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MAXIMAL FUZZY TOPOLOGIES

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In this paper we introduce and study maximal fuzzy P-spaces where P is fuzzy Lindelöf, fuzzy countably compact, fuzzy compact, fuzzy lightly compact or fuzzy strongly compact. Characterizations are given for maximal fuzzy P-spaces where P is fuzzy Lindelöf, fuzzy countably compact, fuzzy compact, or fuzzy strongly compact. Necessary condition is given for maximal fuzzy lightly compact spaces and fuzzy connected spaces.

1. INTRODUCTION

This paper can be considered as a continuation of [4] and it presents interesting relations among various notions derived from maximal fuzzy topologies. A fuzzy topological space [5] with property P is said to be maximal P if there is no strictly larger fuzzy topology on X with property P. In this paper we shall investigate maximal fuzzy P-spaces where P is fuzzy connectedness, fuzzy lightly compactness etc in a manner similar to that of [2].

2. FUZZY CONNECTED SPACES

A fuzzy topological space (X, T) is defined to be fuzzy connected [6] if it has no proper fuzzy clopen set. We define such a fuzzy connected space to be maximal fuzzy connected if any fuzzy connected topology stronger than T necessarily coincides with T. If λ is a fuzzy set in a fuzzy topological space then the closure and the interior of λ will be as usual denoted by $\overline{\lambda}$ and λ^0 respectively. A fuzzy set λ is called fuzzy regular open [4] if $\lambda = (\overline{\lambda})^0$. Given any fuzzy topological space (X, T), the fuzzy regular open sets in T form a base for a unique fuzzy topology T_0 called the fuzzy semi-regular topology on X associated with T. A fuzzy topology T is fuzzy semi-regular [3] $\Leftrightarrow T = T_0$.

Let (X, T_0) be a fuzzy semi-regular space.

Let $E(T_0) = \{T^* | T^*$ is a fuzzy topology on X and $(T^*)_0 = T_0\}$. For any two elements T_1, T_2 in $E(T_0)$ define $T_1 \leq T_2$ if T_1 is weaker than T_2. It can be shown that $E(T_0)$ has a maximal element. A maximal element of $E(T_0)$ is called a submaximal fuzzy topology and X endowed with such a fuzzy topology is referred to as a submaximal fuzzy topological space. In this connection we establish the following results:

Proposition 1. A fuzzy topological space (X, T) is fuzzy connected $\Leftrightarrow$ (X, T_0) is fuzzy connected.

Proof. Suppose T is not fuzzy connected. Then there exists a proper fuzzy set λ which is both open and closed, so λ is regular open and regular closed. Therefore T_0 is not fuzzy connected which is a contradiction. The converse follows since T_0 is weaker than T. □

The following two results can be deduced from the above Proposition 1.

Proposition 2. A maximal fuzzy connected space is submaximal.

Proposition 3. A fuzzy topology T on X is submaximal $\Leftrightarrow$ every fuzzy set λ in X such that $\overline{\lambda} = 1$ belongs to T.

3. FUZZY LIGHTLY COMPACT SPACE

We define the concept of fuzzy lightly compact space based on the corresponding concept in topology given in [2].

Definition. A fuzzy topological space (X, T) is said to be fuzzy lightly compact if for all $\{\lambda_i\}_{i=1}^{\infty} \subset T$ with $\sup\{\lambda_i\} = 1$, there exists an $n_0 \in N$ such that

$$\sup\left\{\overline{\lambda_i}\right\}_{i=1}^{n_0} = 1.$$

In this connection one can prove the following results easily.

Proposition 4. A space (X, T) is fuzzy lightly compact $\Leftrightarrow$ (X, T_0) is fuzzy lightly compact.

Proposition 5. A maximal fuzzy lightly compact space (X, T) is submaximal.

The converse of Proposition 5 is not true as the following example shows.

Example. Let $X = \{x_0, x_1, x_2, \ldots\}$ be a set of points. Let A be any element in $\mathcal{P}(X - x_0)$. Define

$$f_0 : X \longrightarrow [0,1] \quad \text{as} \quad f_0(x) = 0 \quad \text{for all } x \in X,$$

$$f_1 : X \longrightarrow [0,1] \quad \text{as} \quad f_1(x) = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{if } x \neq x_0 \end{cases}$$

$$\text{and} \quad f_A : X \longrightarrow [0,1] \quad \text{as} \quad f_A(x) = \begin{cases} 1 & \text{if } x \in A \cup \{x_0\} \\ 0 & \text{if } x \notin A \cup \{x_0\} \end{cases}.$$

Let $T = \{f_0, f_1, f_A | A \in \mathcal{P}(X - x_0)\}$. Then (X, T) is fuzzy lightly compact and submaximal.

Let us now fix another point $x_1 \in X$ and define

$$g_0 : X \longrightarrow [0,1] \quad \text{as} \quad g_0(x) = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{otherwise} \end{cases}$$

$$g_1 : X \longrightarrow [0,1] \quad \text{as} \quad g_1(x) = \begin{cases} 1 & \text{if } x = x_1 \\ 0 & \text{otherwise.} \end{cases}$$

For all $y \in X \setminus \{x_0, x_1\}$, define

$$\begin{aligned} g_y : X \longrightarrow [0,1] \quad \text{as} \quad g_y(x_0) &= 1 \\ g_y(y) &= 1 \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Let T' be the fuzzy topology generated by the base $\{g_0, g_1, g_y | y \in X \setminus \{x_0, x_1\}\}$. Then (X, T') is fuzzy lightly compact and T' is strictly stronger than T and therefore (X, T) is not maximally fuzzy lightly compact.

4. FUZZY COMPACT (COUNTABLY COMPACT, LINDELÖF) SPACES

In this section we give characterizations for maximal fuzzy compact (Countably compact, Lindelöf) spaces.

Definition. Let (X, T) be any fuzzy topological space. (X, T) is said to be topologically generated [8] fuzzy compact (Countably compact, Lindelöf) space if there exists a compact (Countably compact, Lindelöf) topology $\mathcal{T}$ on X such that $T = \omega(\mathcal{T}) = \mathcal{F}$, where $\mathcal{F}$ is the set of all continuous functions from $(X, \mathcal{T})$ to I.

We make use of the following results from [4] and [8] to prove Proposition 6.

Theorem A. [4] Let (X, T) be a fuzzy countably compact (fuzzy compact or fuzzy Lindelöf) and $\delta \notin T$. Then $(X, T(\delta))$ is fuzzy countably compact (fuzzy compact, fuzzy Lindelöf) $\Leftrightarrow$ $1 - \delta$ is fuzzy countably compact (fuzzy compact or fuzzy Lindelöf).

Theorem B. [8] If (X, T) is a topologically generated fuzzy compact space, then every fuzzy closed set is fuzzy compact.

Theorem C. [8] If $f : (X, T) \longrightarrow (Y, T')$ is fuzzy continuous and λ is fuzzy compact fuzzy set in (X, T), then $f(\lambda)$ is fuzzy compact.

Proposition 6. The following are equivalent for a topologically generated fuzzy compact (Countably compact, Lindelöf) space (X, T).

1. (X, T) is maximal fuzzy compact (Countably compact, Lindelöf) space (X, T).
2. The set of all fuzzy compact (Countably compact, Lindelöf) sets of X coincides with the set of all fuzzy closed sets of X.
3. If Y is topologically generated fuzzy compact (Countably compact, Lindelöf) space and if f is any fuzzy continuous bijection from Y onto X, then f is a fuzzy homeomorphism.

Proof. We prove this proposition for fuzzy compact spaces and the proof is similar for the other two cases.

(1) $\Longrightarrow$ (2). Suppose there exist a fuzzy compact set λ which is not fuzzy closed. Then $1-\lambda \notin T$ and $(X,\, T(1-\lambda))$ where $T(1-\lambda) = \{[(1-\lambda)\wedge\mu]\vee\nu \mid \mu,\, \nu \in T\}$ is fuzzy compact by Theorem A. This is a contradiction to our assumption (1). Therefore every fuzzy compact set is fuzzy closed. Also from Theorem B it follows that every fuzzy closed set of X is fuzzy compact. Hence (1) $\Longrightarrow$ (2).

(2) $\Longrightarrow$ (3). We need to show that f^{-1} is fuzzy continuous. Let λ be a fuzzy closed set in Y. Then $(f^{-1})^{-1}(\lambda) = f(\lambda)$ and as λ is closed in $Y \Rightarrow \lambda$ is a fuzzy compact set in $Y \Rightarrow f(\lambda)$ is a fuzzy compact (by Theorem C) $\Rightarrow f(\lambda)$ is a fuzzy closed set in X (by assumption (2)). This proves f^{-1} is fuzzy continuous. Hence (2) $\Longrightarrow$ (3).

(3) $\Longrightarrow$ (1). Let T' be any topologically generated fuzzy compact topology on X such that $T \leq T'$. Then the identity map $i : (X,T) \longrightarrow (X,T')$ satisfies the condition (3) and so $T = T'$. That is (X,T) is maximally fuzzy compact. □

Definition. [4] Let (X,T) be a fuzzy topological space and λ be a fuzzy set in X. λ is called a fuzzy G_δ-set if

$$\lambda = \bigwedge_{i=1}^{\infty} \lambda_i \quad \text{where each} \quad \lambda_i \in T.$$

A fuzzy topological space (X,T) is called a fuzzy G_δ-space [1] if every fuzzy G_δ-set is fuzzy open.

In this connection, we have the following result.

Proposition 7. Let (X,T) be a topologically generated fuzzy Lindelöf space. If (X,T) is maximal fuzzy Lindelöf, then X is fuzzy G_δ-space.

Proof. Let λ be any G_δ-set of X. Then we can write $\lambda = \bigwedge_{n=1}^{\infty}\lambda_n$ where each $\lambda_n \in T$. Now $1-\lambda = \bigvee_{n=1}^{\infty}(1-\lambda_n)$. From Proposition 6 we find that $1-\lambda$ is closed and so $\lambda \in T$. □

Remark 1. In [7] a new definition for the notion of compactness is introduced viz a fuzzy set λ of (X, T) is said to be "compact" $\Leftrightarrow$ each filter basis $\mathcal{B}$ such that every finite intersection of members of $\mathcal{B}$ is quasi-coincident with λ, $(\wedge\overline{\lambda}) \wedge \lambda \neq 0,\ \lambda \in \mathcal{B}$. To distinguish from the above compactness notion let us denote this by "compact*". In a similar manner one can also define "Lindelöf*". Regarding these notions of compact* and Lindelöf* one can prove the following results. For concepts not defined here we refer to [7].

Result 1. The following are equivalent for a topologically generated fuzzy compact* (Lindelöf*) space (X, T).

1. (X, T) is maximal fuzzy compact* (Lindelöf*).
2. The set of all fuzzy compact* (Lindelöf*) sets of X coincide with the set of all fuzzy closed sets of X.
3. If Y is topologically generated fuzzy compact* (Lindelöf*) space and if f is any fuzzy continuous bijection from Y onto X, then f is a fuzzy homeomorphism.

Result 2. Let (X, T) be a topologically generated fuzzy Lindelöf* space. (X, T) is maximal fuzzy Lindelöf* and fuzzy Hausdorff $\Leftrightarrow$ X is fuzzy Lindelöf*, fuzzy Hausdorff and fuzzy G_δ space. Also $X \times X$ is maximal fuzzy Lindelöf* if $X \times X$ is fuzzy Lindelöf*, X is maximal fuzzy Lindelöf* and fuzzy Hausdorff.

Remark 2. The study of the product of maximal P spaces where P is fuzzy Lindelöf, fuzzy countably compact etc is rendered uninteresting by the fact that these fuzzy topological properties are not productive in general.

5. STRONGLY COMPACT FUZZY TOPOLOGICAL SPACES

Definition. [9] Let λ be a fuzzy subset of a fuzzy topological space X. λ is said to be pre-open if $\lambda < (\overline{\lambda})^0$. The set of all pre-open fuzzy sets of X is denoted by $PO(X)$. Let (Y, S) be another fuzzy topological space. Let T_Φ be a fuzzy topology on X which has $PO(X)$ as a subbase. A mapping $f : X \to Y$ is Φ-continuous if $f : (X, T_\Phi) \longrightarrow (Y, S)$ is continuous. f is said to be Φ'-continuous if $f : (X, T_\Phi) \longrightarrow (Y, S_\Phi)$ is Φ-continuous. A fuzzy topological space X is said to be fuzzy strongly compact if every pre-open cover of X has a finite subcover.

We make use of the following two Theorems from [9] in establishing Proposition 8.

Theorem D. Let (X, T) be a fuzzy topological space which is strongly compact. Then each T_Φ-closed fuzzy set in X is strongly compact.

Theorem E. Let X and Y be fuzzy topological spaces and let $f : X \to Y$ be Φ'-continuous. If a fuzzy subset λ of X is strongly compact relative to X, then $f(\lambda)$ is strongly compact relative to Y.

Proposition 8. For a fuzzy topological space the following are equivalent:

1. (X,T) is maximal fuzzy strongly compact.
2. The class of strongly compact fuzzy sets of X equals the class of T_Φ-closed fuzzy subsets of X.
3. If (Y,S) is a fuzzy strongly compact space and if f is any Φ'-continuous bijection from Y onto X, then f is a fuzzy homeomorphism.

Proof. (1) $\Longrightarrow$ (2). By Theorem A, T_Φ-closed fuzzy sets are strongly compact. Suppose there exists a fuzzy strongly compact fuzzy set λ which is not T_Φ-closed. Then $1-\lambda \notin T_\Phi$ and $(X,\ T(1-\lambda))$ where $T(1-\lambda) = \{[(1-\lambda)\wedge\mu]\vee\nu \mid \mu,\ \nu \notin T\}$ is strongly compact which is such that $T < T(1-\lambda)$ which is a contradiction. Hence (1) $\Longrightarrow$ (2).

(2) $\Longrightarrow$ (3). Let λ be any S_Φ-closed fuzzy set in Y. Then $(f^{-1})^{-1}(\lambda) = f(\lambda)$ and it is sufficient if we show $f(\lambda)$ is T_Φ-closed. Since λ is S_Φ-closed, λ is fuzzy strongly compact in Y by Theorem D and $f(\lambda)$ is fuzzy strongly compact in X by Theorem E. That is $f(\lambda)$ is T_Φ-closed set in X. Hence (2) $\Longrightarrow$ (3).

(3) $\Longrightarrow$ (1). Suppose T' is any strongly compact fuzzy topology on X such that $T' \geq T$. Now the identify map $i:(X,T') \longrightarrow (X,T)$ satisfies the condition (3). Therefore $T=T'$. That is (X,T) is maximal fuzzy strongly compact. □

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