KYBERNETIKA — VOLUME 31 (1995), NUMBER 4, PAGES 413-422

ANTICIPATION IN DISCRETE-TIME LQ CONTROL I: Open-Loop Control

VÁCLAV SOUKUP

Polynomial approach is used to explain the partial improvement in single-input, single-output (SISO) linear quadratic computer control design. It consists of input signals anticipation which is always applicable if external inputs to the process are known in advance. The open-loop structure is considered in this first part while the second part of the work will concern the feedback LQ control.

I.1. INTRODUCTION

Standard control algorithms result into the control actions which start at the same time when an external input change (reference, load disturbance) occurs. Minimizing a chosen performance index state space as well as input-output methods of the control design have been developed.

Nevertheless in many applications we know some time in advance when and how the input signals will turn in the future. Then a control action need not wait for such a determined change but can anticipate it. This possibility has been (probably for the first time) mentioned by Štecha and Havlena [9]. The idea of anticipation may be simply realized in computer-controlled processes using polynomial system and signal description. It is shown for quadratic cost function and SISO linear systems in this work.

Quadratic or least squares control strategy dominates in the control theory for a long time [3, 8, 1]. Also there are many works using algebraic methods in LQ and LQG discrete-time control. The fundamental results in this field have been derived in [7], the comprehensive study concerning SISO systems represents [4].

Unlike in [7] the approach based on the coprime polynomials in SISO problem description is used in this contribution.

At first necessary operations and symbols used in polynomial theory are briefly summarized in Preliminaries. Further particulars can be found in [7]. Following the short survey about the principle of open-loop LQ control the own anticipation problem is solved in Section 4. Some further, completing relations are shown in Section 5 and the work ends by the simple illustrative example.

I.2. PRELIMINARIES

Polynomials in d (one step delay) and recurrent power sequences as polynomial fractions are the main elements in SISO, linear, discrete-time systems polynomial theory.

A polynomial

$$a = \alpha_0 + \alpha_1 d + \cdots + \alpha_\eta d^\eta \tag{I.2.1}$$

has the degree $\deg a = \eta$ if $\alpha_\eta \neq 0$; $\deg 0 = -\infty$, and is causal $a = c^c$ if $\alpha_0 \neq 0$.

Sometimes it is more suitable to consider d as Z-transform complex variable $(d = z^{-1})$ and a polynomial $a = a(d)$ as a function of d.

A polynomial (2.1) can be factorized into

$$a = a^H a^= = a^+ a^- = a^+ a^0 a^=,$$

where all zeros of $a^H(d)$, $a^=(d)$, $a^+(d)$, $a^-(d)$ and $a^0(d)$ have the property $|d_i| \geq 1$, $|d_i| < 1$, $|d_i| > 1$, $|d_i| \leq 1$ and $|d_i| = 1$, respectively.

Further operations are defined and denoted by

$$a_* = \alpha_0 + \alpha_1 d^{-1} + \cdots + \alpha_\eta d^{-\eta},$$
$$\tilde{a} = a_* d^\eta = \alpha_\eta + \alpha_{\eta-1} d + \cdots + \alpha_0 d^\eta, \quad a^* = a^+ a^{-\sim}$$

and

$$a_* a = a^* a^*_* = s_* s,$$

where

$$s = a^* = s^H \quad \text{(spectral factorization)}.$$

Having two polynomials a, b we write (a, b) for their greatest common divisor, $b|a$, $b \neq 0$, if $a = c\,b$, and $b \sim a$ if $a = c\,b$ with $\deg c = 0$.

A recurrent power sequence

$$F = \frac{b}{a} = \varphi_\xi d^\xi + \varphi_{\xi+1} d^{\xi+1} + \ldots \tag{I.2.2}$$

has $\operatorname{ord} F = \xi$ if $\varphi_\xi \neq 0$ and is a) causal if $\xi \geq 0$, b) stable if it is causal and $\varphi_\xi \to 0$ for $\xi \to \infty$.

Provided that $(a, b) \sim 1$, it must be $a = a^c$ for causal sequence and $a = a^+$ for stable sequence in (2.2).

For a stable $F = \varphi_0 + \varphi_1 d + \ldots$ it is defined $F_* = \varphi_0 + \varphi_1 d^{-1} + \ldots$. Then

$$F_* F = \ldots + \gamma_1 d^{-1} + \gamma_0 + \gamma_1 d + \ldots = C_{\gamma *} + \gamma_0 + C_\gamma,$$

where $C_\gamma = \gamma_1 d + \gamma_2 d^2 + \ldots$.

The denotation

$$\gamma_0 = \langle F_* F \rangle = \sum_{\xi=0}^{\infty} \varphi_\xi^2$$

is used.

I.3. STANDARD OPEN-LOOP LQ CONTROL

Open-loop SISO discrete time control problem is sketched in Fig. I.1.

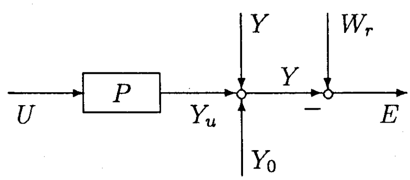

Fig. I.1.

A process output Y should track a reference W_r being affected by a possible load disturbance V at the same time. Current conditions at the control start depending on the system previous behaviour may be modeled by signal Y_0. All the signals are assumed to be described here in discrete-time forms and the discrete-time model P of a continuous-time process includes zero-order hold with a period T.

The aim of LQ optimal open-loop discrete-time control is to determine such a sequence U to minimize the performance index

$$\vartheta = \sum_{k=0}^{\infty} (\psi\, e_k^2 + \varphi\, u_k^2) = \psi\langle E_*\, E\rangle + \varphi\langle U_*\, U\rangle \tag{I.3.1}$$

where stable sequences

$$E = e_0 + e_1\, d + \ldots, \qquad U = u_0 + u_1\, d + \ldots,$$

and $e_k = e(k\,T)$ and/or $u_k = u(k\,T)$ are the error and/or control signal values at time $k\,T$; $k = 0, 1, \ldots$; $\psi \neq 0$, $\varphi \neq 0$ are chosen weighting scalars.

Seeing Fig. I.1 we can write

$$E = W_r - Y = W - Y_u = W - P\,U, \tag{I.3.2}$$

where

$$W = W_r - Y_0 - V = w_0 + w_1\, d + \ldots \tag{I.3.3}$$

represents the only equivalent input. Control action starts by the value u_0 synchronized with the input value w_0 at time $k\,T = 0$.

I.4. ANTICIPATION IN LQ OPEN-LOOP CONTROL

Assume that the input course (3.3) is known in advance, say ν steps before it will really act, i. e., at time $-\nu\,T$. Then the purposeful control signal can be applied and the error signal observed starting at this time. We can write

$$e_{k+\nu} = w_k - y_{u,k+\nu} \quad \text{or} \quad E\, d^{-\nu} = W - Y_u\, d^{-\nu}. \tag{I.4.1}$$

If the numbering of time is shifted by ν steps back then (4.1) obtains the form

$$E = W\,d^{\nu} - Y_u = W\,d^{\nu} - P\,U. \tag{I.4.2}$$

Let

$$P = \frac{b}{a}, \quad (a,b) \sim 1, \tag{I.4.3}$$

$$W = \frac{f}{h}, \quad (h,f) \sim 1, \tag{I.4.4}$$

and

$$a_h = \frac{a}{(a,h)}, \quad h_a = \frac{1}{(a,h)}. \tag{I.4.5}$$

Then LQ open-loop control with anticipation is solved by the following theorem.

Theorem 1. Given a process (4.3) subjected to the equivalent input (4.4), LQ open-loop control minimizing the expression (3.1) and using ν-step anticipation results into the sequence

$$U = \frac{a_h\,y}{h_a\,s}, \tag{I.4.6}$$

where the stable polynomial $s = s^{+}$ follows from the spectral factorization

$$s_*\,s = a_*\,\varphi\,a + b_*\,\psi\,b \tag{I.4.7}$$

and a_h, h_a stand in (4.5). The polynomial y along with x and z satisfies the equations

$$d^{\rho}\,s_*\,y + h\,z = d^{\rho+\nu}\,b_*\,\psi\,f \tag{I.4.8}$$

and

$$d^{\rho}\,s_*\,x - b_f\,h_a\,z = d^{\rho}\,a_*\,\varphi\,a_h\,f_b \tag{I.4.9}$$

with the minimum $\deg z$ where $\rho = \max(\deg a,\,\deg b)$,

$$b_f = \frac{b}{(b,\,f\,d^{\nu})} \quad \text{and} \quad f_b = \frac{f\,d^{\nu}}{(b,\,f\,d^{\nu})}. \tag{I.4.10}$$

The corresponding error signal E is expressed by

$$E = \frac{(b,\,f\,d^{\nu})\,x}{h_a\,s}. \tag{I.4.11}$$

The problem is solvable if and only if $h_a \sim h_a^{+}$. The solution (4.6), if it exists, is unique.

Proof. The proof will be divided into three parts.

1. At first it must be proved that $s = s^{+}$. It has been shown in [2]. Assume that $s = s^{H} = s^{+}\,s^{0}$, i.e., $s(d) = 0$ has a zero $d_i = \exp(-j\,\omega_i\,T)$, $|d_i| = 1$. Then seeing (4.7)

$$s_*(d_i)\,s(d_i) = a(d_i^{-1})\,\varphi\,a(d_i) + b(d_i^{-1})\,\psi\,b(d_i) = \varphi|a(d_i)|^2 + \psi|b(d_i)|^2 = 0$$

and hence $a(d_i) = 0$ as well as $b(d_i) = 0$ would have to be true. But it is excluded by the assumption $(a, b) \sim 1$ and therefore $s^0 = 1$.
2. Secondly the solvability of the equations (4.8) and (4.9) will be treated. Writing these equations in the vector-matrix form

$$C[x\ y\ z]^T = D$$

where
$$C = \begin{bmatrix} d^\rho s_* & 0 & -b_f a_h \\ 0 & d^\rho s_* & h \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} d^\rho a_* \varphi a_h f_b \\ d^{\rho+\nu} b_* \psi f \end{bmatrix},$$

the solution exists if and only if the greatest common divisors of all nonzero minors of C and $[C\ D]$ are the same. Let us write

the first order minors of C: $d^\rho s_*,\ b_f h_a,\ h$;
the first order minors of $[C\ D]$: $d^\rho s_*,\ b_f h_a,\ h,\ d^\rho a_* \varphi a_h f_b,\ d^{\rho+\nu} b_* \psi f$;
the second order minors of C: $d^{2\rho} s_*^2,\ d^\rho s_* h,\ d^\rho s_* b_f h_a$;
and the second order minors of $[C\ D]$: $d^{2\rho} s_*^2,\ d^\rho s_* h,\ d^\rho s_* b_f h_a,\ d^{2\rho} s_* a_* \varphi a_h f_b,$
$d^{2\rho+\nu} s_* b_* \psi f$ and $d^\rho s_* s h_a f_b$,

where the last minor follows from (4.7) and (4.10).

Denoting
$$g_1 = (d^\rho s_*,\ b_f h_a,\ h) = (d^\rho s_*,\ h_a)$$
and
$$g_2 = \left(d^{2\rho} s_*^2,\ d^\rho s_* h,\ d^\rho s_* b_f h_a\right) = d^\rho s_* g_1$$
it must be
$$g_1 | d^\rho a_* \varphi a_h f_b,\ g_1 | d^{\rho+\nu} b_* \psi f \quad \text{and} \quad g_2 | d^\rho s_* s h_a f_b,$$
i.e., $g_1 | s h_a f_b$. Supposing $\deg s = \sigma$ yields $g_1 = (d^{\rho-\sigma} \tilde{s},\ h_a)$. Hence assuming $h_a \sim h_a^+$ the equations (4.8) and (4.10) are always solvable since for $s = s^+$ is $\tilde{s}$ unstable and therefore $g_1 \sim 1$.
3. To prove that the optimal solution is given by Theorem 1 we express using (4.2) and (4.7)

$$\begin{aligned} J &= \psi E_* E + \varphi U_* U = \psi \frac{f_* f}{h_* h} - \psi \frac{b f_*}{a h_*} d^{-\nu} U - \psi \frac{b_* f}{a_* h} d^\nu U_* + \frac{s_* s}{a_* a} U_* U = \\ &= \psi \frac{f_* f}{h_* h} - \psi^2 \frac{f_* f b_* b}{h_* h s_* s} + \left(\psi \frac{b_* f d^\nu}{s_* h} - \frac{s}{a} U\right)_* \left(\psi \frac{b_* f d^\nu}{s_* h} - \frac{s}{a} U\right) = J_a + J_B, \end{aligned} \tag{I.4.12}$$

where
$$J_A = \psi \frac{f_* f}{h_* h} - \psi^2 \frac{f_* f b_* b}{h_* h s_* s} = \psi \varphi \frac{f_* f a_{h*} a_h}{h_{a*} h_a s_* s} \tag{I.4.13}$$
is independent on U and with the finite value $\vartheta_A = \langle J_A \rangle$ for $h_a \sim h_a^+$.

The second term J_B in (4.12) can be arranged into

$$\begin{aligned} J_B &= \left(\psi \frac{b_* f d^\nu}{s_* h} - \frac{s}{a} U\right)_* \left(\psi \frac{b_* f d^\nu}{s_* h} - \frac{s}{a} U\right) = \\ &= \left(\frac{d^\rho s_* y + h z}{d^\rho s_* h} - \frac{s}{a} U\right)_* \left(\frac{d^\rho s_* y + h z}{d^\rho s_* h} - \frac{s}{a} U\right) = \\ &= \left(\frac{y}{h} + \frac{z}{d^\rho s_*} - \frac{s}{a} U\right)_* \left(\frac{y}{h} + \frac{z}{d^\rho s_*} - \frac{s}{a} U\right) = \left(X + \frac{z}{d^\rho s_*}\right)_* \left(X + \frac{z}{d^\rho s_*}\right), \end{aligned} \tag{I.4.14}$$

where
$$X = \frac{y}{h} - \frac{s}{a}U. \qquad \text{(I.4.15)}$$

The opening decomposition yields just the equation (4.8). Its general solution may be written as
$$y = y_0 - h\,t \quad \text{and} \quad z = z_0 + d^\rho\, s_*\, t,$$
where y_0, z_0 is the particular solution with minimum $\deg z$, $\deg z < \rho$, and t an arbitrary polynomial.

Then in accordance with (4.14)
$$\vartheta_B = \langle J_B \rangle = \langle (X+t)_*\,(X+t)\rangle + \left\langle \frac{z_{0*}\, z_0}{s_*\, s} \right\rangle$$
since $\left\langle \left(\frac{z_0}{d^\rho\, s_*}\right)_* (X+t) \right\rangle = 0$ as well as $\left\langle \frac{z_0}{d^\rho\, s_*}(X_* + t_*) \right\rangle = 0$. Hence
$$\min_U \vartheta = \vartheta_A + \min_{X,t} \vartheta_B = \vartheta_A + \left\langle \frac{z_{0*}\, z_0}{s_* s} \right\rangle$$
for $X = 0$ and $t = 0$. Then seeing (4.15) the optimal control sequence U stands in (4.6).

If the equation (4.8) is multiplied by b we obtain
$$d^\rho\, s_*\, b\, y + b\, h\, z = d^{\rho+\nu}\, b_*\, \psi\, b\, f = d^{\rho+\nu}\, f(s_*\, s - a_*\, \varphi\, a)$$
and then
$$d^\rho\, s_*(s\, f\, d^\nu - b\, y) = d^{\rho+\nu}\, a_*\, \varphi\, a\, f + b\, h\, z = (a,h)\,(b, f\, d^\nu)\, d^\rho\, s_*\, x, \qquad \text{(I.4.16)}$$
where the complementary equation (4.9) has been introduced.

Combining (4.2) and (4.16) the error E results into (4.11). The condition $h_a \sim h_a^+$ is necessary for E and U to be stable. Then the equations (4.8) and (4.9) are always solvable as shown above. Since their minimum-degree solution with respect to z is unique the entire problem has the unique solution.

Note. Thus far the exact mathematical system solution of the problem has been shown. Analyzing the results one can see that the polynomial cancellations are assumed in the design:

a) the process factor a_h is cancelled by the same polynomial in control signal U not to appear in the output Y_u;

b) the common process-input factor (a,h) does not appear in the error E being cancelled by the difference $s\, f_b - b_f\, y = (a,h)\, x$.

Of course, small differences between a real process dynamics and its available mathematical model must always be expected in practical tasks. Since such distinctions, if occur between unstable factors, result in unstable error E, open loop LQ control may be applied, in fact, to stable real processes only.

I.5. FURTHER RELATIONS

Let us consider two solutions of the open-loop LQ anticipating control corresponding to two values of anticipation steps $\nu - i$ and ν, $i > 0$.

Using the relations and meanings used and derived above we can find that

$$y_\nu = y_{\nu-i}\, d^i - \frac{h}{d^\rho\, s_*}\left(z_\nu - z_{\nu-i}\, d^i\right), \quad x_\nu = x_{\nu-i} + \frac{b_f\, h_a}{d^\rho\, s_*}\left(z_\nu - z_{\nu-i}\right),$$

$$U_\nu = U_{\nu-i}\, d^i - \frac{a}{d^\rho\, s_*\, s}\left(z_\nu - z_{\nu-i}\, d^i\right),$$

$$E_\nu = \frac{(b,\, f\, d^\nu)}{(b,\, f\, d^{\nu-i})}\, E_{\nu-i} + \frac{b}{d^\rho\, s_*\, s}\left(z_\nu - z_{\nu-i}\right)$$

and

$$\vartheta_\nu = \vartheta_{\nu-i} + \left\langle \frac{z_{\nu *}\, z_\nu}{s_*\, s} \right\rangle - \left\langle \frac{z_{\nu-i,*}\, z_{\nu-i}}{s_*\, s} \right\rangle = \vartheta_{\nu-i} - \Delta\, \vartheta_{\nu,\nu-i},$$

where x_j, y_j, z_j, U_j, E_j and ϑ_j are optimal for j-step anticipating LQ solution, $j = \nu - i$, ν and

$$\Delta\, \vartheta_{\nu,\nu-i} = \left\langle \frac{z_{\nu-i,*}\, z_{\nu-i}}{s_*\, s} \right\rangle - \left\langle \frac{z_{\nu *}\, z_\nu}{s_*\, s} \right\rangle \tag{I.5.1}$$

is the decrease in ϑ if the anticipation steps increase from $\nu - i$ to ν. Intuitively we expect $\Delta\, \vartheta_{\nu,\nu-i} \geq 0$. However the decrease (5.1) depends on the process and input dynamics and cannot be simply expressed without solving both the corresponding cases, i. e., the problem has to be solved for $\nu - i$ as well as ν.

To investigate the theoretical limit case when $\nu \to \infty$ we must return to the primary numbering of time and instead of (4.2) write

$$\hat{E} = W - P\, \hat{U}, \tag{I.5.2}$$

where $\hat{E} = E\, d^{-\nu}$ and $\hat{U} = U\, d^{-\nu}$ are two-side noncausal sequences. Then $\hat{U}$ is required to be determined such that

$$\hat{\vartheta} = \sum_{k=-\infty}^{\infty} \left(\psi\, \hat{e}_k^2 + \varphi\, \hat{u}_k^2\right) = \psi \left\langle \hat{E}_*\, \hat{E} \right\rangle + \varphi \left\langle \hat{U}_*\, \hat{U} \right\rangle \tag{I.5.3}$$

is finite and minimal.

As regards the two-side sequences and corresponding polynomial equations the good references are [5, 6]. We mention here only that if

$$\hat{E} = \ldots + \hat{e}_{-2}\, d^{-2} + \hat{e}_{-1}\, d^{-1} + \hat{e}_0 + \hat{e}_1\, d + \hat{e}_2\, d^2 + \ldots$$

then

$$\hat{E}_* = \ldots + \hat{e}_2\, d^{-2} + \hat{e}_1\, d^{-1} + \hat{e}_0 + \hat{e}_{-1}\, d + \hat{e}_{-2}\, d^2 + \ldots$$

Using (5.2) in (5.3) yields

$$\hat{J} + \psi\, \hat{E}_*\, \hat{E} + \varphi\, \hat{U}_*\, \hat{U} = \hat{J}_A + \hat{J}_B,$$

where

$$\hat{J}_A = \psi \frac{f_*\, f}{h_* h} - \psi^2 \frac{f_*\, f\, b_*\, b}{h_*\, h\, s_*\, s} = \psi\, \varphi \frac{a_{h*}\, a_h\, f_*\, f}{h_a\, s_*\, s} = J_A$$

standing in (4.13) and

$$\hat{J}_B = \left(\frac{b_* \psi f}{h s_*} - \frac{s}{a}\hat{U}\right)_* \left(\frac{b_* \psi f}{h s_*} - \frac{s}{a}\hat{U}\right)$$

with

$$\hat{\vartheta}_B = \min_{\hat{U}} \left\langle \hat{J}_B \right\rangle = 0 \quad \text{for} \quad \hat{U} = \frac{a_h\, b_*\, \psi\, f}{h_a\, s_*\, s}$$

provided that $h_a \sim h_a^+$. Hence

$$\hat{\vartheta} = \lim_{\nu \to \infty} \vartheta = \lim_{\nu \to \infty} (\vartheta_A + \vartheta_B) = \vartheta_A < \vartheta_\nu \quad \text{for any finite } \nu.$$

I.6. EXAMPLE

Given

$$P = \frac{d}{1-d} = \frac{b}{a}, \qquad W_r = \frac{1}{1-d},$$

$$V = \frac{0.17\, d}{1 + 0.5\, d} \qquad \text{and} \quad Y_0 = \frac{0.33\, d}{1-d}.$$

Let us find LQ optimal control sequence assuming the inputs are known ν steps in advance, $\nu = 0,\ 1,\ 2,\ 3$, alternatively and $\psi = \varphi = 1$.

At first we determine

$$W = W_r - Y_0 - V = \frac{f}{h} = \frac{1}{(1-d)\, 1 + 0.5\, d)},$$

$$a_h = 1, \quad h_a = 1 + 0.5\, d = h_a^+, \quad s = 1.618 - 0.618\, d \quad \text{and} \quad \rho = 1.$$

1. For $\nu = 0$ (control without anticipation) we have $b_f = b = d$ and $f_b = f = 1$, the min deg z solution of the equation (4.8)

$$(-0.618 + 1.618\, d)\, y + (1-d)\,(1 + 0.5\, d)\, z = 1$$

is $y = 0.58 + 0.42\, d$ and $z = 1.358$. The optimal control sequence

$$U = \frac{0.58 + 0.42\, d}{(1 + 0.5\, d)\,(1.618 - 0.618\, d)} = 0.358 + 0.217\, d + 0.043\, d^2 + 0.037\, d^3 + \ldots$$

We can determine $x = 1.618 + 0.42\, d$ from the equation (4.9)

$$(-0.618 + 1.618\, d)\, x - d(1 + 0.5\, d)\, z = -1 + d.$$

Then the error sequence

$$E = \frac{1.618 + 0.42\, d}{(1 + 0.5\, d)\,(1.618 - 0.618\, d)} = 1 + 0.142\, d + 0.174\, d^2 + 0.007\, d^3 + \ldots$$

and the performance index $\vartheta = 1.23$.

2. For $\nu = 1$ we have $b_f = f_b = 1$, the equation (4.8)

$$(-0.618 + 1.618\,d)\,y + (1-d)\,(1+0.5\,d)\,z = d$$

is solved by $y = 0.84 + 0.16\,d$ and $z = 0.519$. Hence

$$U = \frac{0.85 + 0.16\,d}{(1+0.5\,d)\,(1.618 - 0.618\,d)} = 0.519 + 0.038\,d + 0.095\,d^2 - 0.015\,d^2 + \ldots;$$

the equation (4.9)

$$(-0.618 + 1.618\,d)\,x - (1+0.5\,d)\,z = -1 + d$$

yields $x = 0.778$.

Then

$$E = \frac{0.778}{(1+0.5\,d)\,(1.618 - 0.618\,d)} = 0.481\,d - 0.057\,d^2 + 0.099\,d^3 - 0.023\,d^4 + \ldots,$$

and $\vartheta = 0.53$.

3. Using $\nu = 2$ yields $b_f = 1,\ f_b = d$, the equation (4.8)

$$(-0.618 + 1.618\,d)\,y + (1-d)\,(1+0.5\,d)\,z = d^2$$

is solved by $y = 0.321 + 0.679\,d$ and $z = 0.198$, the equation (4.9)

$$(-0618 + 1.618\,d)\,x - (1+0.5\,d)\,z = (-1+d)\,d$$

gives $x = -0.321 + 0.618\,d$.

Then

$$\begin{aligned} U &= \frac{0.321 + 0.679\,d}{(1+0.5\,d)\,(1.618 - 0.618\,d)} = 0.198 + 0.396\,d - 0.008\,d^2 + 0.077\,d^3 + \ldots, \\ E &= \frac{(-0.321 + 0.618\,d)\,d}{(1+0.5\,d)\,(1.618 - 0.618\,d)} = -0.198\,d + 0.405\,d^2 - 0.086\,d^3 + \ldots \end{aligned}$$

and $\vartheta = 0.42$.

4. Finally for $\nu = 3$ we get $b_f = 1,\ f_b = d^2$, the equation (4.8) is solved by $y = 0.123 + 0.259\,d + 0.618\,d^2$ and $z = 0.076$. Then $x = -0.123 - 0.382\,d + 0.618\,d^2$,

$$\begin{aligned} U &= \frac{0.123 + 0.259\,d + 0.618\,d^2}{(1+0.5\,d)\,(1.618 - 0.618\,d)} = 0.076 + 0.151\,d + 0.378\,d^2 - 0.016\,d^2 + \ldots, \\ E &= \frac{(-0.123 - 0.382\,d + 0.618\,d^2)\,d}{(1+0.5\,d)\,(1.618 - 0.618\,d)} = -0.076\,d - 0.227\,d^2 + 0.394\,d^3 - 0.090\,d^4 + \ldots \end{aligned}$$

and $\vartheta = 0.41$.

The theoretical minimum for $\nu \to \infty$ is

$$\vartheta_A = \left\langle \frac{1}{(1+0.5\,d^{-1})\,(1+0.5\,d)\,(1.618 - 0.618\,d^{-1})\,(1.618 - 0.618\,d)} \right\rangle = 0.405.$$

(Received October 8, 1992.)

REFERENCES

[1] B. D. O. Anderson and J. B. Moore: Optimal Control – Linear Quadratic Methods. Prentice–Hall, Englewood Cliffs 1989.
[2] K. J. Aström and B. Wittenmark: Computer Controlled Systems. Prentice–Hall, Englewood Cliffs 1984.
[3] S. S. L. Chang: Synthesis of Optimum Control Systems. McGraw–Hill, New York 1961.
[4] K. J. Hunt: Stochastic Optimal Control Thecry with Application in Self Tuning Control. Springer Verlag, Berlin – Heidelberg 1989.
[5] J. Ježek: Symmetric polynomial equations. Part I. Kybernetika *19* (1983), 2, 121–130.
[6] J. Ježek: Conjugated and symmetric polynomial equations. Part II. Kybernetika *19* (1983), 3, 196–211.
[7] V. Kučera: Discrete Linear Control. Wiley, Chichester 1979.
[8] A. P. Sage: Optimum Systems Control. Prentice–Hall, Englewood Cliffs 1968.
[9] J. Štecha and V. Havlena: Coupled tanks. Preprints of International Summer School, ČVUT – ÚTIA, Praha 1992.

Doc. Ing. Václav Soukup, CSc., Elektrotechnická fakulta ČVUT (Faculty of Electrical Engineering – Czech Technical University), Karlovo nám. 13, 121 35 Praha 2. Czech Republic.