) = \tilde{y}_{u_1}(T_1, x_{u_0}(T_0, x_0)) = ce^{T_1 A}x_{u_0}(T_0, x_0) + \int_0^{T_1} ce^{tA}Bu_1(T_1 - t)\,dt$$
$$\geq -\varepsilon + |y_0| - 1 + 2\varepsilon > |y_0| - 1.$$

Hence $y_u(T_0 + T_1, x_0) = \tilde{y}_u(T_0 + T_1, x_0) + y_0 > -1$. Similarly one proves that there exists u such that $y_u(T_0 + T_1, x_0) < 1$.

2. $\Lambda(A) > 0$ and $\omega(\tilde{y}) = \omega(e^{tA})$. Because of Condition 1, this implies that $\tilde{y}(t)$ is oscillating and unbounded. In this case, we can choose u to be identically equal to zero.

3. $\Lambda(A) > 0$ and $\omega(\tilde{y}) \neq \omega(e^{tA})$. Then $y(t) = o(t^{k-1}e^{\alpha t})$ $(t \to \infty)$. It follows that the supremum of the values that $y(t)$ can attain is

$$y_{\sup}(t) \geq \int_0^t \rho_U(ce^{\tau A}B)\,\mathrm{d}\tau - |y(t)| \geq \gamma t^{k-1}e^{\alpha t} - y(t) \to \infty \quad (t \to \infty).$$

Here, we have used Lemma 6.4 and Lemma 6.5. Similarly, $y_{\inf}(t) \to -\infty$ $(t \to \infty)$. Consequently, we can let y assume any value, in particular, we can achieve that $|y(t_0)| < 1$ for some t_0. □

6. APPENDIX: ALMOST PERIODIC FUNCTIONS AND BOHL FUNCTIONS

According to H. Bohr, a continuous function $f : \mathbb{R} \to \mathbb{R}$ is called an *almost periodic function (=: APF)* if

$$\forall_{\varepsilon>0}\,\exists_{\ell>0}\,\forall_{T\geq 0}\,\exists_{\tau\in[T,T+\ell]}\forall_{t\geq 0}\,|f(t+\tau) - f(t)| < \varepsilon.$$

It is easily seen that periodic functions are APF's. The fundamental result of the theory of APF's we are going to use is the theorem that the sum of two APF's is also an APF. Hence the space of APF's is linear. In particular, any finite sum of periodic functions is an APF. This is the type of APF we are going to encounter. It is well known (see [1]) that an APF is bounded and uniformly continuous. We will need two more properties of APF's:

Lemma 6.1. Let f be an APF. Then

$$\limsup_{t\to\infty} f(t) = \sup_{t\geq 0} f(t), \quad \liminf_{t\to\infty} f(t) = \inf_{t\geq 0} f(t).$$

The second result we need is:

Lemma 6.2. Let f be a nonzero APF. Then:

$$\exists_{\delta>0}\,\exists_{\ell>0}\,\forall_{t_0\in\mathbb{R}} \int_{t_0}^{t_0+\ell} |f(t)|\,\mathrm{d}t > \delta.$$

Both results are proved in [3].

A *Bohl function* is a function $y : \mathbb{R} \to \mathbb{R}^n$ of the form:

$$y(t) = \sum_{\ell} p_\ell(t)\, e^{\lambda_\ell t}, \tag{2}$$

where the sum is finite and the $p_\ell(t)$'s are polynomials. For such a Bohl function y, where we assume that the λ_ℓ's are distinct and the p_ℓ's are nonzero, we define the following quantities:

- The *spectrum* $\bar{\sigma}(y)$: the set of λ_ℓ's occurring in (2). The elements of $\bar{\sigma}(y)$ will be called the *exponents* of y,
- The *spectral abscis* $\bar{\Lambda}(y) := \max\{\operatorname{Re}\lambda | \lambda \in \bar{\sigma}(y)\}$,
- The *index* $\nu(y)$: the highest degree of the $p_\ell(t)$'s occurring in the terms of (2) corresponding to the values of λ_ℓ such that $\operatorname{Re}\lambda_\ell = \bar{\Lambda}(y)$,
- The *order* $\omega(y)$: the pair $(\bar{\Lambda}(y), \nu(y))$.
- The *dominant spectrum* $\bar{\sigma}^*(y)$: the set of λ_ℓ's in (2) for which there exists a term $p_\ell(t)e^{\lambda_\ell t}$ with $(\operatorname{Re}\lambda_\ell, \deg p_\ell) = \omega(y)$.

We will use these notations for scalars, vectors and matrices. Note that $\sigma(A) = \bar{\sigma}(e^{tA})$, $\sigma^*(A) = \bar{\sigma}^*(e^{tA})$ and $\Lambda(A) = \bar{\Lambda}(e^{tA})$. The following decomposition of a Bohl function plays an important role in this paper.

Theorem 6.3. Let y be a real nonzero Bohl function. Then y can be written as

$$y(t) = t^\mu e^{\alpha t}(f(t) + g(t)),$$

where $(\alpha, \mu) = \omega(y)$, and f, g are real functions such that f is nonzero and almost periodic, and $g(t) \to 0$ $(t \to \infty)$. If y is scalar and $\alpha \notin \bar{\sigma}^*(y)$, then

- $\liminf_{t\to\infty} f(t) < 0$,
- $\limsup_{t\to\infty} f(t) > 0$.

Proof. Rewriting (2) as $y(t) = \sum_{j,\ell} a_{j\ell} t^j e^{\lambda_\ell t}$, we collect the terms $\alpha_{j\ell} t^j e^{\lambda_\ell t}$ for which $\lambda_\ell \in \bar{\sigma}^*(y)$ and $j = \nu(y)$. Because the function y is real, these terms can be combined to $y_1 = t^\mu e^{\alpha t} f(t)$, where $f(t)$ is an expression of the form $f(t) = \sum_k \gamma_k \cos(\omega_k t + \phi_k)$. The function $f(t)$ is obviously nonzero and almost periodic. Furthermore, by the definition of $\bar{\sigma}^*(y)$, it follows that $g(t) := t^{-\mu} e^{-\alpha t} y(t) - f(t) \to 0$ $(t \to \infty)$. If $\alpha \notin \bar{\sigma}^*(y)$, none of the the ω's in the sum defining $f(t)$ is zero. We show that this implies that $\liminf_{t\to\infty} f(t) < 0$. If $\liminf_{t\to\infty} f(t) \geq 0$, Lemma 6.1 implies $f(t) \geq 0$ for all $t \geq 0$. Then the function F defined by $F(t) := \sum_k \gamma_k \omega_k^{-1} \sin(\omega_k t + \phi_k)$ satisfies $F'(t) = f(t)$ and hence is increasing. However, this function F is also an APF. In particular, F is bounded, which implies that $\lim_{t\to\infty} F(t)$ exists. But now Lemma 6.1 implies that $\inf F(t) = \sup F(t) = \lim F(t)$, and hence that $F(t)$ is constant. That is, $f(t)$ is identically zero, which is a contradiction. Similarly, the assumption $\limsup_{t\to\infty} f(t) \leq 0$ leads to a contradiction. □

A scalar Bohl function that satisfies the condition $\bar{\Lambda}(y) \notin \bar{\sigma}^*(y)$ is called *oscillating*. It follows from the above theorem that such a function has infinitely many zeroes on the positive real axis. We will also need the following estimate:

Lemma 6.4. Let y be a nonzero Bohl function. Then there exists $\delta > 0$ and T_0 such that

$$\int_0^T |y(t)|\, dt \geq \delta T^\mu e^{\alpha T}$$

for all $T \geq T_0$, where $(\alpha, \mu) := \omega(y)$.

Proof. We decompose $y(t)$ according to the previous lemma as $y(t) = t^{\mu}e^{\alpha t}(f(t) + g(t))$. Then, according to Lemma 6.2, there exists numbers $\delta_1 > 0, \ell > 0$ such that for all $t_0 \in \mathbb{R}$, the inequality $\int_{t_0}^{t_0+\ell} |f(t)|\,dt > \delta_1$ holds. For $T \geq \ell + 1$, the quantity $(T-\ell)^{\mu}e^{\alpha(T-\ell)}/(T^{\mu}e^{\alpha T})$ is bounded from below by, say, $\gamma_1 > 0$. Also, because $g(t) \to 0$ $(t \to \infty)$, there exists $t_1 > 1$ such that $|g(t)| < \varepsilon := \gamma_1\delta_1/(2\ell)$ $(t \geq t_1)$. Then we find for $T \geq t_1 + \ell$

$$\int_0^T |y(t)|\,dt \geq \int_{T-\ell}^T |y(t)|\,dt \geq \int_{T-\ell}^T t^{\mu}e^{\alpha t}|f(t)|\,dt - \int_{T-\ell}^T t^{\mu}e^{\alpha t}|g(t)|\,dt$$

$$\geq (T-\ell)^{\mu}e^{\alpha(T-\ell)}\delta_1 - T^{\mu}e^{\alpha T}\ell\varepsilon \geq T^{\mu}e^{\alpha T}(\gamma_1\delta_1 - \ell\varepsilon) = T^{\mu}e^{\alpha T}\delta,$$

where $\delta := \gamma_1\delta_1/2$. □

Finally we derive a result that relates to linear systems:

Theorem 6.5. Let (c, A, B) be a system and let $\lambda \in \sigma^*(A)$ be a controllable and observable eigenvalue. Then

1. $\omega(ce^{tA}B) = \omega(e^{tA})$,
2. There exists a vector v such that $\bar{\sigma}(ce^{tA}v) = \{\lambda\}$ and $\omega(ce^{tA}v) = \omega(e^{tA})$.

Proof.

1. It is rather obvious that, if $(\alpha, \mu) = \omega(Ce^{tA}B)$, then $\bar{\Lambda}(e^{tA}) \geq \alpha$, and if $\bar{\Lambda}(e^{tA}) = \alpha$, then $\nu(e^{tA}) \geq \mu$. In order to prove the inverse inequalities, we show that $\lambda \in \bar{\sigma}(ce^{tA}B)$ and that the multiplicity of λ as an eigenvalue of A equals $\nu(ce^{tA}B)$. Using a state transformation we achieve that the matrices c, A, B take the form

$$\bar{A} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \bar{c} = [\bar{c_1},\ \bar{c_2}],$$

where the $q \times q$ matrix $\sigma(A_1) = \{\lambda\}$ and $\lambda \notin \sigma(A_2)$. Also, A_1 has the form

$$A_1 = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & \lambda \end{bmatrix}.$$

Note that A contains only one Jordan block corresponding to the eigenvalue λ, because of the (c, A)-observability of λ. If $\bar{c}_1 = [c_1, \ldots, c_q]$, then observability of λ implies that $c_1 \neq 0$. Similarly, the last row of B_1, say b_q is nonzero. Therefore, a short computation yields that $ce^{tA_1}B = e^{\lambda t}(c_1 b_q t^{q-1}/(q-1)! +$ terms of lower degree). This implies the required result.

2. Take in the above decomposition, $\bar{v} = [v_1', 0]'$, where $v_1 := [0, \ldots, 1]'$, and apply the previous result.

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REFERENCES

[1] H. Bohr: Almost Periodic Functions, Chelsea, New York 1947.
[2] R. Koplon: Linear Systems with Constrained Outputs and Transitions. Thesis, Dept. of Math. New Brunswick, New Jersey, October 1994.
[3] R. Koplon, E. D. Sontag and M. L. J. Hautus: Observability of linear systems with saturated outputs. Linear Algebra Appl. *205/206* (1994), 909–936.
[4] R. Schwarzschild, E. D. Sontag and M. L. J. Hautus: Output-saturated systems. In: Proc. Amer. Control Conf., Chicago 1992, pp. 2504–2509.
[5] E. D. Sontag: An algebraic approach to bounded controllability of linear systems. Internat. J. Control *39* (1984), 181–188.

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