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REGIONAL POLE PLACEMENT

VLADIMÍR KUČERA AND FRANTIŠEK J. KRAUS

The problem of pole placement is studied for a fixed plant with rational transfer function and unsharply given desired poles. All output feedback controllers are parametrized which will place the poles within a subset of the complex plane that corresponds to a polytope in the coefficient space.

1. INTRODUCTION

The pole placement has proved to be one of the most successful design methods for linear control systems. Ultimately, *each* design specification leads, either directly or indirectly, to a particular assignment of the poles of the system.

The practical designs, however, rarely call for *exact* pole positions. The poles are required instead to lie within a subset of the complex plane.

Well known examples of the subsets are half–planes or sectors used to achieve a desired stability margin. These subsets are defined directly through the desired pole loci in the complex plane.

We propose here an alternative: the subset is defined in an implicit manner by specifying a desired region for the coefficients of the characteristic polynomial. Preferably it willl be a polytope. This approach has been employed [1] to solve the robust pole placement, where the plant lives in a polytope also. The set of controllers that achieve the desired pole placement, however, was not parametrized in [1].

2. FIXED POLES

We briefly review a solution to the problem of assigning a *fixed* characteristic polynomial. We are given a plant with strictly proper real–rational transfer function

$$G(s) = \frac{B(s)}{A(s)}$$

where $A(s)$ and $B(s)$ are coprime polynomials. We suppose that the plant is a minimal realization of $G(s)$.

Further given is a polynomial $C(s)$ and we seek to determine all output feedback controllers with proper rational transfer function, say $K(s)$, such that the closed-loop system has characteristic polynomial $C(s)$.

Let $X(s)$, $Y(s)$ be any polynomials that satisfy the equation

$$A(s)X(s) + B(s)Y(s) = C(s) \tag{1}$$

as well as the constraint

$$\deg X(s) \geq \deg Y(s). \tag{2}$$

Then [2]

$$K(s) = -\frac{Y(s)}{X(s)}.$$

If

$$\deg C(s) \geq 2 \deg A(s) - 1 \tag{3}$$

then polynomials satisfying (1) and (2) exist no matter which polynomial $C(s)$ is chosen. Denote $X'(s), Y'(s)$ the solution pair of (1) that satisfies $\deg Y'(s) < \deg A(s)$; it is the least-degree solution pair of (1) with respect to $Y(s)$. Then the solution set of (1) having the property (2) is given as [3]

$$\begin{aligned} X(s) &= X'(s) - B(s)T(s) \\ Y(s) &= Y'(s) + A(s)T(s) \end{aligned}$$

where either $T(s) = 0$ or $T(s)$ is a free polynomial parameter of degree

$$\deg T(s) \leq \deg C(s) - 2 \deg A(s).$$

Polynomials satisfying (1) and (2) can exist even for polynomials $C(s)$ having degree lower than that indicated in (3). This situation, however, is not generic.

3. POLES WITHIN A REGION

When the desired poles are not fixed but limited to a certain region in the complex plane, one can proceed as follows [4].

Call D the region; as a matter of fact, it can be any subset of the complex plane not including the point $s = \infty$. Let $R_D(s)$ be the set of rational functions having no pole outside D. Then $R_D(s)$ is a ring, in particular a principal ideal domain [4].

Recall that the plant transfer function, $G(s)$, is a strictly proper rational function and write it as

$$G(s) = \frac{\bar{B}(s)}{\bar{A}(s)}$$

where $\bar{A}(s)$ and $\bar{B}(s)$ are coprime functions from $R_D(s)$. Consider the equation

$$\bar{A}(s)\bar{X}(s) + \bar{B}(s)\bar{Y}(s) = 1 \tag{4}$$

in $R_D(s)$ and denote $\bar{X}'(s)$, $\bar{Y}'(s)$ any particular solution of (4). Then the solution set of (4) is given as

$$\begin{aligned}\bar{X}(s) &= \bar{X}'(s) - \bar{B}(s)\bar{T}(s)\\ \bar{Y}(s) &= \bar{Y}'(s) + \bar{A}(s)\bar{T}(s)\end{aligned}$$

where $\bar{T}(s)$ is a free parameter ranging over $R_D(s)$. Then [4]

$$K(s) = -\frac{\bar{Y}(s)}{\bar{X}(s)}$$

defines the set of all output feedback controllers that achieve the desired pole placement: all closed–loop poles are within D.

This general result covers the case where neither the position nor the *number* of poles is specified within D. In many practical situations, however, the number of poles is either fixed or at least limited from above. It is not a simple task to identify the subset of $\bar{T}(s)$'s that parametrize the controllers under this constraint. Instead we shall propose an alternative approach. The (largest) degree of the characteristic polynomials to be designed being fixed, we specify the pole positions by constraining the coefficients to a polytope in the coefficient space. A simple parametrization of the corresponding controllers can then be obtained.

4. PROBLEM FORMULATION

We shall now investigate the pole placement problem for a *polytope* of characteristic polynomials.

We consider a fixed plant that gives rise to a strictly proper rational transfer function

$$G(s) = \frac{B(s)}{A(s)}$$

where $A(s)$ and $B(s)$ are coprime polynomials. We suppose that the plant is a minimal realization of $G(s)$.

Also given is a polytope of polynomials

$$\Gamma_C = \left\{ C(s) | C(s) = \sum_{i=1}^{n} \lambda_i C_i(s),\ \lambda_i \geq 0,\ \sum_{i=1}^{n} \lambda_i = 1 \right\}$$

where $C_i(s)$, $i = 1, 2, ..., n$ are the corner polynomials of the desired region.

We seek to find the set of output feedback controllers with proper rational transfer function, say $K(s)$, such that the characteristic polynomial of the closed–loop system, $C(s)$, ranges over Γ_C.

5. SOLUTION

The problem will be solved by attending to the corners. Denote $X_i'(s)$, $Y_i'(s)$ the least–degree solution with respect to $Y_i(s)$ of the polynomial equation

$$A(s)X_i(s) + B(s)Y_i(s) = C_i(s)$$

for $i = 1, 2, \ldots, n$. Then for any $C(s) \in \Gamma_C$ we obtain for the associated set of multipliers λ_i, $i = 1, 2, \ldots, n$

$$A(s)\sum_{i=1}^{n}\lambda_i X_i'(s) + B(s)\sum_{i=1}^{n}\lambda_i Y'(s) = \sum_{i=1}^{n}\lambda_i C_i(s).$$

Therefore the solution set of equation (1) reads

$$\begin{aligned} X(s) &= \sum_{i=1}^{n}\lambda_i X_i'(s) - B(s)T(s) \\ Y(s) &= \sum_{i=1}^{n}\lambda_i Y_i'(s) + A(s)T(s) \end{aligned} \tag{5}$$

and, under the proviso that (3) holds for any polynomial $C(s) \in \Gamma_C$, the polynomials (5) will satisfy the constraint (2) if and only if either $T(s) = 0$ or $\deg T(s) \leq \deg C(s) - 2\deg A(s)$. Should (3) fail to hold for some $C(s) \in \Gamma_C$, the least-degree solution pair corresponding to $T(s) = 0$ in (5) can still satisfy (2).

The set of desirable controllers is then given by the transfer functions

$$K(s) = -\frac{Y(s)}{X(s)}. \tag{6}$$

It is a doubly parametrized family. The polynomial $T(s)$ parametrizes all controllers that assign the same $C(s)$ while the n-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ parametrizes all the $C(s)$ allowed.

The parameter $T(s) = 0$ yields the controllers of least McMillan degree that achieve the pole placement desired. If one allows a higher degree, then $T(s)$ can be used to meet further design specifications beyond the pole placement.

6. EXAMPLE

We consider a water tank with the inflow–to–level transfer function

$$G(s) = \frac{1}{s+1}.$$

To counteract level disturbances without offset, a PI controller is to be used,

$$K(s) = k_P + \frac{k_I}{s}.$$

We seek to tune the controller so that the closed–loop poles are placed within the disk

$$|s+3| < 1.$$

The closed–loop characteristic polynomial will have degree 2,

$$C(s) = s^2 + c_1 s + c_0$$

and the polytope Γ_C in the coefficient space that corresponds to the disk above is given by the inequalities

$$c_0 - 4c_1 + 16 > 0$$

$$c_0 - 3c_1 + 8 < 0$$

$$c_0 - 2c_1 + 4 > 0.$$

The three corner polynomials are

$$C_1(s) = s^2 + 4s + 4$$

$$C_2(s) = s^2 + 6s + 8$$

$$C_3(s) = s^2 + 8s + 16.$$

We now solve the polynomial equations

$$(s+1)X_i(s) + Y_i(s) = C_i(s)$$

for $i = 1, 2, 3$ to obtain the following least–degree solution pairs

$$X_1'(s) = s + 3, \quad Y_1'(s) = 1$$

$$X_2'(s) = s + 5, \quad Y_2'(s) = 3$$

$$X_3'(s) = s + 7, \quad Y_3'(s) = 9.$$

The solution set of equation (1) that satisfies (2) is given, in parametric form, by formula (5) in which $T(s) = \tau$, a real constant:

$$\begin{aligned} X(s) &= s + (3\lambda_1 + 5\lambda_2 + 7\lambda_3) - \tau \\ Y(s) &= (\lambda_1 + 3\lambda_2 + 9\lambda_3) + (s+1)\tau \end{aligned}$$

and $\lambda_1, \lambda_2, \lambda_3$ are real non–negative numbers such that $\lambda_1 + \lambda_2 + \lambda_3 = 1$.

Now τ can be chosen so as to obtain a PI controller. The choice

$$\tau = 3\lambda_1 + 5\lambda_2 + 7\lambda_3$$

yields

$$K(s) = -(3\lambda_1 + 5\lambda_2 + 7\lambda_3) - \frac{4\lambda_1 + 8\lambda_2 + 16\lambda_3}{s},$$

see (6).

7. CONCLUSIONS

The pole placement problem with unsharply given desired poles has been considered. The problem can easily be solved using polynomial techniques when the pole region is specified as a polytope in the coefficient space. The set of controllers that achieve the desired pole locations has been obtained in a simple *parametric* form. The parameters can be chosen so as to meet additional design specifications.

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Doc. Ing. Vladimír Kučera, DrSc., Ústav teorie informace a automatizace AV ČR (Institute of Information Theory and Automation – Academy of Sciences of the Czech Republic), Pod vodárenskou věží 4, 182 08 Praha 8. Czech Republic.

Dr. Ing. František J. Kraus, Automatic Control Laboratory, Federal Institute of Technology, ETH–Zentrum, CH–8092 Zürich. Switzerland.