

CRITICAL VALUES OF FISHER'S AND SIEGEL'S TEST<sup>1</sup>

JAROMÍR ANTOCH

Fisher's and Siegel's tests of periodicity belong among classical tools in the time series analysis. Unfortunately, since now the corresponding critical values and their approximations were published in rather limited scope. Main aim of present paper is to give extended tables of critical values for them based both on exact distribution of the test statistic and several related approximations.

## 1. FISHER'S TEST OF PERIODICITY

Testing periodicity in time series belongs among classical statistical problems. The solution based on periodogram has been proposed and thoroughly studied by Fisher [3, 4, 5]. Since that time this test can be found in (almost) all books devoted to the time series analysis. Nevertheless, it is worth of noticing that critical values and their approximations available in the literature were calculated only for rather limited scope of sample sizes and levels (which almost did not change since the publication of Fisher's papers).

Let  $X_1, \dots, X_n$ ,  $n = 2m+1$ , be a series of random variables. Define periodogram by

$$I(\lambda) = \frac{1}{2\pi n} \left| \sum_{t=1}^n X_t e^{-it\lambda} \right|^2 \quad (1.1)$$

and normed periodogram by

$$Y_r = \frac{I(\lambda_r)}{\sum_{j=1}^m I(\lambda_j)}, \quad r = 1, \dots, m, \quad (1.2)$$

where  $\lambda_r = 2\pi r/n$ .

Assume that  $X_1, \dots, X_n$  are i.i.d.  $N(0, \sigma^2)$  variables. For  $Y = \max_{1 \leq i \leq m} Y_i$  Fisher proved that

$$P(Y > y) = 1 - \sum_{j=0}^m (-1)^j \binom{m}{j} (1 - jy)_+^{m-1}, \quad 0 < y < 1, \quad (1.3)$$

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where  $x_+ = x I[x > 0]$ . Denote the exact critical value on the level  $\alpha$  based on (1.3) by  $a_m^*(\alpha)$ .

Following two approximations of (1.3) are often used in practice, i.e.

$$P(Y > y) \cong m(1 - y)^{m-1}, \quad 0 < y < 1, \quad (1.4)$$

and

$$\lim_{m \rightarrow \infty} P\left(Y > \frac{z + \ln m}{m}\right) = 1 - \exp\{-e^{-z}\}, \quad 0 < z < \infty. \quad (1.5)$$

While (1.4), suggested already by Fisher [3], is based on the use of first two terms of the sum in (1.3), the proof of (1.5) can be found in Anděl [1], e.g.

It is evident that the critical values  $a_{m,1}^*(\alpha)$  and  $a_{m,2}^*(\alpha)$  on the level  $\alpha$  based on (1.4) and (1.5) are given by

$$a_{m,1}^*(\alpha) = 1 - \left(\frac{\alpha}{m}\right)^{1/(m-1)} \quad \text{and} \quad a_{m,2}^*(\alpha) = \frac{1}{m} \left\{ -\ln(-\ln(1-\alpha)) + \ln m \right\}. \quad (1.6)$$

Both exact critical values  $a_m^*(\alpha)$  and corresponding approximations  $a_{m,1}^*(\alpha)$  and  $a_{m,2}^*(\alpha)$  on the levels  $\alpha = 0.05$ ,  $\alpha = 0.025$  and  $\alpha = 0.01$  can be found in Tables 1a-1c.

**Table 1a.** Critical values  $a_m^*(\alpha)$  and their approximations  $a_{m,1}^*(\alpha)$  and  $a_{m,2}^*(\alpha)$  of Fisher's test on the level  $\alpha = 0.05$ .

| $m$  | $a_m^*(0.05)$ | $a_{m,1}^*(0.05)$ | $a_{m,2}^*(0.05)$ |
|------|---------------|-------------------|-------------------|
| 10   | 0.4449525606  | 0.4449526922      | 0.5272780342      |
| 15   | 0.3346119552  | 0.3346306865      | 0.3785496966      |
| 20   | 0.2704042202  | 0.2704593863      | 0.2982963761      |
| 25   | 0.2280456085  | 0.2281322130      | 0.2475628429      |
| 30   | 0.1978406246  | 0.1979495607      | 0.2123797543      |
| 35   | 0.1751294860  | 0.1752531872      | 0.1864440945      |
| 40   | 0.1573827449  | 0.1575157410      | 0.1664768675      |
| 45   | 0.1431035999  | 0.1432420859      | 0.1505968386      |
| 50   | 0.1313472721  | 0.1314886262      | 0.1376443650      |
| 60   | 0.1130887896  | 0.1132310308      | 0.1177423301      |
| 70   | 0.0995257984  | 0.0996653978      | 0.1031241498      |
| 80   | 0.0890258722  | 0.0891612068      | 0.0919027735      |
| 90   | 0.0806402063  | 0.0807706034      | 0.0830000546      |
| 100  | 0.0737781851  | 0.0739034577      | 0.0757536543      |
| 150  | 0.0522136336  | 0.0523158530      | 0.0532055369      |
| 200  | 0.0407367370  | 0.0408220288      | 0.0413425630      |
| 250  | 0.0335542444  | 0.0336271973      | 0.0339666246      |
| 300  | 0.0286124993  | 0.0286761722      | 0.0289132590      |
| 350  | 0.0249931178  | 0.0250495880      | 0.0252232240      |
| 400  | 0.0222217969  | 0.0222725246      | 0.0224041494      |
| 450  | 0.0200280827  | 0.0200741285      | 0.0201765396      |
| 500  | 0.0182461407  | 0.0182882977      | 0.0183696066      |
| 600  | 0.0155221997  | 0.0155582707      | 0.0156118748      |
| 700  | 0.0135334433  | 0.0135649701      | 0.0136018222      |
| 800  | 0.0120144724  | 0.0120424764      | 0.0120685087      |
| 900  | 0.0108145470  | 0.0108397397      | 0.0108584333      |
| 1000 | 0.0098415283  | 0.0098644251      | 0.0098779505      |

**Table 1b.** Critical values  $a_m^*(\alpha)$   
and their approximations  $a_{m,1}^*(\alpha)$  and  $a_{m,2}^*(\alpha)$   
of Fisher's test on the level  $\alpha = 0.025$ .

| $m$  | $a_m^*(0.025)$ | $a_{m,1}^*(0.025)$ | $a_{m,2}^*(0.025)$ |
|------|----------------|--------------------|--------------------|
| 10   | 0.4860957335   | 0.4860957335       | 0.597883350        |
| 15   | 0.3667695215   | 0.3667712494       | 0.425619306        |
| 20   | 0.2965837815   | 0.2965944508       | 0.333598765        |
| 25   | 0.2500836274   | 0.2501057906       | 0.275804233        |
| 30   | 0.2168602548   | 0.2168925847       | 0.235914213        |
| 35   | 0.1918565532   | 0.1918968075       | 0.206617091        |
| 40   | 0.1723108490   | 0.1723569406       | 0.184128678        |
| 45   | 0.1565828876   | 0.1566331351       | 0.166286832        |
| 50   | 0.1436348652   | 0.1436879786       | 0.151765052        |
| 60   | 0.1235318804   | 0.1235880632       | 0.129509636        |
| 70   | 0.1086074861   | 0.1086645327       | 0.113210071        |
| 80   | 0.0970612742   | 0.0971179626       | 0.100728236        |
| 90   | 0.0878462548   | 0.0879019128       | 0.090845769        |
| 100  | 0.0803105941   | 0.0803648642       | 0.082814744        |
| 150  | 0.0566679818   | 0.0567142358       | 0.057912503        |
| 200  | 0.0441177478   | 0.0441571794       | 0.044872231        |
| 250  | 0.0362794075   | 0.0363135713       | 0.036790327        |
| 300  | 0.0308952288   | 0.0309253046       | 0.031266657        |
| 350  | 0.0269571714   | 0.0269840106       | 0.027240154        |
| 400  | 0.0239453463   | 0.0239695690       | 0.024169795        |
| 450  | 0.0215636619   | 0.0215857295       | 0.021745440        |
| 500  | 0.0196307562   | 0.0196510197       | 0.019781107        |
| 600  | 0.0166793667   | 0.0166967822       | 0.016788281        |
| 700  | 0.0145273903   | 0.0145426607       | 0.014610679        |
| 800  | 0.0128855788   | 0.0128991760       | 0.012951737        |
| 900  | 0.0115898527   | 0.0116021083       | 0.011642355        |
| 1000 | 0.0105400275   | 0.0105511835       | 0.010584025        |

## 2. SIEGEL'S TEST OF PERIODICITY

Siegel [11] suggested a new test statistic

$$T = \sum_{j=1}^m (Y_j - \zeta a_m^*)_+ \quad (2.1)$$

where  $\zeta \in (0, 1)$  is a constant. In our calculations we used  $\zeta = 0.6$  following the suggestions from the literature. In the same paper Siegel derived exact distribution of  $T$  under  $H_0$  that no periodic components are presented, i.e.

$$P(T > t) = \sum_{l=1}^m \binom{m}{l} \sum_{j=0}^{l-1} \binom{l-1}{j} \binom{m-1}{j} (-1)^{j+l+1} t^j (1 - \zeta l a_m^* - t)_+^{m-j-1} \quad (2.2)$$

and its first two moments

$$E T = (1 - \zeta a_m^*)^m \quad (2.3)$$

and

$$E T^2 = \frac{2(1 - \zeta a_m^*)^{m+1} + (m-1)(1 - 2\zeta a_m^*)^{m+1}}{m+1}. \quad (2.4)$$

Denote the exact critical value on the level  $\alpha$  based on (2.2) by  $s_m^*(\alpha)$ .

**Table 1c.** Critical values  $a_m^*(\alpha)$   
and their approximations  $a_{m,1}^*(\alpha)$  and  $a_{m,2}^*(\alpha)$   
of Fisher's test on the level  $\alpha = 0.01$ .

| $m$  | $a_m^*(0.01)$ | $a_{m,1}^*(0.01)$ | $a_{m,2}^*(0.01)$ |
|------|---------------|-------------------|-------------------|
| 10   | 0.5358411166  | 0.5358411166      | 0.6902734319      |
| 15   | 0.4068884773  | 0.4068885083      | 0.4872132951      |
| 20   | 0.3297109241  | 0.3297117919      | 0.3797940750      |
| 25   | 0.2781931506  | 0.2781961963      | 0.3127610020      |
| 30   | 0.2412432645  | 0.2412490163      | 0.2667115536      |
| 35   | 0.2133757652  | 0.2133841290      | 0.2330142082      |
| 40   | 0.1915648429  | 0.1915754614      | 0.2072257170      |
| 45   | 0.1740019966  | 0.1740144630      | 0.1868180381      |
| 50   | 0.1595381266  | 0.1595520615      | 0.1702434446      |
| 60   | 0.1370779591  | 0.1370939019      | 0.1449082298      |
| 70   | 0.1204057783  | 0.1204228441      | 0.1264092066      |
| 80   | 0.1075120344  | 0.1075296481      | 0.1122771982      |
| 90   | 0.0972263567  | 0.0972441552      | 0.1011106544      |
| 100  | 0.0888194946  | 0.0888372438      | 0.0920531941      |
| 150  | 0.0624811196  | 0.0624972682      | 0.0640718968      |
| 200  | 0.0485340067  | 0.0485482178      | 0.0494923329      |
| 250  | 0.0398407595  | 0.0398533072      | 0.0404864405      |
| 300  | 0.0338793190  | 0.0338905052      | 0.0343464390      |
| 350  | 0.0295252165  | 0.0295352890      | 0.0298802353      |
| 400  | 0.0261992670  | 0.0262084199      | 0.0264790344      |
| 450  | 0.0235719967  | 0.0235803793      | 0.0237986595      |
| 500  | 0.0214418074  | 0.0214495377      | 0.0216295146      |
| 600  | 0.0181931054  | 0.0181997919      | 0.0183284648      |
| 700  | 0.0158277202  | 0.0158336104      | 0.0159303279      |
| 800  | 0.0140252680  | 0.0140305313      | 0.0141059511      |
| 900  | 0.0126042461  | 0.0126090031      | 0.0126694933      |
| 1000 | 0.0114539580  | 0.0114582978      | 0.0115079045      |

### $\chi_0^2$ -approximation

Best up to now proposed approximation to the exact distribution is that based on the non-central chi-square distribution with zero degrees of freedom  $\chi_0^2(\lambda)$  suggested by Siegel [9, 10]. More precisely, he suggested to approximate the exact distribution of  $T$  by that of  $c\chi_0^2(\lambda)$  where

$$P(\chi_0^2(\lambda) \leq t) = 1 - e^{-(\lambda+t)/2} \sum_{l=1}^{\infty} \frac{(\lambda/2)^l}{l!} \sum_{j=0}^{l-1} \frac{(t/2)^j}{j!}, \quad t \geq 0, \quad (2.5)$$

$$c = \frac{2(1 - \zeta a_m^*)^{m+1} + (m-1)(1 - 2\zeta a_m^*)^{m+1} - (m+1)(1 - \zeta a_m^*)^{2m}}{4(m+1)(1 - \zeta a_m^*)^m} \quad (2.6)$$

and

$$\lambda = \frac{(1 - \zeta a_m^*)^m}{c}. \quad (2.7)$$

The choice of constants  $c$  and  $\lambda$  ensure that both  $T$  and  $c\chi_0^2(\lambda)$  have the same first two moments given by (2.3) and (2.4). Denote the critical value on the level  $\alpha$  based on (2.5)–(2.7) by  $s_{m,1}^*(\alpha)$ . We can see from the results presented in Section 3

that this approximation gives excellent fit to the exact distribution even for small values of  $m$ . Its basic disadvantage is that it is computationally rather complex and "costly".

Numerical problems connected with the calculation of the cumulative distribution function and quantiles of  $\chi_0^2(\lambda)$ , which usually can be neither found in general statistical packages nor calculated with desired precision lead Anděl [2] to a proposal of several simpler approximations based only on the quantiles of standard normal and central chi-square distributions which are easily available. We shall present here only resulting formulas while for their derivation we refer to Anděl [2].

### $\chi^2$ -approximation

The distribution function  $F$  of the statistic  $c\chi_0^2$  is composed from two parts. More precisely,  $F = p_1 F_1 + p_2 F_2$ , where  $F_1$  is the distribution function of the random variable degenerated in zero and  $F_2$  is a distribution function of a continuous distribution. It was shown by Anděl [2] that  $F_2$  can be reasonably approximated by  $h\chi^2$ , where  $\chi^2$  has classical  $\chi^2$ -distribution with  $q$  degrees of freedom. Therefore, approximate critical value  $s_{m,2}^*(\alpha)$  for  $s_m^*(\alpha)$  on the level  $\alpha$  based on this, so called  $\chi^2$ -approximation, is given by

$$s_{m,2}^*(\alpha) = h\chi_q^2 \left( 1 - \frac{\alpha}{p_2} \right), \quad (2.8)$$

where  $\chi_q^2(\beta)$  is the  $\beta$ -quantile of the chi-square distribution with  $q$  degrees of freedom,

$$p_2 = 1 - e^{-\lambda/2}, \quad \nu = \frac{c\lambda}{p_2}, \quad \sigma^2 = \frac{c^2\lambda(4+\lambda)}{p_2} - \nu^2, \quad h = \frac{\sigma^2}{2\nu}, \quad q = \frac{2\nu^2}{\sigma^2}, \quad (2.9)$$

$c$  and  $\lambda$  are given by (2.6) and (2.7).

### Wilson – Hilmerty approximation

Applying well known Wilson – Hilmerty transformation between the quantiles of normal and  $\chi^2$ -distributions in (2.8) (see Johnson and Kotz [6] for details), we obtain immediately another simple approximation for  $s_m^*(\alpha)$  of the form

$$s_{m,3}^*(\alpha) = hq \left( \frac{u_{1-\alpha/p_2}}{3} \sqrt{\frac{2}{q}} + 1 - \frac{2}{9q} \right)^3, \quad (2.10)$$

where  $u_\beta$  is the  $\beta$ -quantile of the standard normal distribution and  $h$ ,  $p_2$  and  $q$  are given by (2.9).

### $N$ -approximation

If we approximate  $F_2$  by the normal distribution  $N(\nu, \sigma^2)$  instead by  $h\chi^2$ , we can arrive to the following approximation of  $s_m^*(\alpha)$  on the level  $\alpha$ , so called  $N$ -approximation  $s_{m,4}^*(\alpha)$ , which is given by

$$s_{m,4}^*(\alpha) = \nu + \sigma\Phi^{-1} \left( 1 - \frac{\alpha}{p_2} \right), \quad (2.11)$$

where  $\nu$ ,  $p_2$  and  $\sigma^2$  are given by (2.9).

### Normal approximation

It can be shown that statistic  $T$  is asymptotically normal, see Siegel [9] for details. Unfortunately, this approximation is rather bad. Nevertheless, for completeness of our results we present in Tables 2a–2c also approximate critical values  $s_{m,5}^*(\alpha)$  on the level  $\alpha$  based on it.

**Table 2a.** Critical values  $s_m^*(\alpha)$  and their approximations  $s_{m,1}^*(\alpha) - s_{m,5}^*(\alpha)$  of Siegel's test on the level  $\alpha = 0.05$ .

| $m$  | $s_m^*(0.05)$  | $s_{m,1}^*(0.05)$ | $s_{m,2}^*(0.05)$ | $s_{m,3}^*(0.05)$ | $s_{m,4}^*(0.05)$ | $s_{m,5}^*(0.05)$ |
|------|----------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| 10   | 0.181 291 6218 | 0.177 537 9601    | 0.176 636 0292    | 0.175 077 2516    | 0.168 347 5348    | 0.150 727 7932    |
| 15   | 0.140 175 1436 | 0.137 746 0573    | 0.137 045 7805    | 0.135 834 7966    | 0.130 677 1967    | 0.116 907 2311    |
| 20   | 0.115 943 4232 | 0.114 206 1432    | 0.113 627 4325    | 0.112 628 8196    | 0.108 138 4704    | 0.097 056 2457    |
| 25   | 0.099 732 5432 | 0.098 411 3018    | 0.097 915 4549    | 0.097 061 2739    | 0.092 958 6304    | 0.083 776 4515    |
| 30   | 0.088 019 8043 | 0.086 971 9250    | 0.086 537 0686    | 0.085 788 2471    | 0.081 954 7503    | 0.074 169 9790    |
| 35   | 0.079 106 0851 | 0.078 249 1355    | 0.077 861 5229    | 0.077 193 3724    | 0.073 567 1110    | 0.066 847 2738    |
| 40   | 0.072 063 4801 | 0.071 346 0562    | 0.070 996 3860    | 0.070 392 2268    | 0.066 935 7694    | 0.061 051 5241    |
| 45   | 0.066 339 1976 | 0.065 727 3658    | 0.065 408 9978    | 0.064 856 9851    | 0.061 545 1937    | 0.056 332 3676    |
| 50   | 0.061 581 9252 | 0.061 052 2712    | 0.060 760 2684    | 0.060 251 6611    | 0.057 066 3270    | 0.052 403 6678    |
| 60   | 0.054 101 0364 | 0.053 689 9335    | 0.053 440 0989    | 0.052 999 8122    | 0.050 028 4358    | 0.046 210 9737    |
| 70   | 0.048 458 1309 | 0.048 127 4977    | 0.047 910 0866    | 0.047 521 3199    | 0.044 726 9675    | 0.041 525 6146    |
| 80   | 0.044 029 7152 | 0.043 756 6785    | 0.043 565 0964    | 0.043 216 6970    | 0.040 573 0418    | 0.037 838 5710    |
| 90   | 0.040 449 3552 | 0.040 219 2530    | 0.040 048 7954    | 0.039 732 9531    | 0.037 220 0896    | 0.034 850 1791    |
| 100  | 0.037 486 6267 | 0.037 289 5754    | 0.037 136 7506    | 0.036 847 7675    | 0.034 450 0507    | 0.032 371 6728    |
| 150  | 0.027 934 6469 | 0.027 828 8584    | 0.027 733 5263    | 0.027 530 2669    | 0.025 555 9642    | 0.024 339 1141    |
| 200  | 0.022 651 7317 | 0.022 586 5412    | 0.022 522 9130    | 0.022 366 0731    | 0.020 668 8386    | 0.019 862 2645    |
| 250  | 0.019 246 0210 | 0.019 203 3929    | 0.019 159 7306    | 0.019 032 1755    | 0.017 534 0659    | 0.016 959 4325    |
| 300  | 0.016 845 0268 | 0.016 816 6401    | 0.016 786 5700    | 0.016 679 2307    | 0.015 332 7663    | 0.014 903 4318    |
| 350  | 0.015 049 6264 | 0.015 030 9901    | 0.015 010 6620    | 0.014 918 1374    | 0.013 691 8600    | 0.013 360 0344    |
| 400  | 0.013 649 7353 | 0.013 638 1563    | 0.013 625 0706    | 0.013 543 8763    | 0.012 415 6859    | 0.012 152 6231    |
| 450  | 0.012 523 5715 | 0.012 517 3090    | 0.012 509 7559    | 0.012 437 5097    | 0.011 391 1913    | 0.011 178 4719    |
| 500  | 0.011 595 3645 | 0.011 593 2284    | 0.011 589 9911    | 0.011 524 9907    | 0.010 548 2408    | 0.010 373 4784    |
| 600  | 0.010 149 3846 | 0.010 153 1745    | 0.010 156 1094    | 0.010 102 1240    | 0.009 237 6913    | 0.009 115 3019    |
| 700  | 0.009 069 2380 | 0.009 077 0036    | 0.009 083 9992    | 0.009 037 9859    | 0.008 260 7215    | 0.008 171 7714    |
| 800  | 0.008 227 6189 | 0.008 238 1689    | 0.008 247 9205    | 0.008 207 9385    | 0.007 500 5942    | 0.007 434 1523    |
| 900  | 0.007 550 8836 | 0.007 563 4402    | 0.007 575 0978    | 0.007 539 8333    | 0.006 890 0012    | 0.006 839 3243    |
| 1000 | 0.006 993 2836 | 0.007 007 3134    | 0.007 020 3011    | 0.006 988 8237    | 0.006 387 2442    | 0.006 347 9570    |

### 3. CRITICAL VALUES AND CONCLUSIONS

This section presents critical values calculated using all just described methods. If we take a look on (1.3) and especially (2.2), we can immediately see computational problems connected with the evaluation of exact critical values due especially to plenty of binomial coefficients involved. If one wishes to obtain “really exact” critical values, it is necessary to use either programming language enabling to work in arbitrary precision (like in C) or some of higher level programming languages offering this feature. In our case we have used for the calculation program *Mathematica®* v. 2.2. The main reason was that in such a case we might use aside the arbitrary precision several build in special functions. Moreover, resulting code which is available from the author on request is more “readable and understandable”.

Tables 1a-1c contain both exact critical values of Fisher's test of periodicity  $a_m^*(\alpha)$  and two approximations described by (1.6). We can see immediately that while the values of  $a_{m,1}^*(\alpha)$  give reasonable approximation even for small values of  $m$ , this is not the case of  $a_{m,2}^*(\alpha)$ , which is surprisingly imprecise even for very large sample sizes.

Tables 2a-2c contain both exact critical values  $s_m^*(\alpha)$  of Siegel's test of periodicity and five approximations described in Section 2. We can see immediately that while values of the approximations of  $s_m^*(\alpha)$  based on both central and non-central chi-square distribution are quite precise even for small values of  $m$ , this is not the case of both  $N$  and normal approximations. *Globally said, we can conclude that both  $\chi^2$ -approximation and Wilson-Hilferty approximation based on standard  $\chi^2$  distribution give excellent counterpart to both the exact critical values and their  $\chi_0^2$ -approximation even for small sample sizes. Their main advantage is that they are much more easily available and more easily calculable.*

**Table 2b.** Critical values  $s_m^*(\alpha)$  and their approximations  $s_{m,1}^*(\alpha) - s_{m,5}^*(\alpha)$  of Siegel's test on the level  $\alpha = 0.025$ .

| $m$  | $s_m^*(0.025)$  | $s_{m,1}^*(0.025)$ | $s_{m,2}^*(0.025)$ | $s_{m,3}^*(0.025)$ | $s_{m,4}^*(0.025)$ | $s_{m,5}^*(0.025)$ |
|------|-----------------|--------------------|--------------------|--------------------|--------------------|--------------------|
| 10   | 0.195 354 459 2 | 0.193 978 768 6    | 0.193 587 746 2    | 0.191 918 215 4    | 0.172 586 990 1    | 0.141 595 046 9    |
| 15   | 0.149 780 318 3 | 0.148 866 541 7    | 0.148 559 824 7    | 0.147 264 093 5    | 0.132 849 459 0    | 0.108 277 306 2    |
| 20   | 0.123 003 606 3 | 0.122 369 312 9    | 0.122 118 890 2    | 0.121 057 699 9    | 0.109 093 091 0    | 0.089 110 333 1    |
| 25   | 0.105 152 946 2 | 0.104 697 224 6    | 0.104 486 345 5    | 0.103 585 583 2    | 0.093 142 127 6    | 0.076 432 952 8    |
| 30   | 0.092 303 881 4 | 0.091 967 151 5    | 0.091 785 801 6    | 0.091 001 989 1    | 0.081 622 394 1    | 0.067 334 484 5    |
| 35   | 0.082 561 265 1 | 0.082 307 142 5    | 0.082 148 871 2    | 0.081 454 329 1    | 0.072 875 126 8    | 0.060 440 859 6    |
| 40   | 0.074 890 535 9 | 0.074 695 787 2    | 0.074 556 171 4    | 0.073 532 155 7    | 0.065 985 144 3    | 0.055 011 335 0    |
| 45   | 0.068 676 067 3 | 0.068 525 240 9    | 0.068 401 094 1    | 0.067 834 297 7    | 0.060 404 122 9    | 0.050 608 441 2    |
| 50   | 0.063 527 207 6 | 0.063 409 662 5    | 0.063 298 599 6    | 0.062 779 217 6    | 0.055 782 511 4    | 0.046 955 893 3    |
| 60   | 0.055 463 766 7 | 0.055 392 196 6    | 0.055 302 166 0    | 0.054 856 935 8    | 0.048 553 123 0    | 0.041 223 400 2    |
| 70   | 0.049 412 383 1 | 0.049 370 038 3    | 0.049 296 271 3    | 0.048 906 491 5    | 0.043 138 104 6    | 0.036 907 674 4    |
| 80   | 0.044 684 941 5 | 0.044 662 109 4    | 0.044 601 354 9    | 0.044 254 687 0    | 0.038 916 652 1    | 0.033 525 378 7    |
| 90   | 0.040 878 431 6 | 0.040 869 131 8    | 0.040 819 056 0    | 0.040 506 912 0    | 0.035 524 771 2    | 0.030 793 503 9    |
| 100  | 0.037 740 257 1 | 0.037 740 630 6    | 0.037 699 493 1    | 0.037 415 645 0    | 0.032 734 256 6    | 0.028 534 572 1    |
| 150  | 0.027 707 933 6 | 0.027 730 293 4    | 0.027 718 404 2    | 0.027 523 514 2    | 0.023 859 755 0    | 0.021 260 453 3    |
| 200  | 0.022 228 237 1 | 0.022 257 134 9    | 0.022 261 352 5    | 0.022 113 563 8    | 0.019 052 537 2    | 0.017 241 678 8    |
| 250  | 0.018 728 767 9 | 0.018 760 057 8    | 0.018 774 313 9    | 0.018 655 743 6    | 0.016 002 494 7    | 0.014 651 800 6    |
| 300  | 0.016 280 265 3 | 0.016 312 514 7    | 0.016 333 484 8    | 0.016 234 815 8    | 0.013 879 636 6    | 0.012 826 075 5    |
| 350  | 0.014 460 881 1 | 0.014 493 512 7    | 0.014 519 178 0    | 0.014 434 932 4    | 0.012 309 047 0    | 0.011 460 763 3    |
| 400  | 0.013 049 974 5 | 0.013 082 729 7    | 0.013 111 776 5    | 0.013 038 458 5    | 0.011 095 499 1    | 0.010 396 087 6    |
| 450  | 0.011 920 339 1 | 0.011 953 092 2    | 0.011 984 622 2    | 0.011 919 864 2    | 0.010 126 882 1    | 0.009 539 465 3    |
| 500  | 0.010 993 205 9 | 0.011 025 892 1    | 0.011 059 267 4    | 0.011 001 391 5    | 0.009 334 019 8    | 0.008 833 306 2    |
| 600  | 0.009 556 689 8 | 0.009 589 150 9    | 0.009 624 935 7    | 0.009 577 427 7    | 0.008 109 532 5    | 0.007 732 971 6    |
| 700  | 0.008 490 479 1 | 0.008 522 661 8    | 0.008 559 769 5    | 0.008 519 686 1    | 0.007 203 982 2    | 0.006 910 757 2    |
| 800  | 0.007 664 263 8 | 0.007 696 146 1    | 0.007 733 919 7    | 0.007 699 403 5    | 0.006 504 261 3    | 0.006 269 919 4    |
| 900  | 0.007 003 088 3 | 0.007 034 659 5    | 0.007 072 683 3    | 0.007 042 488 2    | 0.005 945 585 9    | 0.005 754 487 8    |
| 1000 | 0.006 460 618 1 | 0.006 491 872 2    | 0.006 529 875 4    | 0.006 503 125 6    | 0.005 488 054 7    | 0.005 329 688 6    |

**Table 2c.** Critical values  $s_m^*(\alpha)$  and their approximations  $s_{m,1}^*(\alpha) - s_{m,5}^*(\alpha)$  of Siegel's test on the level  $\alpha = 0.01$ .

| $m$  | $s_m^*(0.01)$   | $s_{m,1}^*(0.01)$ | $s_{m,2}^*(0.01)$ | $s_{m,3}^*(0.01)$ | $s_{m,4}^*(0.01)$ | $s_{m,5}^*(0.01)$ |
|------|-----------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| 10   | 0.214 407 337 9 | 0.216 973 362 1   | 0.216 965 516 5   | 0.215 545 159 7   | 0.178 803 034 8   | 0.128 116 592 5   |
| 15   | 0.163 715 457 7 | 0.164 871 142 5   | 0.164 839 468 9   | 0.163 700 504 9   | 0.136 913 065 9   | 0.096 039 859 1   |
| 20   | 0.133 711 589 5 | 0.134 448 128 5   | 0.134 419 773 2   | 0.133 484 311 9   | 0.111 771 397 0   | 0.078 170 486 7   |
| 25   | 0.113 703 477 4 | 0.114 254 610 5   | 0.114 232 511 4   | 0.113 442 825 5   | 0.094 886 996 4   | 0.066 546 331 7   |
| 30   | 0.099 323 723 6 | 0.099 772 677 8   | 0.099 756 614 3   | 0.099 075 231 2   | 0.082 711 380 6   | 0.058 292 764 0   |
| 35   | 0.088 444 769 6 | 0.088 828 759 2   | 0.088 818 032 0   | 0.088 219 973 8   | 0.073 486 620 8   | 0.052 087 728 4   |
| 40   | 0.079 900 498 2 | 0.080 238 977 5   | 0.080 232 898 7   | 0.079 700 835 6   | 0.066 238 804 2   | 0.047 229 979 3   |
| 45   | 0.072 995 766 8 | 0.073 300 161 7   | 0.073 298 135 4   | 0.072 819 594 7   | 0.060 383 235 9   | 0.043 310 046 0   |
| 50   | 0.067 289 252 1 | 0.067 566 862 3   | 0.067 568 389 7   | 0.067 134 109 5   | 0.055 546 887 4   | 0.040 071 526 0   |
| 60   | 0.058 383 708 8 | 0.058 621 339 3   | 0.058 628 801 9   | 0.058 263 415 1   | 0.048 009 461 4   | 0.035 014 010 6   |
| 70   | 0.051 730 440 0 | 0.051 939 171 0   | 0.051 951 394 7   | 0.051 637 117 5   | 0.042 390 714 8   | 0.031 227 568 4   |
| 80   | 0.046 554 325 2 | 0.046 740 915 7   | 0.046 757 049 5   | 0.046 482 164 5   | 0.038 029 832 4   | 0.028 273 409 9   |
| 90   | 0.042 402 440 6 | 0.042 571 408 7   | 0.042 590 815 1   | 0.042 347 195 8   | 0.034 540 269 9   | 0.025 896 320 2   |
| 100  | 0.038 991 588 8 | 0.039 146 146 6   | 0.039 168 331 1   | 0.038 950 111 9   | 0.031 680 289 2   | 0.023 937 096 1   |
| 150  | 0.028 176 762 6 | 0.028 286 046 3   | 0.028 317 552 8   | 0.028 177 408 6   | 0.022 665 383 8   | 0.017 670 573 5   |
| 200  | 0.022 342 804 5 | 0.022 428 244 5   | 0.022 464 984 0   | 0.022 364 631 4   | 0.017 848 149 6   | 0.014 239 801 7   |
| 250  | 0.018 652 777 5 | 0.018 723 636 4   | 0.018 763 622 4   | 0.018 687 157 7   | 0.014 824 084 2   | 0.012 042 663 5   |
| 300  | 0.016 091 154 5 | 0.016 152 270 7   | 0.016 194 381 9   | 0.016 133 723 0   | 0.012 737 704 1   | 0.010 501 164 8   |
| 350  | 0.014 200 326 1 | 0.014 254 531 7   | 0.014 298 072 1   | 0.014 248 572 3   | 0.011 205 646 9   | 0.009 352 822 2   |
| 400  | 0.012 742 444 2 | 0.012 791 528 1   | 0.012 836 039 0   | 0.012 794 792 3   | 0.010 029 632 8   | 0.008 460 214 7   |
| 450  | 0.011 581 123 2 | 0.011 626 281 4   | 0.011 671 445 8   | 0.011 636 520 4   | 0.009 096 463 0   | 0.007 744 015 5   |
| 500  | 0.010 632 318 6 | 0.010 674 386 1   | 0.010 719 977 8   | 0.010 690 028 1   | 0.008 336 655 7   | 0.007 155 041 4   |
| 600  | 0.009 170 811 5 | 0.009 208 351 7   | 0.009 254 343 0   | 0.009 231 676 6   | 0.007 171 277 8   | 0.006 240 093 2   |
| 700  | 0.008 093 626 1 | 0.008 128 033 5   | 0.008 174 040 7   | 0.008 156 399 7   | 0.006 316 615 9   | 0.005 558 838 1   |
| 800  | 0.007 263 926 4 | 0.007 296 051 0   | 0.007 341 845 7   | 0.007 327 844 1   | 0.005 661 025 3   | 0.005 029 458 2   |
| 900  | 0.006 603 470 5 | 0.006 633 862 4   | 0.006 679 304 6   | 0.006 668 036 3   | 0.005 140 979 2   | 0.004 604 779 3   |
| 1000 | 0.006 064 143 5 | 0.006 093 176 8   | 0.006 138 180 0   | 0.006 129 023 1   | 0.004 717 578 1   | 0.004 255 576 7   |

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*RNDr. Jaromír Antoch, CSc., Matematicko-fyzikální fakulta University Karlovy (Faculty of Mathematics and Physics – Charles University), Sokolovská 83, 18600 Praha 8. Czech Republic.*