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SIMULTANEOUS STABILIZATION BASED ON OUTPUT MEASUREMENT

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Based on a recent convex programming algorithm for simultaneous stabilization by linear state feedback, we propose two types of control law for stabilizing a family of systems, when either a simultaneously stabilizing state feedback gain or a simultaneously stabilizing output injection matrix exists, and complete state information is not available. The proposed control laws are illustrated by a numerical example.

1. INTRODUCTION

The problem of simultaneously stabilizing a whole family of plants has received considerable attention for many years. The present work was motivated by a recent result [1] that provides a relatively simple algorithm for solving the following problem: Given a family of plants in state space representation (Φ_i, Γ_i), $i = 1, \ldots, M$, find a linear state feedback gain F such that $(\Phi_i + \Gamma_i F)$ is stable for $i = 1, \ldots, M$, or determine that no such F exists. This method is based on mapping the set of all simultaneously stabilizing linear feedback gains into a convex set, and employing a cutting plane technique involving a sequence of linear programming problems. In this note, we discuss ways of utilizing this approach for the case where complete state information is not available.

Dynamic Compensators

One way of approaching the simultaneous stabilization problem with incomplete state information is to use observer-based control laws, i.e. dynamic compensators. Necessary and sufficient conditions for the existence of simultaneously stabilizing compensators were given in [2], using coprime factorization. However, these conditions are tractable only for the case of two systems. In [3] it is in fact shown that for three or more systems the existence of a stabilizing compensator is "rationally undecidable" (i.e. there exists no explicit criterion).

Turning to the convex programming approach, the problem with observer-based controllers is that state feedback and state estimation cannot be separated in face of the uncertainty represented by a whole family of systems. Assuming that a simultaneously stabilizing state feedback gain has been found, it is possible to use the above-mentioned algorithm to search for a simultaneously stabilizing full order observer gain, but this search is dependent on the state feedback gain previously obtained. If no stabilizing observer for this state feedback exists, nothing can be said because there may exist stabilizing observers for different feedback gains.

In order to search directly for the compensator parameters, the problem can be transformed into an equivalent static output feedback problem; the difficulty in this case is that the set of all stabilizing output feedback gains cannot be mapped into a convex set. A way of attacking this problem is to search for a matrix which belongs to a convex set, and whose inverse belongs to another convex set. In [4] an algorithm for this problem is discussed, but this algorithm involves solving a sequence of convex programming problems, and its convergence is not guaranteed in general.

Periodic Output Feedback and Fast Output Sampling

Our approach taken here is based on the well known fact that if a system is controllable and observable, the poles of the system discretized at output sampling rate can be arbitrarily assigned by piecewise constant periodic output feedback, provided the number of gain changes during one output sampling interval is not less than the systems controllability index [5]. We show that the existence of a simultaneously stabilizing output injection matrix generically implies the existence of a simultaneously stabilizing piecewise constant periodic output feedback gain. The algorithm proposed in [1] can be used to search for such an output injection matrix, and any such matrix defines a set of stabilizing feedback gains, namely those which realize this output injection for the whole family of systems. Naturally it is desirable to choose within this set a feedback gain that yields the 'best performance' in some sense. Moreover, the condition that an admissible gain realize the same output injection for every system of the family is unnecessarily restrictive, and we propose an optimization procedure that allows to relax this condition and therefore to search for the optimal gain over a larger set.

The effect of using piecewise constant periodic output feedback can be viewed as increasing the number of inputs of an associated discrete-time system, such that the range space of that systems input matrix becomes the whole state space. This approach is different from simultaneous stabilization by periodic dynamic compensators proposed e. g. in [6]. The latter are essentially based on dividing an output sampling interval into as many subintervals as there are plants to stabilize, and to include a deadbeat controller for each plant. Unlike the approach taken here, in this case no attention is paid to performance considerations.

In addition to periodic output feedback for the case where a simultaneously stabilizing output injection matrix exists, we consider the dual case where a simultaneously stabilizing state feedback gain can be found. The dual approach then requires to increase the row rank of the measurement matrix of an associated discretized system, which can be achieved by sampling the output several times during one input sampling interval, and constructing the control signal from these output samples. We give conditions for the existence of a simultaneously stabilizing control law of this type.

This paper is organized as follows. Problem definition and preliminary results are given in Section 2. Section 3 presents results on periodic output feedback, and in Section 4 fast output sampling is discussed. In Section 5, the results are illustrated by a simple numerical example.

2. PROBLEM FORMULATION, PRELIMINARY RESULTS

We consider the problem of stabilizing simultaneously the collection of systems $S = \{A_i, B_i, C_i\}$, defined by

$$\begin{aligned} \dot{x}(t) &= A_i x(t) + B_i u(t) \\ y(t) &= C_i\, x(t), \qquad i = 1, \ldots, M. \end{aligned} \tag{1}$$

with $A \in \Re^{n\times n}, B \in \Re^{n\times m}, C \in \Re^{p\times n}$.

We assume that each (A_i, B_i, C_i) is controllable and observable.

Two types of control laws will be considered.

a) Periodic Output Feedback

Output measurements are available at time instants $t = k\tau,\ k = 0, 1, \ldots$. The control signal is generated according to

$$u(t) = K_l\, y(k\tau), \quad k\tau + l\Delta \le t < k\tau + (l+1)\,\Delta, \quad K_{l+N} = K_l, \tag{2}$$

for $l = 0, 1, \ldots$, where a sampling interval τ is divided into N subintervals $\Delta = \tau/N$. Note that the sequence of gain matrices $\{K_0, K_1, \ldots, K_{N-1}\}$, when substituted into (2), generates a time-varying, piecewise constant output feedback gain $K(t)$ for $0 \le t < \tau$.

b) Fast Output Sampling

Here output measurements are taken at time instants $t = l\Delta,\ l = 0, 1, \ldots$, whereas a constant control signal is applied over a period τ. The control signal is generated according to

$$u(t) = [L_0\ L_1 \ldots L_{N-1}] \begin{bmatrix} y(k\tau - \tau + \Delta) \\ y(k\tau - \tau + 2\Delta) \\ \vdots \\ y(k\tau) \end{bmatrix}, \quad k\tau < t \le (k+1)\tau. \tag{3}$$

Convex Programming

We briefly summarize the convex programming approach proposed in [1], in its version for discrete-time systems.

Let $\Phi_i \in \Re^{n\times n},\ \Gamma_i \in \Re^{n\times r},\ q = n + r$. Given a family of systems

$$S = \{\Phi_i, \Gamma_i\}, \quad i = 1, \ldots, M,$$

one can define a convex cone $\mathcal{C}(S) \subset \Re^{q\times q}$ and a mapping $g : \Re^{q\times q} \longrightarrow \Re^{r\times n}$ with the properties

$$\exists F : \quad \rho(\Phi_i + \Gamma_i F) < 1, \quad i = 1, \dots, M$$
$$\Longleftrightarrow \qquad \mathcal{C}(S) \neq \emptyset$$

and

$$W \in \mathcal{C}(S), \quad F = g(W) \qquad \Longrightarrow \qquad \rho(\Phi_i + \Gamma_i F) < 1, \quad i = 1, \dots, M,$$

where $\rho()$ denotes spectral radius. Moreover, by defining a convex function $\Re^{q\times q} \longrightarrow \Re$ with the property

$$f(W) \geq \|F\|,$$

one can formulate the search for a simultaneously stabilizing gain F as a convex programming problem

$$\min_{W \in \mathcal{C}(S)} f(W),$$

where the minimization yields a matrix with an upper bound on its norm minimized. This problem can then be solved by Kelley's cutting plane algorithm [7], i.e. by solving a sequence of linear programming problems. Because of the structure of the constraint region, computing separating hyperplanes is particularly simple in this case and involves only an eigenvalue problem.

The above algorithm is guaranteed to converge, if a solution exists, or otherwise the linear programming problem becomes unfeasible after a finite number of iterations.

3. PERIODIC OUTPUT FEEDBACK

Consider a system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{4}$$

and let $\Phi = e^{A\Delta}$, $\Gamma = \int_0^\Delta e^{As} B \,\mathrm{d}s$. Applying periodic output feedback (2) yields a closed loop system that satisfies

$$x(k\tau + \tau) = (\Phi^N + \mathbf{\Gamma}\mathbf{K}C)\, x(k\tau), \tag{5}$$

where

$$\mathbf{\Gamma} = [\Phi^{N-1}\Gamma \dots \Gamma],$$
$$\mathbf{K}^T = [K_0^T \dots K_{N-1}^T].$$

Note that asymptotic stability of (5) implies asymptotic stability of (4). Turning to the problem of stabilizing a family of systems $\{A_i, B_i, C_i\}_{i=1}^M$, equation (5) suggests the following: search for an output injection matrix G with the property

$$\rho(\Phi_i^N + GC_i) < 1, \qquad i = 1, \dots, M,$$

and, if it exists, find $\mathbf{K}$ such that

$$\mathbf{\Gamma}_i \mathbf{K} = G, \qquad i = 1, \ldots, M. \tag{6}$$

Any $\mathbf{K}$ that satisfies (6) yields a simultaneously stabilizing periodic output fedback gain when the corresponding matrix blocks are substituted into

$$K(t) = K_l, \qquad l\Delta \le t < (l+1)\,\Delta, \qquad l = 0, 1, \ldots$$

Existence of a Simultaneously Stabilizing Periodic Output Feedback Gain

We show that the existence of a simultanously stabilizing output injection matrix generically implies the existence of a simultaneously stabilizing output feedback gain K. By 'generically' we mean the following [8]: denote by $\mathcal{G}$ the set of all controllable families of systems S for which a simultaneously stabilizing output injection matrix exists, and by $\tilde{\mathcal{G}}$ the subset of $\mathcal{G}$ for which a stabilizing periodic output gain exists. Then $\tilde{\mathcal{G}}$ is open and dense in $\mathcal{G}$. (Here we consider $\mathcal{G}$ and $\tilde{\mathcal{G}}$ as subsets of $\Re^{M(n^2+nm+np)}$.)

To prove the above claim, assume G is simultaneously stabilizing, and define

$$\tilde{\Phi} = \begin{bmatrix} \Phi_1 & & 0 \\ & \ddots & \\ 0 & & \Phi_M \end{bmatrix}, \quad \tilde{\Gamma} = \begin{bmatrix} \Gamma_1 \\ \vdots \\ \Gamma_M \end{bmatrix}, \quad \tilde{G} = \begin{bmatrix} G \\ \vdots \\ G \end{bmatrix}, \tag{7}$$

then the linear equation

$$\left[\tilde{\Phi}^{N-1}\tilde{\Gamma} \ldots \tilde{\Gamma}\right] \begin{bmatrix} K_0 \\ \vdots \\ K_{N-1} \end{bmatrix} = \tilde{G} \tag{8}$$

has a solution if $(\tilde{\Phi}, \tilde{\Gamma})$ is controllable with controllability index $\tilde{\nu}_c$, and $N \ge \tilde{\nu}_c$; and any solution satisfies (6).

The following Lemma completes the proof.

Lemma 3.1. Controllability of (Φ_i, Γ_i), $i = 1, \ldots, M$, generically implies controllability of $(\tilde{\Phi}, \tilde{\Gamma})$.

Proof. See the appendix. □

Thus we have established the following

Theorem 3.1. For a family S of controllable systems, existence of a simultaneously stabilizing output injection matrix generically implies the existence of a simultaneously stabilizing periodic output feedback gain.

Remark. For the case where the matrices $\Phi_1 \dots \Phi_M$ have no common eigenvalue, it follows from the proof of Lemma 3.1 that existence of a simultaneously stabilizing output injection matrix always implies the existence of a simultaneously stabilizing periodic output feedback gain.

Closed Loop Performance

The above result is concerned with the existence of simultaneously stabilizing output feedback gains. Now we consider the closed loop performance under a simultaneously stabilizing control law. This point is crucial because a time-varying feedback gain may, even if stabilizing, cause excessive control action.

Assume that for a given family of plants a simultaneously stabilizing output injection matrix exists. Fix $N \geq \tilde{\nu}_c$, then the solutions of (8) form a set of simultaneously stabilizing gains. Within this set, we wish to find a gain that minimizes a performance index. But this set is unnecessarily narrow. If G is obtained by convex programming as discussed in Section 2, then it corresponds to an interior point of the convex cone $\mathcal{C}(S)$. By the nature of the cutting plane technique employed, a solution point will always lie on the boundary of a closed convex set contained in $\mathcal{C}(S)$, thus having a certain distance from the 'stability boundary'. Therefore, by not insisting that a feedback gain achieve the same right hand side in (6) for every system of the family, minimization can be carried out over a larger set, thereby improving performance. Requirement for simultaneous stability is of course that the G_i in

$$\left[\tilde{\Phi}^{N-1}\tilde{\Gamma}\dots\tilde{\Gamma}\right]\begin{bmatrix} K_0 \\ \vdots \\ K_{N-1} \end{bmatrix} = \begin{bmatrix} G_1 \\ \vdots \\ G_M \end{bmatrix} \tag{9}$$

are all stabilizing.

To accommodate the above considerations, we define a performance index as follows. Consider the auxiliary discrete-time system $(\tilde{\Phi}, \tilde{\Gamma}, \tilde{C})$, with $\tilde{\Phi}$ and $\tilde{\Gamma}$ defined as in (7), and

$$\tilde{C} = \frac{1}{M}[C_1 \dots C_M].$$

Denote the auxiliary state by ξ_l, i.e.

$$\xi_{l+1} = \tilde{\Phi}\xi_l + \tilde{\Gamma}u_l,$$

and consider the N-periodic output feedback law

$$u_{kN+l} = K_l\tilde{C}\xi_{kN}, \qquad K_{l+N} = K_l.$$

Choose weight matrices

$$R, \quad \tilde{Q} = \mathrm{diag}(Q_1 \dots Q_M), \quad \tilde{P} = \mathrm{diag}(P_1 \dots P_M),$$

whith $R \in \Re^{m\times m}$, Q_i, $P_i \in \Re^{n\times n}$ positive definite and symmetric. Let ξ^*_{kN} denote the state that would be reached at instant $l = kN$, given $\xi_{(k-1)N}$, if the gain K would satisfy (8), i.e.

$$\xi^*_{kN} = (\tilde{\Phi}^N + \tilde{G}\tilde{C})\xi_{(k-1)N}.$$

Then we want to find the gain that minimizes

$$J(K) = \sum_{l=0}^{\infty}[\xi_l^T \; u_l^T]\begin{bmatrix} \tilde{Q} & 0 \\ 0 & R \end{bmatrix}\begin{bmatrix} \xi_l \\ u_l \end{bmatrix} + \sum_{k=1}^{\infty}(\xi_{kN} - \xi_{kN}^*)^T \tilde{P}(\xi_{kN} - \xi_{kN}^*). \quad (10)$$

Before we show how to (approximately) minimize this cost function, we discuss the effect the two cost terms and its weights have on the solution. Roughly speaking, the first term represents 'averaged' state and control energy of all systems of the family, whereas the second term penalizes deviation of the G_i's in (9) from G. As stated in Corollary 3.1 below, $G_i \to G$ for $i = 1, \dots, M$ as $\tilde{P} \to \infty$.

To see the relation between the first term and the performance of the systems to be stabilized, partition $\xi^T = [x^{(1)^T} \dots x^{(M)^T}]^T$, and consider the closed loop solution of the auxiliary system

$$\begin{bmatrix} x_l^{(1)} \\ \vdots \\ x_l^{(M)} \end{bmatrix} = \left(\begin{bmatrix} \Phi_1^l & & \\ & \ddots & \\ & & \Phi_M^l \end{bmatrix} \right.$$

$$\left. + \frac{1}{M}\begin{bmatrix} \Phi_1^{l-1}\Gamma_1 & \dots & \Gamma_1 \\ \vdots & & \vdots \\ \Phi_M^{l-1}\Gamma_M & \dots & \Gamma_M \end{bmatrix}\begin{bmatrix} K_0 \\ \vdots \\ K_{l-1} \end{bmatrix}[C_1 \dots C_M]\begin{bmatrix} x_0^{(1)} \\ \vdots \\ x_0^{(M)} \end{bmatrix}\right)$$

$$= \begin{bmatrix} \Phi_1^l + \frac{1}{M}\mathbf{F}_1^l\mathbf{K}C_1 & \frac{1}{M}\mathbf{F}_1^l\mathbf{K}C_2 & \dots & \frac{1}{M}\mathbf{F}_1^l\mathbf{K}C_M \\ \frac{1}{M}\mathbf{F}_2^l\mathbf{K}C_1 & \Phi_2^l + \frac{1}{M}\mathbf{F}_2^l\mathbf{K}C_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{1}{M}\mathbf{F}_M^l\mathbf{K}C_1 & \dots & \dots & \Phi_M^l + \frac{1}{M}\mathbf{F}_M^l\mathbf{K}C_M \end{bmatrix}\begin{bmatrix} x_0^{(1)} \\ \vdots \\ x_0^{(M)} \end{bmatrix},$$

where $\mathbf{F}_i^l = [\Phi_i^{l-1}\Gamma_i \dots \Gamma_i]$. Assuming that the individual systems all start at the same initial state, it follows that

$$x_l^{(i)} = (\Phi_i^l + \mathbf{F}_i^l\mathbf{K}\bar{C})x_0,$$

$$\bar{C} = \frac{1}{M}\sum_{i=1}^{M} C_i.$$

This shows that summing over $\xi^T\tilde{Q}\xi$ gives the combined state energy of all systems to be stabilized, with the measurement matrices C_i replaced by their mean-value $\bar{C}$.

Now let $\tilde{\Phi}_{cl}$ be the matrix that satisfies

$$\xi_{kN+N} = \tilde{\Phi}_{cl}\xi_{kN},$$

define $\tilde{P}_0 = \text{diag}(P_0 \dots P_0)$, $P_0 = x_0x_0^T$, and let Σ be the solution of the discrete-time Lyapunov equation

$$\tilde{\Phi}_{cl}\Sigma\tilde{\Phi}_{cl}^T - \Sigma + \tilde{P}_0 = 0. \quad (11)$$

Then we have

Theorem 3.2. The periodic output feedback gain $K = \{K_0, \dots K_{N-1}\}$ that minimizes (10) is given by

$$K_l = -R^{-1}\tilde{\Gamma}^T \Lambda_l, \qquad l = 0, \dots, N-1 \tag{12}$$

where Λ_l is obtained from the solution of the two-point boundary value problem

$$\begin{bmatrix} \Omega_{l+1} \\ \Lambda_{l+1} \end{bmatrix} = H \begin{bmatrix} \Omega_l \\ \Lambda_l \end{bmatrix} + \begin{bmatrix} 0 \\ \gamma_l \end{bmatrix},$$

with

$$H = \begin{bmatrix} \tilde{\Phi} & -\tilde{\Gamma} R^{-1} \tilde{\Gamma}^T \\ -\tilde{\Phi}^{-T} \tilde{Q} \tilde{\Phi} & \tilde{\Phi}^{-T}(I + \tilde{Q}\tilde{\Gamma} R^{-1} \tilde{\Gamma}^T) \end{bmatrix},$$

$$\gamma_l = -\tilde{\Phi}^{-T} \tilde{Q} \tilde{\Phi}^{l+1} \Sigma \tilde{C}^T (\tilde{C} \Sigma \tilde{C}^T)^{-1},$$

and boundary conditions

$$\Omega_0 = 0, \qquad \Lambda_N = \tilde{\Phi}^{-1} \tilde{P} (\Omega_N - \tilde{G}).$$

Proof. See the appendix. □

Remark 1. The forcing term in the above two-point boundary value problem depends on $\tilde{\Phi}_{\text{cl}}$ (via Σ), which is itself dependent on K. In order to obtain an approximate solution, one can replace $\tilde{\Phi}_{\text{cl}}$ by $(\tilde{\Phi}^N + \tilde{G}\tilde{C})$. To justify this, consider that the rationale behind the cost function (10) is to allow the G_i's in (9) to move around in a neighborhood of G. So $\|G_i - G\|$ will be small, and for this case the above will be a close approximaton of $\tilde{\Phi}_{\text{cl}}$. Moreover, as $\tilde{P} \to \infty$, this approximation becomes exact (see Corollary 3.1).

Remark 2. The boundary condition on Λ_N is expressed in terms of Ω_N. To compute Λ_N, partition H^l as

$$H^l = \begin{bmatrix} h_{11}^l & h_{12}^l \\ h_{21}^l & h_{22}^l \end{bmatrix} \tag{13}$$

and define

$$\Pi_i^l \gamma = \sum_{j=0}^{l-1} h_{i2}^{l-j-1} \gamma_j, \qquad \tilde{h}_2 = h_{22}^N (h_{12}^N)^{-1}, \qquad \tilde{g} = \tilde{h}_2 \Pi_1^N \gamma - \Pi_2^N \gamma,$$

then

$$\Lambda_N = \tilde{h}_2 \Omega_N - \tilde{g}. \tag{14}$$

Combining this and the boundary condition yields

$$\Lambda_N = (I - \tilde{h}_2 \tilde{P}^{-1} \tilde{\Phi})^{-1} (\tilde{h}_2 \tilde{G} - \tilde{g}). \tag{15}$$

Comparing (14) and (15), immediately gives the following

Corollary 3.1. For K as in (12), the G_i on the right hand side of (9) satisfy

$$G_i \to G \quad \text{as} \quad \tilde{P} \to \infty.$$

Also, for $\tilde{P} \to \infty$, the solution of the two-point boundary value problem simplifies to

$$\Lambda_l = h_{21}^{l-N}\tilde{G} - \sum_{j=l}^{N-1} h_{22}^{l-j-1}\gamma_j. \tag{16}$$

To sum up the results of this section: We have shown that existence of a simultaneously stabilizing output injection matrix for a family of systems generically implies the existence of a simultaneously stabilizing periodic output feedback gain. The algorithm in Section 2, applied to $\{(\Phi_i^N)^T, C_i^T\}$, can be used to determine if such a matrix exists, and if so to find one. If it exists, Theorem 3.2 provides a way to compute an output feedback gain. One can start with $\tilde{P}$ large, which forces $G_i \to G$, $i = 1, \ldots, M$. To relax this condition and allow searching over a larger set of gains (and at the same time put more relative weight on performance), one can solve the two-point boundary value problem for decreasing values of $\tilde{P}$, until a satisfactory performance is achieved, or the solution ceases to be simultaneously stabilizing.

4. FAST OUTPUT SAMPLING

In the previous section, we considered the case where a simultaneously stabilizing output injection matrix exists, and a periodic output feedback law (2) is used to increase the column rank of the input matrix of the system discretized at output sampling rate. In this section, we consider the dual case, where a simultaneously stabilizing state feedback gain exists, and fast output sampling (3) is used to increase the row rank of the measurement matrix of the system discretized at input sampling rate.

Consider again a continuous-time system (A, B, C). Let (Φ, Γ, C) denote this system sampled at rate $1/\Delta$, and $(\Phi_\tau, \Gamma_\tau, C)$ the same system sampled at rate $1/\tau$. Assume F is a state feedback gain such that $(\Phi_\tau + \Gamma_\tau F)$ is stable and has no poles at the origin. Let $\mathbf{y}_{k\tau}^T = [y^T(k\tau-\tau+\Delta) \ \ldots \ y^T(k\tau)]^T$, and use $\mathbf{y}_{k\tau+\tau} = (\mathbf{C}_0 + \mathbf{D}F)x_{k\tau}$ and (3) to obtain the closed loop system

$$x_{k\tau+\tau} = (\Phi_\tau + \Gamma_\tau \mathbf{LC})\, x_{k\tau} = (\Phi_\tau + \Gamma_\tau F)\, x_{k\tau}, \tag{17}$$

where

$$\mathbf{L} = [L_0 \ldots L_{N-1}],$$
$$\mathbf{C} = (\mathbf{C}_0 + \mathbf{D}F)\,(\Phi_\tau + \Gamma_\tau F)^{-1}, \tag{18}$$

and

$$\mathbf{C}_0 = \begin{bmatrix} C\Phi \\ C\Phi^2 \\ \vdots \\ C\Phi^N \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} C\Gamma \\ C\Phi\Gamma \\ \vdots \\ C\sum_{j=0}^{N-1}\Phi^j\Gamma \end{bmatrix}.$$

A matrix $\mathbf{L}$ that satisfies $\mathbf{LC} = F$ exists if $F^T \in \mathrm{Im}\,(\mathbf{C}^T)$. Introduce the following condition:

Definition 4.1. For an observable system (Φ, Γ, C), let

$$T_l = [(\Phi^{l-1})^T C^T \quad \ldots C^T],$$

then a gain matrix F is said to satisfy condition $(*)$ for this system if

$$(\Phi^l)^T C^T + F^T \Gamma^T T_l \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} \notin \ \mathrm{Im}\, T_l, \qquad l = 1, \ldots, \nu_o,$$

where ν_o is the systems observability index.

The following is straightforward to verify.

Lemma 4.1. An output feedback gain matrix $\mathbf{L}$ that satisfies (17) exists if (Φ, C) is observable and F satisfies condition $(*)$.

Now we turn to the problem of simultaneously stabilizing a family of systems $\{A_i, B_i, C_i\}_{i=1}^M$. Assume there exists F such that $(\Phi_{\tau.i} + \Gamma_{\tau.i}F)$ is stable and has no poles at the origin for $i = 1, \ldots, M$. Define

$$\tilde{\Phi} = \mathrm{diag}(\Phi_1 \ldots \Phi_M), \qquad \tilde{\Phi}_\tau = \mathrm{diag}(\Phi_{1.\tau} \ldots \Phi_{M.\tau}),$$
$$\tilde{\Gamma} = \mathrm{diag}(\Gamma_1 \ldots \Gamma_M), \qquad \tilde{\Gamma}_\tau = \mathrm{diag}(\Gamma_{1.\tau} \ldots \Gamma_{M.\tau}),$$
$$\tilde{C} = [C_1 \ldots C_M],$$
$$\tilde{F} = \mathrm{diag}(F \ldots F).$$

It follows from Lemma 3.1 that observability of (Φ, C) generically implies observability of $(\tilde{\Phi}, \tilde{C})$. Thus, assume that $(\tilde{\Phi}, \tilde{C})$ is observable with observability index $\tilde{\nu}_o$, fix $N \geq \tilde{\nu}_o$, and let $\tilde{\mathbf{C}}$ be the matrix obtained when substituting the above into (18). Then we have

Theorem 4.1. If $\tilde{F}$ satisfies condition $(*)$ for $(\tilde{\Phi}, \tilde{\Gamma}, \tilde{C})$, then

$$\mathbf{L}\tilde{\mathbf{C}} = [F \ \ldots \ F]$$

has a solution $\mathbf{L}$, and any such solution simultaneously stabilizes $\{A_i, B_i, C_i\}$ when substituted into (3).

Remark. Note that the state feedback gain F is assumed to be such that there is no closed loop pole at the origin for the whole family of systems to be stabilized. When using the convex programming algorithm introduced in Section 2 to search for a simultaneously stabilizing F, it is straightforward to change the definition of the convex cone $\mathcal{C}(S)$ such that $(\Phi_{\tau.i} + \Gamma_{\tau.i}F)$ has all roots *inside* the unit disc and *outside* a disc with radius ϵ around the origin for $i = 1, \ldots, M$.

5. NUMERICAL EXAMPLE

In this section we illustrate the two proposed types of control law with a simple numerical example.

We consider the problem of stabilizing the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & \alpha_i \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

$$y = [\zeta_i \;\; 0]x, \qquad x_0^T = [1.0 \;\; 1.0],$$

under two operating points

$$\alpha_1 = 0.5, \qquad \zeta_1 = 3.0,$$

and

$$\alpha_2 = 1.5, \qquad \zeta_2 = 1.0.$$

Discretized at a sampling interval $\tau = 1.0$, state and input matrices for these operating points are

$$\Phi_{\tau.1} = \begin{bmatrix} 0.760 & 0.459 \\ -0.919 & 0.760 \end{bmatrix}, \qquad \Gamma_{\tau.1} = \begin{bmatrix} 0.240 \\ 0.919 \end{bmatrix},$$

$$\Phi_{\tau.2} = \begin{bmatrix} 0.339 & 1.152 \\ -0.768 & 0.339 \end{bmatrix}, \qquad \Gamma_{\tau.2} = \begin{bmatrix} 0.661 \\ 0.768 \end{bmatrix}.$$

Periodic Output Feedback

Using the convex programming algorithm presented in Section 2, a simultaneously stabilizing output injection matrix obtained after 15 iterations is

$$G^T = [0.1061 \;\; -0.1817].$$

For $N = 4$, $(\tilde{\Phi}, \tilde{\Gamma})$ is controllable. We choose

$$R = 1, \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad P = p\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\tilde{Q} = \mathrm{diag}(Q, Q), \qquad \tilde{P} = \mathrm{diag}(P, P).$$

For $p = 10^6$, application of Theorem 3.2 yields a gain sequence

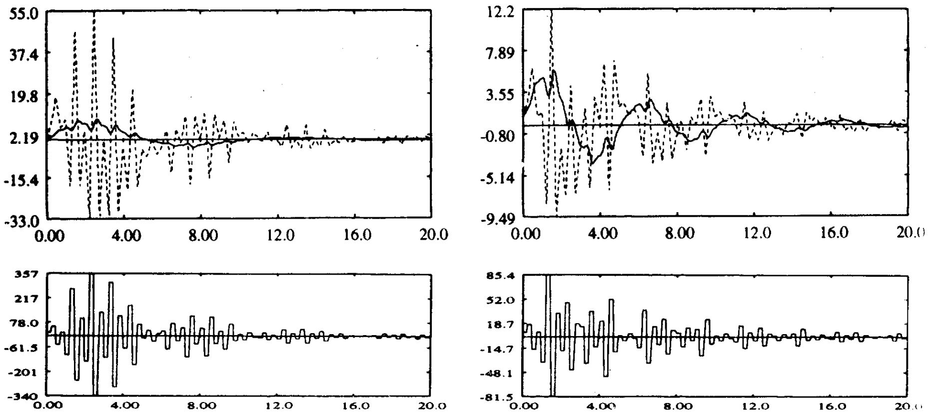
$$\{K_l\} = \{6.77, -17.06, 16.27, -6.42\}.$$

The closed loop response to x_0 under the control law (2) with this gain is shown in Fig. 1.

Reducing the terminal cost to $p = 10^4$, gives a gain sequence

$$\{K_l\} = \{0.95, -0.25, -0.39, -0.76\}.$$

The corresponding closed loop response is shown in Fig. 2.



a) System 1 b) System 2

Fig. 1. Closed-loop response with periodic output feedback, $p = 10^6$. Above: states, below: control signal.

Fast Output Sampling

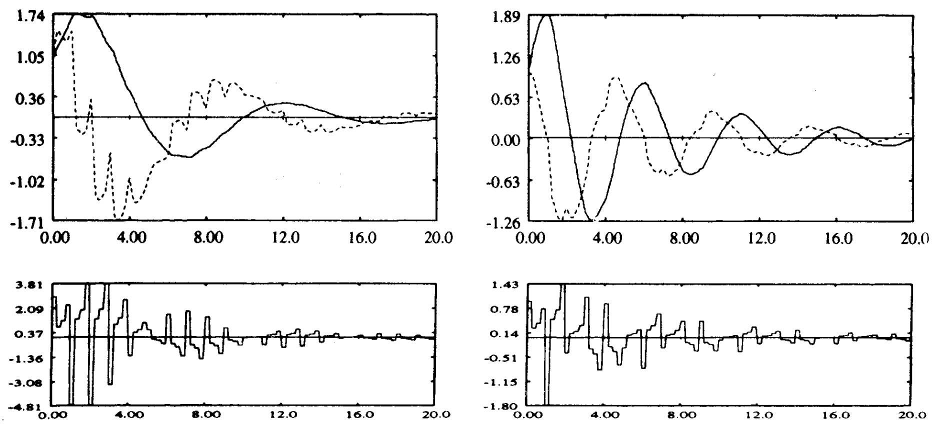
By convex programming, a simultaneously stabilizing state feedback gain obtained after 14 iterations is

$$F = [-0.2467 \;\; 0.5100].$$

For $N = 4$, $(\tilde{\Phi}, \tilde{C})$ is observable. Using Theorem 4.1, an output gain is computed as

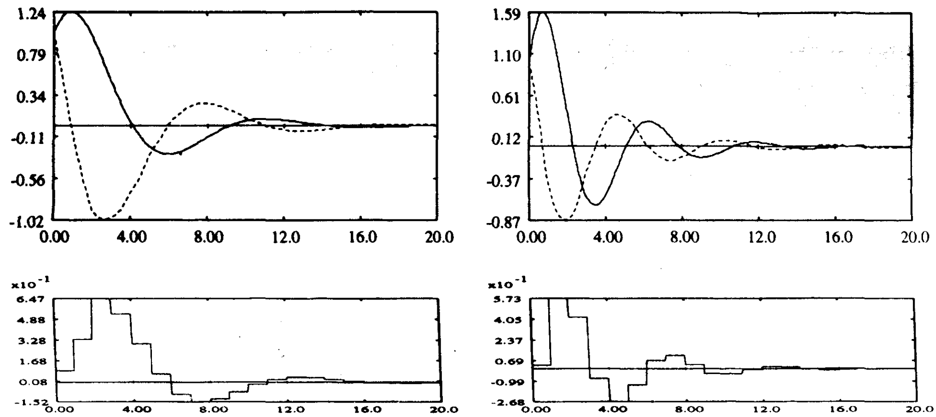
$$L = [7.45 \;\; -25.75 \;\; 30.53 \;\; -12.20].$$

The closed loop response to x_0 under the control law (3) with this gain is shown in Fig. 3.



a) System 1 b) System 2

Fig. 2. Closed-loop response with periodic output feedback, $p = 10^4$. Above: states, below: control signal.



a) System 1 b) System 2

Fig. 3. Closed-loop response with fast output sampling. Above: states, below: control signal.

APPENDIX

Proof of Lemma 3.1.

Let $\mathcal{G}$ be the set of families $S = \{\Phi_i, \Gamma_i\}$ of controllable systems for which a simultaneously stabilizing output injection matrix exists, and $\tilde{\mathcal{G}}$ the subset of $\mathcal{G}$ for which $(\tilde{\Phi}, \tilde{\Gamma})$ is controllable. We have to show that $\tilde{\mathcal{G}}$ i open and dense in $\mathcal{G}$, or equivalently, that the following holds

$$\text{(i)}\ \forall S \in \tilde{\mathcal{G}},\ \exists \epsilon > 0 : \mathcal{B}_\epsilon(S) \subset \tilde{\mathcal{G}}$$

$$\text{(ii)}\ \forall S \notin \tilde{\mathcal{G}},\ \forall \epsilon > 0 : \mathcal{B}_\epsilon(S) \cap \tilde{\mathcal{G}} \neq \emptyset,$$

where $\mathcal{B}_\epsilon(S)$ denotes a neighbourhood of S with radius ϵ.

We show that (i) holds. Consider any $S \in \tilde{\mathcal{G}}$. Controllability of $(\tilde{\Phi}, \tilde{\Gamma})$ implies that no left eigenvector q of $\tilde{\Phi}$ is orthogonal to $\tilde{\Gamma}$

$$q(\lambda)\,\tilde{\Phi} = \lambda q(\lambda) \quad \Longrightarrow \quad q(\lambda)\,\tilde{\Gamma} \neq [0\dots0]. \tag{19}$$

Let $\nu_j(\lambda)$, $j = 1, \dots, \mu(\lambda)$ be the indices of those matrices Φ_{ν_j} of which λ is an eigenvalue. Partition $q = [q_1 \dots q_M]$, then (19) is equivalent to

$$q_{\nu_j}(\lambda)\,\Phi_{\nu_j} = \lambda q_{\nu_j}(\lambda) \quad \Longrightarrow$$

$$\sum_{j=1}^{\mu(\lambda)} q_{\nu_j}(\lambda)\,\Gamma_{\nu_j} \neq [0\dots0], \qquad \forall\,\lambda \in \sigma(\tilde{\Phi}). \tag{20}$$

Define

$$c(\lambda) = \left\| \sum_{j=1}^{\mu(\lambda)} q_{\nu_j}(\lambda)\,\Gamma_{\nu_j} \right\| > 0.$$

Now let S be perturbed to $S + \Delta S$; assume first that the perturbation affects only the input matrix $\tilde{\Gamma}$. $(\tilde{\Phi}, \tilde{\Gamma} + \Delta\tilde{\Gamma})$ is controllable if every left eigenvector q of $\tilde{\Phi}$ satisfies

$$\sum_{j=1}^{\mu(\lambda)} q_{\nu_j}(\lambda)\,(\Gamma_{\nu_j} + \Delta\Gamma_{\nu_j}) \neq [0 \ldots 0], \qquad \forall\,\lambda \in \sigma(\tilde{\Phi})$$

This clearly holds if

$$\left\| \sum_{j=1}^{\mu(\lambda)} q_{\nu_j}(\lambda)\,\Delta\Gamma_{\nu_j} \right\| < c(\lambda), \qquad \forall\,\lambda \in \sigma(\tilde{\Phi}),$$

which shows that there exists a neighborhood of $\tilde{\Gamma}$ for which $(\tilde{\Phi}, \tilde{\Gamma} + \Delta\tilde{\Gamma})$ is controllable. (For multiple eigenvalues, the argument can be adjusted by considering invariant subspaces.)

Next, consider a perturbation of the state matrix $\tilde{\Phi}$ to $\tilde{\Phi} + \Delta\tilde{\Phi}$. Let Δq_{ν_j} denote the change in the eigenvector partition with index ν_j as defined above, caused by this perturbation. Then $(\tilde{\Phi} + \Delta\tilde{\Phi}, \tilde{\Gamma})$ is controllable if every left eigenvector of $\tilde{\Phi} + \Delta\tilde{\Phi}$ satisfies

$$\sum_{j=1}^{\mu(\lambda)} (q_{\nu_j}(\lambda) + \Delta q_{\nu_j}(\lambda))\,\Gamma_{\nu_j} \neq [0 \ldots 0], \qquad \forall\,\lambda \in \sigma(\tilde{\Phi})$$

This holds if

$$\left\| \sum_{j=1}^{\mu(\lambda)} \Delta q_{\nu_j}(\lambda)\,\Gamma_{\nu_j} \right\| < c(\lambda), \qquad \forall\,\lambda \in \sigma(\tilde{\Phi}),$$

which shows that there exists a neighborhood of $\tilde{\Phi}$ for which $(\tilde{\Phi} + \Delta\tilde{\Phi}, \tilde{\Gamma})$ is controllable.

Combining these two results shows that there exists a neighborhood of S which is contained in $\tilde{\mathcal{G}}$; since S was arbitrary, this proves (i).

To show that (ii) holds, consider any family S for which $(\tilde{\Phi}, \tilde{\Gamma})$ is not controllable. Then there exist some eigenvectors q of $\tilde{\Phi}$ such that $q\tilde{\Gamma} = [0 \ldots 0]$. Density of $\tilde{\mathcal{G}}$ follows from the fact that it is possible to choose an arbitrary small perturbation that makes $q\Delta\tilde{\Gamma} \neq [0 \ldots 0]$. This completes the proof. □

Remark. The fact that controllability is a generic property, is well known; see [9], where a somewhat different definition of genericity is used. What has been shown here is that generically, the parallel connection of controllable systems as in (7) is also controllable. It follows from the proof that the parallel connection is always controllable if the systems share no common eigenvalues. Moreover, if they do share eigenvalues, the families of systems whose parallel connection is not controllable, can be identified as belonging to a union of μ_s linear varieties determined by (20), where μ_s is the number of shared eigenvalues.

Proof of Theorem 3.2.

It must be shown that K given by (12) minimizes J. We drop the tilde on system and cost matrices.

The cost function (10) can be rewritten as

$$J = \operatorname{tr} \sum_{l=0}^{N-1} [(\Phi^l + \Omega_l C)^T \quad (K_l C)^T] \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} [\ \Phi^l + \Omega_l C K_l C\ ]\,\Sigma + \operatorname{tr} C^T(\Omega_N - G)^T P(\Omega_N - G)\,C\Sigma, \tag{21}$$

where

$$\Omega_l = \sum_{j=0}^{l-1} \Phi^{l-j-1}\Gamma K_j, \tag{22}$$

and Σ is the solution of the discrete-time Lyapunov equation (11).

Denote by $\mathcal{S}_N$ the space of matrix sequences $\{M_0, M_1 \dots M_{N-1}\}$. Define an inner product on S_N

$$\langle M, N\rangle = \sum_{l=0}^{N-1} \operatorname{tr} M_l^T N_l$$

and the operator $L : \mathcal{S}_N(\Re^{m\times p}) \to \mathcal{S}_N(\Re^{n\times p})$ by

$$LM = \{L_0 M,\ L_1 M, \dots,\ L_{N-1}M\},$$

$$L_l M = \sum_{j=0}^{l-1} \Phi^{l-j-1}\Gamma M_j, \qquad L_0 = 0.$$

With these definitions, (21) can be written as

$$J = \langle\{\Phi^l\} + LKC, Q(\{\Phi^l\} + LKC)\,\Sigma\rangle + \langle KC, RKC\Sigma\rangle + \operatorname{tr} C^T(\Omega_N - G)^T P(\Omega_N - G)\,C\Sigma, \tag{23}$$

where K stands for the sequence $\{K_l\}$, and $\{\Phi^l\}$ for $\{I,\ \Phi, \dots,\ \Phi^{N-1}\}$.

Employing the adjoint operator L^*, and introducing the positive definite operator $Y : \mathcal{S}_N(\Re^{m\times p}) \to \mathcal{S}_N(\Re^{m\times p})$ and the sequence $X \in \mathcal{S}_N(\Re^{m\times n})$ defined by

$$Y_l = R + L_l^* Q L,$$

$$X_l = L_l^* Q\{\Phi^l\},$$

obtain

$$J = \langle KC, (YKC + 2X)\,\Sigma\rangle + \operatorname{tr} C^T(\Omega_N - G)^T P(\Omega_N - G)\,C\Sigma + (\text{terms independent of } K). \tag{24}$$

For K to minimize (24), the first variation δJ must be zero for all possible perturbations $K + \epsilon\tilde{K}$

$$\delta J = \left\langle \tilde{K}C, 2(YKC + X + \Gamma^T\{\Phi^{N-l-1}\}^T P(L_N K - G)C)\Sigma \right\rangle = 0,$$

which holds if

$$YK = -X\Sigma\Psi - \Gamma^T\{\Phi^{N-l-1}\}^T P(L_N K - G)$$

where

$$\Psi = C^T(C\Sigma C^T)^{-1}.$$

Substituting for X and Y gives

$$K_l = -R^{-1}\left(L_l^* Q\{L_j K + \Phi^j\Sigma\Psi\} + \Gamma^T(\Phi^{N-l-1})^T P(L_N K - G)\right).$$

Using

$$L_l^* M = \Gamma^T \sum_{j=l+1}^{N-1} (\Phi^T)^{j-l-1} M_j$$

and

$$\Omega_j = L_j K,$$

we get

$$K_l = -R^{-1}\Gamma^T\Lambda_l,$$

where

$$\Lambda_l = \sum_{j=l+1}^{N-1} (\Phi^T)^{j-l-1} Q(\Omega_j + \Phi^j\Sigma\Psi) + (\Phi^{N-l-1})^T P(\Omega_N - G). \qquad (25)$$

Λ_l and Ω_l as given by (25) and (22), are solutions of the difference equations

$$\Lambda_l = \Phi^T\Lambda_{l+1} + Q(\Omega_{l+1} + \Phi^{l+1}\Sigma\Psi),$$

$$\Omega_{l+1} = \Phi\Omega_l + \Gamma K_l,$$

with boundary conditions

$$\Omega_0 = 0, \quad \Lambda_N = \Phi^{-T} P(\Omega_N - G).$$

Rearrangement yields the statement of the theorem.

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