KYBERNETIKA — VOLUME 31 (1995), NUMBER 4, PAGES 359–373

A CONTRIBUTION TO BOOTSTRAPPING AUTOREGRESSIVE PROCESSES[1]

ZUZANA PRÁŠKOVÁ

A sequence of random vectors elements of which depend on time-delayed observations of an autoregressive process is considered and the distribution of smooth functions of the sample mean of such vectors is studied asymptotically. Both classical approximation based on the Edgeworth expansion and the bootstrap distribution are developed. It is shown that the accuracy of bootstrap approximation is $o(n^{-\frac{1}{2}})$ and therefore better than that of the normal one. Examples of studentized statistics that can appear in the analysis of autoregressive models are shown.

1. INTRODUCTION

Consider an autoregressive process AR(p_0) of order p_0,

$$X_t = b_1 X_{t-1} + \ldots + b_{p_0} X_{t-p_0} + Y_t, \quad t = 0, \pm 1, \ldots \tag{1}$$

and suppose that the following assumptions hold:

A.1 Y_t are iid with $EY_t = 0$, $\operatorname{Var} Y_t = \tau^2 > 0$ and $EY_t^8 < \infty$.

A.2 The vector $\boldsymbol{Y} = (Y_1, Y_1^2)'$ satisfies Cramér condition, i.e. $\forall\, d > 0 \;\; \exists\, \delta > 0$ such that $\sup_{\|\boldsymbol{u}\| \geq d} |E \exp(i\boldsymbol{u}'\boldsymbol{Y})| < 1 - \delta$, where $\|.\|$ denotes the Euclidean norm.

A.3 Roots of the polynomial $\lambda^{p_0} - b_1 \lambda^{p_0 - 1} - \ldots - b_{p_0}$ lie within the unit circle.

It is known that commonly used statistics like sample mean, autocorrelations, least-squares estimations of the autoregression coefficients and their studentized versions are functions of statistics

$$\frac{1}{n}\sum_{t=1}^{n} X_t, \quad \frac{1}{n}\sum_{t=1}^{n} X_t X_{t-k}, \quad \frac{1}{n}\sum_{t=1}^{n} X_{t-k}^2, \quad k = 0, 1, \ldots, p$$

for some integer $p > 0$, that need not be equal to p_0.

[1] Partly supported by the Grant Agency of the Czech Republic under Grant No. 201/94/0472.

Consider vectors

$$\boldsymbol{Z}_t = (X_t, X_{t-1}Y_t, \ldots, X_{t-p}Y_t, Y_t^2 - \tau^2)', \quad t = 1, 2, \ldots \tag{2}$$

and

$$\overline{\boldsymbol{Z}}_n = \frac{1}{n}\sum_{t=1}^{n} \boldsymbol{Z}_t. \tag{3}$$

It can be shown that statistics mentioned above can be written in the form $T_n = H(\overline{\boldsymbol{Z}}_n)$, where H is a function of $p+2$ variables, possibly dependent on parameters $b_1, \ldots, b_{p_0}$ and τ^2. Obviously, $E\overline{\boldsymbol{Z}}_n = \boldsymbol{0}$. Denote $\boldsymbol{\Sigma}_n = \mathrm{Var}(\sqrt{n}\,\overline{\boldsymbol{Z}}_n)$ and assume further that

A.4 $\boldsymbol{\Sigma}_n \to \boldsymbol{\Sigma}$ as $n \to \infty$, where $\boldsymbol{\Sigma}$ is a positive definite matrix.

A.5 H is a real function of $p+2$ variables three times continuously differentiable in a neighbourhood of zero.

Denote

$$\boldsymbol{l} = \left(\frac{\partial H}{\partial x_i}, i = 1, \ldots, p+2\right)'\Bigg|_{\boldsymbol{x}=\boldsymbol{0}},$$

$$\boldsymbol{D} = \left(\frac{\partial^2 H}{\partial x_i \partial x_j}, i, j = 1, \ldots, p+2\right)\Bigg|_{\boldsymbol{x}=\boldsymbol{0}}.$$

In this paper we shall deal with bootstrap approximation of the distribution of $H(\overline{\boldsymbol{Z}}_n)$. Most literature on bootstrap technique is concerned with independent observations. Recently, some papers appeared solving the problem of validity of bootstrap approximation for certain statistics of interest in linear autoregressive (AR) and autoregressive-moving average (ARMA) models.

Freedman [8] studied bootstrap in a stationary model involving dependent observations which also covers an AR model with exogenous variables. Bose in [6] proved by using Edgeworth correction that bootstrap works for least-squares estimates of parameters in AR models. Basawa et al. [3] established the asymptotic validity of the bootstrap estimate in explosive AR(1) process. Paparoditis and Streitberg [13] proved that bootstrap works for sample vector autocorrelations in ARMA models. Kreiss and Franke [12] obtained bootstrap approximation for M-estimates of parameters in ARMA models, Prášková [14] developed Edgeworth expansion for bootstrap sample mean in the first-order autoregression. Some of these mentioned results can be considered as special cases of those for $H(\overline{\boldsymbol{Z}}_n)$ as stated in next sections.

We shall prove that bootstrap approximation works for $H(\overline{\boldsymbol{Z}}_n)$ and establish the accuracy of this approximation. To do it we have to develop the Edgeworth expansion for the distribution of $H(\overline{\boldsymbol{Z}}_n)$ and for its bootstrap version as well.

2. EDGEWORTH EXPANSION

Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$ be a strictly stationary sequence of k-variate random vectors with zero mean. It can be shown that under some conditions the distribution of $\boldsymbol{S}_n = n^{-1/2}(\boldsymbol{X}_1 + \ldots + \boldsymbol{X}_n)$ is asymptotically normal with zero mean and a variance matrix

$\boldsymbol{\Sigma}$ (see [10] for references). Higher order approximations of the distribution of $\boldsymbol{S}_n$ are based on the Edgeworth expansion. Denote by $\psi_{n,s}$ a signed measure associated with the Edgeworth expansion. Typically, in case of independent random vectors we have for $s \geq 3$

$$\psi_{n,s} = \sum_{r=0}^{s-2} n^{-r/2} P_r \tag{4}$$

where P_0 is normal distribution with zero mean and the variance matrix $\boldsymbol{\Sigma}$ and for $r = 1, \ldots, s-2$, P_r is a finite signed measure the density of which with respect to P_0 is a polynomial independent on n with coefficients uniquely determined by the moments of $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ up to order $r+2$. For more detailed discussion on Edgeworth expansion for independent random vectors, both identically and non-identically distributed, see the monograph by Bhattacharya and Rao [4].

Götze and Hipp [10] established the Edgeworth expansion for sums of weakly dependent k-variate random vectors under the following conditions.

Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$ be a sequence of k-variate random vectors on a space $(\Omega, \mathcal{A}, P)$. Let there be σ-fields $\mathcal{D}_j$ and $\mathcal{D}_a^b = \sigma(\bigcup_{j=a}^b \mathcal{D}_j)$. Assume that

B.1 $E\boldsymbol{X}_j = \boldsymbol{0}$, $j = 1, 2, \ldots$

B.2 $E\|\boldsymbol{X}_j\|^{s+1} \leq \beta_{s+1} < \infty$ for some $s \geq 3$ and $j = 1, 2, \ldots$

B.3 There exists a positive constant d such that for $n, m = 1, 2, \ldots$ with $m > d^{-1}$ there exists a k-variate, $\mathcal{D}_{n-m}^{n+m}$ -measurable vector $\boldsymbol{Y}_{nm}$ for which

$$E\|\boldsymbol{X}_n - \boldsymbol{Y}_{nm}\| \leq d^{-1} e^{-dm}.$$

B.4 There exists $d > 0$ such that for all $m, n = 1, 2, \ldots, A \in \mathcal{D}_{-\infty}^n$, $B \in \mathcal{D}_{n+m}^{\infty}$,

$$|P(A \cap B) - P(A)P(B)| \leq d^{-1} e^{-dm}.$$

B.5 There exists $d > 0$ such that for $m, n = 1, 2, \ldots, d^{-1} < m < n$, and $\boldsymbol{t} \in \mathbb{R}^k$ with $\|t\| \geq d$

$$E\left|E\left(\exp\left(it' \sum_{k=n-m}^{n+m} \boldsymbol{X}_k\right) \middle| \mathcal{D}_j, j \neq n\right)\right| \leq e^{-d}.$$

B.6 There exists $d > 0$ such that for all m, n, $r = 1, 2, \ldots, A \in \mathcal{D}_{n-r}^{n+r}$

$$E|P(A|\mathcal{D}_j : j \neq n) - P(A|\mathcal{D}_j : 0 < |n-j| \leq m+r)| \leq d^{-1} e^{-dm}.$$

B.7 $\boldsymbol{\Sigma} = \lim_{n\to\infty} \operatorname{Var}\left(n^{-\frac{1}{2}} \sum_{t=1}^n \boldsymbol{X}_t\right)$ exists and is positive definite.

Let $s_0 \leq s$ be an integer equal to s if s is even and to $s-1$ if s is odd. Let $\psi_{n,s}$ be the Edgeworth expansion of $\boldsymbol{S}_n = n^{-1/2}(\boldsymbol{X}_1 + \ldots + \boldsymbol{X}_n) = \sqrt{n}\,\overline{\boldsymbol{X}}_n$ (for formal definition see Götze and Hipp [10], p. 217.)

Theorem 1. Let $f : \mathbb{R}^k \to \mathbb{R}$ be a measurable function such that $|f(\boldsymbol{x})| \le M(1 + \|\boldsymbol{x}\|^{s_0})$ for every $\boldsymbol{x} \in \mathbb{R}^k$ and a constant M. Assume that B.1 – B.7 hold. Then there exists a positive constant δ not depending on f and M, and for arbitrary $k > 0$ there exists a positive constant c depending on M but not on f such that

$$\left| Ef(\boldsymbol{S}_n) - \int f(\boldsymbol{x})\,\mathrm{d}\psi_{n,s}(\boldsymbol{x}) \right| \le o(n^{-(s-2+\delta)/2}) + c\omega(f, n^{-k}),$$

where

$$\omega(f, n^{-k}) = \int \sup \left\{ |f(\boldsymbol{x} + \boldsymbol{y}) - f(\boldsymbol{x})| : \|\boldsymbol{y}\| \le n^{-k} \right\} \phi_{\boldsymbol{\Sigma}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \tag{5}$$

and $\phi_{\boldsymbol{\Sigma}}$ stands for the normal density function with zero mean and the variance matrix $\boldsymbol{\Sigma}$. The term $o(n^{-(s-2+\delta/2)})$ depends on f through M only.

Proof. See Götze and Hipp [10], Theorem 2.8. □

If $\{\boldsymbol{X}_t\}$ is a strictly stationary sequence, then $\psi_{n,s}$ is of the form (4) (see Remark 2.12 in [10]). Thus, the density of one-term Edgeworth expansion is

$$\left[1 + n^{-\frac{1}{2}} p(\boldsymbol{x})\right] \phi_{\Sigma}(\boldsymbol{x}),$$

where $p(\boldsymbol{x})$ is a polynomial depending on moments of $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ up to order 3.

Now, we can establish the Edgeworth expansion for the vectors $\boldsymbol{Z}_t$ and $\overline{\boldsymbol{Z}}_n$ given by (2) and (3), respectively.

Theorem 2. Let $f : \mathbb{R}^{p+2} \to \mathbb{R}$ be a measurable function such that $|f(\boldsymbol{x})| \le M(1 + \|\boldsymbol{x}\|^2)$ for every $\boldsymbol{x} \in \mathbb{R}^{p+2}$ and a constant M. Assume that A.1 – A.4 hold. Then there exists a positive constant δ not depending on f and M, and for arbitrary $k > 0$ there exists a positive constant c depending on M but not on f such that

$$\left| Ef(\sqrt{n}\,\overline{\boldsymbol{Z}}_n) - \int f(\boldsymbol{x})[1 + n^{-\frac{1}{2}} p(\boldsymbol{x})]\,\phi_{\boldsymbol{\Sigma}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \right| \le o\left(n^{-\frac{1+\delta}{2}}\right) + c\omega(f, n^{-k}),$$

where $\omega(f, n^{-k})$ is defined by (5) and $p(\boldsymbol{x})$ is a polynomial in $\boldsymbol{x} = (x_1, \ldots, x_{p+2})$ depending only on moments of $\boldsymbol{Z}_t$ of order at most three.

Corollary. Under the assumptions of Theorem 2

$$P(\sqrt{n}\,\overline{\boldsymbol{Z}}_n \in C) = \int_C [1 + n^{-\frac{1}{2}} p(\boldsymbol{x})]\,\phi_{\boldsymbol{\Sigma}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + o\left(n^{-\frac{1}{2}}\right)$$

uniformly for any convex set $C \subseteq \mathbb{R}^{p+2}$.

Proof of Theorem 2. The assertion follows from Theorem 1 if we consider vectors $\boldsymbol{Z}_t$ given by (2). We only need to check that assumptions A.1 – A.4 imply

B.1 – B.7 with $s = 3$. The proving method is very close to that developed by Bose [6] who applied Theorem 1 to the vectors

$$(X_{t-1}Y_t, \dots, X_{t-p_0}Y_t, Y_t^2 - \tau^2)'.$$

Here we shall prove that the statement holds true for the vectors (2) and for any integer $p > 0$.

It is well known (see, e. g., Anderson [1], Chap. 5.2.2) that under assumption A.3 the process (1) has an infinite series representation

$$X_t = \sum_{k=0}^{\infty} c_k Y_{t-k}, \tag{6}$$

where $|c_k| < c|\rho|^k$ for a positive constant c and $|\rho| < 1$.

Let $\mathcal{D}_j$ be the σ-field generated by Y_j. Then it can be easily checked that under assumptions A.1 – A.4, conditions B.1 – B.2, B.4 and B.6 – B.7 hold for $\boldsymbol{Z}_t$ given by (2). The assumption B.3 will be satisfied if we choose

$$Y_{nm} = \left(\sum_{k=0}^{m} c_k Y_{n-k}, \sum_{k=0}^{m-1} c_k Y_{n-k-1} Y_n, \dots, \sum_{k=0}^{m-p} c_k Y_{n-k-p} Y_n, Y_n^2 - \tau^2 \right)'.$$

To verify B.5, we can proceed similarly as Bose [6], pp. 1713–1714. From infinite linear representation (6) we get

$$X_j = \begin{cases} c_{j-n}Y_n + B & i \geq n, \\ B & j < n, \end{cases}$$

$$X_{j-i}Y_j = \begin{cases} c_{j-i-n}Y_jY_n + B & j - i \geq n, \\ Y_n B & j = n, \\ B & j - i < n, \end{cases}$$

where B means a random variable independent of Y_n (it can change even in the same formula). Now, denote

$$\begin{aligned} A_{in} &= \sum_{k=0}^{\infty} c_k Y_{n-i-k}, & i = 1, \dots, p, \\ B_{inm} &= \sum_{k=0}^{m-i} c_k Y_{k+i+n}, & i = 1, \dots, p, \\ C_m &= \sum_{k=0}^{m} c_k, \\ \boldsymbol{V}_{nm} &= (C_m, A_{in} + B_{inm}, i = 1, \dots, p)'. \end{aligned}$$

Assumptions A.1 and A.3 and the geometrical boundedness of c_k imply that for all i, $j = 1, \dots, p$, A_{in}, B_{jnm} are independent random variables and for $m, n \to \infty$

$$\boldsymbol{V}_{nm0} = (A_{in} + B_{inm}, i = 1, \dots, p)' \xrightarrow{\mathcal{L}} \boldsymbol{Z} = (Z_{i1} + Z_{i2}, i = 1, \dots, p)'$$

where Z_{i1}, Z_{i2} are independent random variables with the same distribution and same as that of $\sum_{k=0}^{\infty} c_k Y_{i-k}$. Further, we can calculate that

$$\mathrm{Cov}(Z_{i1}, Z_{j1}) = \mathrm{Cov}(Z_{i2}, Z_{j2}) = \mathrm{Cov}(X_i, X_j).$$

Thus, the dispersion matrix $\boldsymbol{\Gamma}$ of the vector $\boldsymbol{Z}$ is positive definite (see [7], Theorem 5.1.1).

Now, it holds for a vector $\boldsymbol{t} = (t_1, \ldots, t_{p+2})'$

$$E\left|E\exp\left(i\boldsymbol{t}'\sum_{k=n-m}^{n+m}\boldsymbol{Z}_k\right)\middle|\mathcal{D}_j, j\neq n\right| = \tag{7}$$

$$E|E\exp(i(\boldsymbol{t}_0'\boldsymbol{V}_{nm}Y_n + t_{p+2}Y_n^2 + B)|Y_j, j\neq n|,$$

where $\boldsymbol{t}_0 = (t_1, \ldots, t_{p+1})'$.

Denote

$$\boldsymbol{t}_{nm} = (\boldsymbol{t}_0'\boldsymbol{V}_{nm}, t_{p+2})', \quad \boldsymbol{Y}_n = (Y_n, Y_n^2)'.$$

Then we have from (7) and A.2

$$E\left|E\exp\left(i\boldsymbol{t}'\sum_{k=n-m}^{n+m}\boldsymbol{Z}_k\right)\middle|\mathcal{D}_j, j\neq n\right| \leq E|E\exp(i\boldsymbol{t}_{nm}'\boldsymbol{Y}_n)|Y_j, j\neq n|$$

$$\leq e^{-\delta}P(\|\boldsymbol{t}_{nm}\|\geq d) + P(\|\boldsymbol{t}_{nm}\| < d).$$

Let $t_1^2 + \ldots + t_{p+1}^2 + t_{p+2}^2 \geq d_1^2 > d^2/l^2$ for $0 < l < 1$. First suppose that $\|\boldsymbol{t}_0\| \neq 0$ and put $\boldsymbol{a} = (a_1, \ldots, a_{p+1})' = \boldsymbol{t}_0\|\boldsymbol{t}_0\|^{-1}$. Denote $\boldsymbol{a}_0 = (a_2, \ldots, a_{p+1})'$. We get

$$\begin{aligned} P(\|\boldsymbol{t}_{nm}\|^2 \geq d^2) &= P((\boldsymbol{t}_0'\boldsymbol{V}_{nm})^2 + t_{p+2}^2 \geq d^2) \\ &\geq P(|\boldsymbol{a}'\boldsymbol{V}_{nm}|^2 \geq l^2) = P(|a_1 c_m + \boldsymbol{a}_0'\boldsymbol{V}_{nm0}|^2 \geq l^2). \end{aligned}$$

If $\|\boldsymbol{a}_0\| \neq 0$, then

$$a_1 C_m + \boldsymbol{a}_0'\boldsymbol{V}_{nm0} \xrightarrow{\mathcal{L}} a_1 C + \boldsymbol{a}_0'\boldsymbol{Z}$$

where $a_1 C$ is a finite constant and $\boldsymbol{a}_0'\boldsymbol{Z}$ has finite variance $\boldsymbol{a}_0'\boldsymbol{\Gamma}\boldsymbol{a}_0$. Thus, there exist at least two points $x_1 \neq x_2 \in \mathbb{R}^1$ and $r > 0$ such that $P(I_1) > 0$, $P(I_2) > 0$, where

$$I_1 = \{\omega \in \Omega : a_1 C + \boldsymbol{a}_0'\boldsymbol{Z} \in (x_1 - r, x_1 + r)\},$$

$$I_2 = \{\omega \in \Omega : a_1 C + \boldsymbol{a}_0'\boldsymbol{Z} \in (x_2 - r, x_2 + r)\}.$$

For l sufficiently small the interval $(-l, l)$ does not intersect at least one from the intervals $(x_1 - r, x_1 + r), (x_2 - r, x_2 + r)$ and thus

$$P(|a_1 C + \boldsymbol{a}_0'\boldsymbol{Z}|^2 \geq l^2) \geq \min(P(I_1), P(I_2)) > 0.$$

Hence, we can conclude that for $m \geq m_0, n \geq n_0$, $P(\|\boldsymbol{t}_{nm}\|^2 \geq d^2) \geq \epsilon > 0$. The cases $\|\boldsymbol{a}_0\| = 0$ and $\|\boldsymbol{t}_0\| = 0$ are trivial. □

P r o o f of Corollary. It follows from Theorem 2 if we put $f(\boldsymbol{x}) = \chi_C(\boldsymbol{x})$ (the indicator function of a convex set C), from the proof of Corollary 2.6 and from the Corollary 3.2 in Bhattacharya and Rao [4]. □

Lemma 3. Let assumptions A.1 – A.5 be satisfied. Then

$$P\left\{\sqrt{n}(H(\overline{\boldsymbol{Z}}_n) - H(\mathbf{0})) < x(\boldsymbol{l}'\boldsymbol{\Sigma}\boldsymbol{l})^{\frac{1}{2}}\right\} = \Phi(x) + n^{-\frac{1}{2}}Q(x)\phi(x) + o\left(n^{-\frac{1}{2}}\right) \tag{8}$$

uniformly for all x, where Φ, ϕ denote the distribution function, respectively the density of $\mathcal{N}(0,1)$ and Q is a polynomial with coefficients dependent on moments of $\boldsymbol{Z}_t$.

Proof. Let λ be the largest eigenvalue of the matrix $\boldsymbol{\Sigma}$. On the set $\{\|\overline{\boldsymbol{Z}}_n\| < n^{-\frac{1}{2}}(\lambda \log n)^{\frac{1}{2}}\}$ we have from the Taylor expansion that

$$H(\overline{\boldsymbol{Z}}_n) - H(\mathbf{0}) = \boldsymbol{l}'\overline{\boldsymbol{Z}}_n + \frac{1}{2}\overline{\boldsymbol{Z}}_n'\boldsymbol{D}\overline{\boldsymbol{Z}}_n + R_n,$$

where $\boldsymbol{l}$ and $\boldsymbol{D}$ are as in A.5 and

$$R_n = O_p\left((n^{-1}\log n)^{\frac{3}{2}}\right).$$

From Theorem 2 we get, if we use properties of the normal density, that

$$P\left(\|\overline{\boldsymbol{Z}}_n\| \geq n^{-\frac{1}{2}}(\lambda \log n)^{\frac{1}{2}}\right) = o\left(n^{-\frac{1}{2}}\right).$$

Thus, if we put $\sqrt{n}\,\overline{\boldsymbol{Z}}_n = \boldsymbol{\zeta}$ we can write

$$\begin{aligned} &P\left\{\sqrt{n}(H(\overline{\boldsymbol{Z}}_n) - H(\mathbf{0})) < x(\boldsymbol{l}'\boldsymbol{\Sigma}\boldsymbol{l})^{\frac{1}{2}}\right\} = \\ &P\left\{\boldsymbol{l}'\boldsymbol{\zeta} + \tfrac{1}{2\sqrt{n}}\boldsymbol{\zeta}'\boldsymbol{D}\boldsymbol{\zeta} < x(\boldsymbol{l}'\boldsymbol{\Sigma}\boldsymbol{l})^{\frac{1}{2}}\right\} + o\left(n^{-\frac{1}{2}}\right). \end{aligned} \tag{9}$$

Further, by using of Theorem 2 we obtain

$$P\left(\boldsymbol{l}'\boldsymbol{\zeta} + \frac{1}{2\sqrt{n}}\boldsymbol{\zeta}'\boldsymbol{D}\boldsymbol{\zeta} < x(\boldsymbol{l}'\boldsymbol{\Sigma}\boldsymbol{l})^{\frac{1}{2}}\right) = \int_M [1 + n^{-\frac{1}{2}}p(\boldsymbol{z})]\phi_{\boldsymbol{\Sigma}}(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z} + o\left(n^{-\frac{1}{2}}\right), \tag{10}$$

where

$$M = \left\{\boldsymbol{z} : \boldsymbol{l}'\boldsymbol{z} + \frac{1}{2\sqrt{n}}\boldsymbol{z}'\boldsymbol{D}\boldsymbol{z} < x(\boldsymbol{l}'\boldsymbol{\Sigma}\boldsymbol{l})^{\frac{1}{2}}\right\}.$$

According to Lemma 3 in Babu and Singh [2] there exists a polynomial q in one variable whose coefficients are continuous functions of the elements of $\boldsymbol{l}$, $\boldsymbol{D}$, $\boldsymbol{\Sigma}$ and the coefficients of $p(\boldsymbol{z})$, such that

$$\begin{aligned} \int_M \left[1 + n^{-\frac{1}{2}}p(\boldsymbol{z})\right] \phi_{\boldsymbol{\Sigma}}(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z} &= \int_{-\infty}^{x} \left[1 + n^{-\frac{1}{2}}q(y)\right] \phi(y)\,\mathrm{d}y + o\left(n^{-\frac{1}{2}}\right) \\ &= \Phi(x) + n^{-\frac{1}{2}}Q(x)\phi(x) + o\left(n^{-\frac{1}{2}}\right). \end{aligned}$$

□

3. BOOTSTRAP DISTRIBUTION

Suppose that $X_{1-p_0}, \dots, X_0, X_1, \dots, X_n$ are observations of the series $\{X_t\}$ given by (1). Let $\hat{b}_1, \dots, \hat{b}_{p_0}$ be consistent estimators of $b_1, \dots, b_{p_0}$. Put

$$\hat{Y}_t = X_t - \sum_{i=1}^{p_0} \hat{b}_i X_{t-i}, \quad t = 1, \dots, n. \tag{11}$$

Let $\overline{Y}$ denote the arithmetic mean of $\hat{Y}_1, \dots \hat{Y}_n$ and $F_n^*(x)$ be the empirical distribution function based on $\hat{Y}_1 - \overline{Y}, \dots, \hat{Y}_n - \overline{Y}$. Given X_t, $t = 1 - p_0, \dots, n$, let Y_t^*, $t = 1 - p, \dots, n$ be i.i.d. with the distribution function F_n^*. Define $X_t^* = 0$ for $t \leq -p$ and for $t \geq 1 - p$ generate bootstrap values

$$X_t^* = \hat{b}_1 X_{t-1}^* + \ldots + \hat{b}_{p_0} X_{t-p_0}^* + Y_t^* \tag{12}$$

and consider vectors

$$\boldsymbol{Z}_t^* = \left(X_t^*, X_{t-1}^* Y_t^*, \dots, X_{t-p}^* Y_t^*, Y_t^{*2} - \tau^{*2}\right)',$$

where τ^{*2} denotes the variance of Y_t^* with respect to the probability induced by F_n^*. In the sequel, we shall denote this probability by P^*; the asterisk will also denote moments and other characteristics of the bootstrap distribution.

In our next considerations we shall suppose that

A.6 Estimators $\hat{b}_1, \dots, \hat{b}_{p_0}$ used in (11) and (12) are strictly consistent.

Then the following results can be proved.

Lemma 4. Under assumptions A.1 – A.4 and A.6

$$\left| E^* f(\sqrt{n}\, \overline{\boldsymbol{Z}}_n^*) - \int f(\boldsymbol{x})[1 + n^{-1/2} p^*(\boldsymbol{x})]\, \phi_{\boldsymbol{\Sigma}^*}(\boldsymbol{x}))\, \mathrm{d}\boldsymbol{x} \right| \leq o\left(n^{-\frac{1}{2}}\right) + c\omega^*(f, n^{-k})$$

holds for almost all sequences $\{X_t\}$; f, c and $k > 0$ are as in Theorem 2, $p^*(\boldsymbol{x})$ is the bootstrap version of $p(\boldsymbol{x})$ and ω^* means ω where $\phi_{\boldsymbol{\Sigma}}$ is replaced by $\phi_{\boldsymbol{\Sigma}}^*$.

Proof. To prove the result we need to introduce a special truncation function defined in Götze and Hipp [10]. Let $g \in \mathcal{C}^\infty(0, \infty)$ satisfy $g(x) = x$ if $x \leq 1, g(x) = 2$ for $x \geq 2$ and g is increasing. For $\boldsymbol{x} \in \mathbb{R}^{p+2}$ define

$$T(\boldsymbol{x}) = \begin{cases} \boldsymbol{x}, & ||\boldsymbol{x}|| \leq n^\gamma, \quad \gamma > \frac{1}{8}, \\ \boldsymbol{x} n^\gamma g(||\boldsymbol{x}|| n^{-\gamma}) / ||\boldsymbol{x}||) & \text{otherwise} \end{cases}$$

and put $\boldsymbol{U}_t^* = T(\boldsymbol{Z}_t^*) - E^*(T(\boldsymbol{Z}_t^*))$.

Let G_n^* be the characteristic function of $n^{-\frac{1}{2}} \sum_{t=1}^n \boldsymbol{U}_t^*$, let $\hat{\psi}_n(\boldsymbol{v})$ denote the Fourier transform of

$$\psi_n^*(\boldsymbol{x}) = \left(1 + n^{-\frac{1}{2}} p^*(\boldsymbol{x})\right) \phi_{\boldsymbol{\Sigma}^*}(\boldsymbol{x}).$$

By using Lemma 3.3 in Götze and Hipp [10] with the function $\hat{K}(n^{-k}\boldsymbol{v}) = 0$ for $\|\boldsymbol{v}\| > \eta^{-1}n^{\frac{1}{2}}$ and a properly chosen $\eta > 0$, and following the proof of (15.34) – (15.37), Theorem 15.1 in Bhattacharya and Rao [4], we get that

$$\left| E^* f(\sqrt{n}\,\overline{\boldsymbol{Z}}_n^*) - \int f(\boldsymbol{x})\,\psi_n^*(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x} \right|$$

$$\le c_1(M) \max_{|\alpha|\le r+3} \int_{\|\boldsymbol{v}\|<\eta^{-1}n^{\frac{1}{2}}} \left| D^\alpha(G_n^*(\boldsymbol{v}) - \hat{\psi}_n(\boldsymbol{v}))\right| \mathrm{d}\boldsymbol{v} \tag{13}$$

$$+c_2(M)\,\omega^*(f, n^{-k}) + o\left(n^{-\frac{1}{2}}\right),$$

where $r = p+2$ is the dimension of the vector $\overline{\boldsymbol{Z}}_n$, $\alpha = (\alpha_1, \ldots, \alpha_r)$, $|\alpha| = \alpha_1 + \ldots + \alpha_r$, D^α stands for the differential operator and constants $c_1(M), c_2(M)$ depend on M but not on f.

The integral on the righthand-side of (13) can be bounded from above by the sum $J_1 + J_2 + J_3 + J_4$, where

$$J_1 = \int_{\|\boldsymbol{v}\|\le Cn^\varepsilon} |D^\alpha(G_n^*(\boldsymbol{v}) - \hat{\psi}_n(\boldsymbol{v}))|\,\mathrm{d}\boldsymbol{v},$$

$$J_2 = \int_{Cn^\varepsilon<\|\boldsymbol{v}\|\le C_1 n^{1/2}} |D^\alpha G_n^*(\boldsymbol{v})|\,\mathrm{d}\boldsymbol{v},$$

$$J_3 = \int_{C_1 n^{1/2}<\|\boldsymbol{v}\|\le\eta^{-1}n^{1/2}} |D^\alpha G_n^*(\boldsymbol{v})|\,\mathrm{d}\boldsymbol{v},$$

$$J_4 = \int_{Cn^\varepsilon<\|\boldsymbol{v}\|\le\eta^{-1}n^{1/2}} |D^\alpha \hat{\psi}_n(\boldsymbol{v})|\,\mathrm{d}\boldsymbol{v}$$

and $0 < \varepsilon < \frac{1}{2}$.

On the set $\|\boldsymbol{v}\| \le Cn^\varepsilon$ we get from Lemma 3.33 in [10] that

$$|D^\alpha(G_n^*(\boldsymbol{v}) - \hat{\psi}_n(\boldsymbol{v}))| \le c\left(1+\beta_4^*)(1+\|\boldsymbol{v}\|^{6+|\alpha|}\right)\exp\left(-c|\boldsymbol{v}\|^2\right)\, n^{-\frac{1}{2}-\varepsilon},$$

where β_4^* is a bound of $E^*\|\boldsymbol{Z}_t^*\|^4$. From here and from the fact that $E^*\|\boldsymbol{Z}_t^*\|^4 \to E\|\boldsymbol{Z}_t\|^4$ a.s. (see Appendix) we can conclude that $J_1 = o\left(n^{-\frac{1}{2}}\right)$ a.s. Proceeding as in the proof of Lemma 3.5 and Lemma 3.6 in Bose [6] we get that J_2 and J_3, respectively, are of order $o\left(n^{-\frac{1}{2}}\right)$ a.s. By a straightforward calculation we also obtain $J_4 = o\left(n^{-\frac{1}{2}}\right)$ a.s. □

Lemma 5. Under assumptions A.1 – A.6

$$P^*\left(\sqrt{n}(H(\overline{\boldsymbol{Z}}_n^*) - H(\boldsymbol{0})\right) < x\left(\boldsymbol{l}'\boldsymbol{\Sigma}^*\boldsymbol{l})^{\frac{1}{2}}\right) = \Phi(x) + n^{-\frac{1}{2}}Q^*(x)\,\phi(x) + o\left(n^{-\frac{1}{2}}\right) \tag{14}$$

holds uniformly in x and for almost all sequences $\{X_t\}$. The polynomial Q^* depends on the moments of $\boldsymbol{Z}_t^*$.

Proof. It follows from Lemma 4 in this paper and Lemma 3 in Babu and Singh [2]. □

Theorem 6. Under assumptions A.1–A.6

$$\sup_x \left|P^*\left(\sqrt{n}(H(\overline{\boldsymbol{Z}}_n^*)-H(\mathbf{0}))<x(\boldsymbol{l}'\boldsymbol{\Sigma}^*\boldsymbol{l})^{\frac{1}{2}}\right)-P\left(\sqrt{n}(H(\overline{\boldsymbol{Z}}_n)-H(\mathbf{0}))<x(\boldsymbol{l}'\boldsymbol{\Sigma}\boldsymbol{l})^{\frac{1}{2}}\right)\right|$$
$$= o\left(n^{-\frac{1}{2}}\right)$$

holds almost surely.

Proof. It follows easily from (8), (14) and the fact that $Q^*(x) \to Q(x)$ a.s. (see Appendix). □

Remark. If we consider an AR(p_0) model with nonzero mean μ,

$$X_t - \mu = \sum_{j=1}^{p_0} b_j(X_{t-j} - \mu) + Y_t,$$

where Y_t satisfy A.1, then Theorem 2 and Lemma 3 hold true for the vectors

$$\boldsymbol{Z}_t = (X_t - \mu, (X_{t-1} - \mu)Y_t, \ldots, (X_{t-p} - \mu)Y_t, Y_t^2 - \tau^2)'.$$

Let $\hat{\mu}, \hat{b}_1, \ldots, \hat{b}_{p_0}$ be strictly consistent estimators of $\mu, b_1, \ldots, b_{p_0}$. Then the bootstrap values can be generated by

$$X_t^* - \hat{\mu} = \sum_{j=1}^{p_0} \hat{b}_j(X_{t-j}^* - \hat{\mu}) + Y_t^*,$$

where (for given values X_t) Y_t^* are drawn from the empirical distribution function based on centered residuals

$$\hat{Y}_t = X_t - \hat{\mu} - \sum_{j=1}^{p_0} \hat{b}_j(X_{t-j} - \hat{\mu}).$$

Then Theorem 6 holds for the vectors

$$\boldsymbol{Z}_t^* = \left(X_t^* - \hat{\mu}, (X_{t-1}^* - \hat{\mu})\,Y_t^*, \ldots, (X_{t-p}^* - \hat{\mu})\,Y_t^*, Y_t^* - \tau^{*2}\right)'.$$

4. EXAMPLES

Consider an AR(1) model

$$X_t = bX_{t-1} + Y_t, \quad |b| < 1, \tag{15}$$

where Y_t are independent, identically and continuously distributed with zero mean and finite moments up to order 8. Denote $\operatorname{Var} Y_t = \tau^2$, $EY_t^3 = m_3$, $EY_t^4 = m_4$. By the straightforward calculation we get that the variance matrix of $\sqrt{n}\,\overline{\boldsymbol{Z}}_n$ where $\boldsymbol{Z}_t$ are the vectors $(X_t, X_{t-1}Y_t, Y_t^2 - \tau^2)'$ converges to the matrix

$$\boldsymbol{\Sigma} = \begin{pmatrix} \frac{\tau^2}{(1-b)^2} & 0 & \frac{m_3}{1-b} \\ 0 & \frac{\tau^4}{1-b^2} & 0 \\ \frac{m_3}{1-b} & 0 & m_4 - \tau^4 \end{pmatrix}.$$

Suppose that τ^2, m_3 and m_4 are such that $\boldsymbol{\Sigma}$ is positive definite. For given observations $X_0, X_1, \ldots, X_n$ let $\hat{b}$ be the least-squares estimator of b,

$$\hat{b} = \sum_{t=1}^{n} X_{t-1}X_t \left(\sum_{t=1}^{n} X_{t-1}^2 \right)^{-1} \tag{16}$$

and $X_0^*, X_1^*, \ldots, X_n^*$ be the bootstrap values generated according to (12), $X_t^* = 0$ for $t < 0$. We obtain bootstrap version $\hat{b}^*$ if we replace X_t by X_t^* in (16).

Example 1. Studentized mean

It is known that under the above assumptions the statistic $n^{\frac{1}{2}}\overline{X} = n^{-\frac{1}{2}} \sum_{t=1}^{n} X_t$ is asymptotically normal with zero mean and the variance $\sigma^2 = \tau^2(1-b)^{-2}$ (consult, e.g., [9], Theorem 6.3.3.) In [14] it was shown that bootstrap works for $n^{\frac{1}{2}}\sigma^{-1}\overline{X}$ with the accuracy $o\left(n^{-\frac{1}{2}}\right)$ almost surely. Now, consider the studentized statistic

$$t = \sqrt{n}\frac{\overline{X}}{\hat{\sigma}},$$

where

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{t=1}^{n} X_t^2 \frac{1+\hat{b}}{1-\hat{b}}.$$

In the sequel, $\sum$ will denote the summation over $t = 1, \ldots, n$. From (15) and (16) we have

$$\hat{\sigma}^2 = \frac{1}{n}\sum X_t^2 \frac{(1+b)\sum X_{t-1}^2 + \sum X_{t-1}Y_t}{(1-b)\sum X_{t-1}^2 + \sum X_{t-1}Y_t}, \tag{17}$$

$$\sum X_{t-1}^2 = (1-b^2)^{-1}\left(\sum (Y_t^2 - \tau^2) + n\tau^2 + 2b\sum X_{t-1}Y_t + X_0^2 - X_n^2\right). \tag{18}$$

Denote

$$\overline{Z}_1 = \frac{1}{n}\sum X_t, \quad \overline{Z}_2 = \frac{1}{n}\sum X_{t-1}Y_t, \quad \overline{Z}_3 = \frac{1}{n}\sum (Y_t^2 - \tau^2).$$

By using of Theorem 2 we can find that each from the following inequalities

$$|\overline{Z}_2| > n^{-\frac{1}{2}}\log n; \quad |\overline{Z}_3| > n^{-\frac{1}{2}}\log n$$

$$X_0^2 > n^{\frac{1}{4}} \log n; \quad X_n^2 > n^{\frac{1}{4}} \log n$$

hold true with probability $o\left(n^{-\frac{1}{2}}\right)$. Thus on the set of probability $1 - o\left(n^{-\frac{1}{2}}\right)$ we get combining (17) and (18)

$$t = n^{\frac{1}{2}}(1-b)\,\tau^{-1}\overline{Z}_1\left(1 - \frac{1}{2\tau^2}\overline{Z}_3\right) + o_p(n^{-\frac{1}{2}}).$$

Similar result holds for the bootstrap version $t^* = \sqrt{n}\,\overline{X}^*/\hat{\sigma}^*$ where

$$\overline{X}^* = \frac{1}{n}\sum X_t^*, \quad \hat{\sigma}^{*2} = \frac{1}{n}\sum X_t^{*2}\frac{1+\hat{b}^*}{1-\hat{b}^*}.$$

Now we can apply Theorem 6 with the function

$$H(z_1, z_2, z_3; b, \tau^2) = (1-b)\,\tau^{-1}z_1\left(1 - \frac{1}{2\tau^2}z_3\right)$$

and show that

$$\sup_x |P(t < x) - P^*(t^* < x)| = o\left(n^{-\frac{1}{2}}\right) \quad \text{a.s.}$$

Example 2. Studentized LS-estimator of b

It is known that asymptotic distribution of $n^{\frac{1}{2}}(1-b^2)^{-\frac{1}{2}}(\hat{b}-b)$ is $\mathcal{N}(0,1)$ (see [7], Section 8.8). Bose [6] proved that bootstrap works for $n^{\frac{1}{2}}(1-b^2)^{-\frac{1}{2}}(\hat{b}-b)$ under the assumption $\tau^2 = 1$. Here we consider the studentized statistic

$$t_b = \frac{\hat{b}-b}{\hat{\tau}}\sqrt{\sum X_{t-1}^2} = \frac{\hat{b}-b}{S_n},$$

where

$$\hat{\tau}^2 = \frac{1}{n}\sum(X_t - \hat{b}X_{t-1})^2 \quad \text{and} \quad S_n^2 = \hat{\tau}^2\left(\sum X_{t-1}^2\right)^{-1}.$$

Similarly as in Example 1 we have

$$t_b = \sqrt{n}\frac{\sum X_{t-1}Y_t}{(\sum Y_t^2 \sum X_{t-1}^2 - (\sum X_{t-1}Y_t)^2)^{\frac{1}{2}}}$$

and utilizing the Taylor expansion we get

$$t_b = \sqrt{n}\frac{(1-b^2)^{\frac{1}{2}}}{\tau^2}\overline{Z}_2\left(1 - \frac{b}{\tau^2}\overline{Z}_2 - \frac{1}{\tau^2}\overline{Z}_3\right) + R_n,$$

where

$$P\left(|R_n| > \varepsilon n^{-\frac{1}{2}}\right) = o\left(n^{-\frac{1}{2}}\right).$$

Similar representation can be used for the bootstrap version

$$t_b^* = \frac{\hat{b}^* - \hat{b}}{\hat{\tau}^*}\sqrt{\sum X_{t-1}^{*2}} = \frac{\hat{b}^* - \hat{b}}{S_n^*},$$

where $\hat{\tau}^*$ and S_n^* are the bootstrap counterparts of $\hat{\tau}$ and S_n. Now, if we denote

$$H(z_1, z_2, z_3; b, \tau^2) = \frac{\sqrt{(1-b^2)}}{\tau^2} z_2 \left(1 - \frac{b}{\tau^2} z_2 - \frac{1}{\tau^2} z_3\right)$$

we get that

$$\sup_x |P(t_b < x) - P^*(t_b^* < x)| = o\left(n^{-\frac{1}{2}}\right) \quad \text{a.s.}$$

Bootstrap one-sided confidence interval for b of level $1-\alpha$ is

$$(\hat{b} - h_{1-\alpha}^* S_n, \quad +\infty),$$

where h_γ^* stands for γ-quantile of the conditional distribution of $(\hat{b}^* - \hat{b})/S_n^*$ given by $X_0, X_1, \ldots, X_n$. The coverage error of this interval is

$$P(b < \hat{b} - h_{1-\alpha}^* S_n) = 1 - P^*\left(\frac{\hat{b}^* - \hat{b}}{S_n^*} < h_{1-\alpha}^*\right) + o\left(n^{-\frac{1}{2}}\right) = \alpha + o\left(n^{-\frac{1}{2}}\right),$$

while the coverage error of the confidence interval based on the normal approximation is

$$P(b < \hat{b} - u_{1-\alpha} S_n) = 1 - \left[\Phi(u_{1-\alpha}) + n^{-\frac{1}{2}} Q(u_{1-\alpha})\,\phi(u_{1-\alpha}) + o\left(n^{-\frac{1}{2}}\right)\right] = \alpha + O\left(n^{-\frac{1}{2}}\right).$$

Here u_γ stands for γ-quantile of the standard normal distribution. We can also construct two-sided bootstrap interval for b,

$$\left(\hat{b} - h_{1-\frac{\alpha}{2}}^* S_n, \quad \hat{b} - h_{\frac{\alpha}{2}}^* S_n\right).$$

Some recent simulation studies dealing with AR(p) models (see for example Kreiss and Franke [12]) confirm that the bootstrap approximation is considerably better than the corresponding normal approximation for relatively small sample sizes.

APPENDIX

Suppose that Y_t appeared in (1) have common distribution function F and recall that F_n^* denotes the empirical distribution function based on estimated centered residuals $\hat{Y}_t - \overline{Y}, \hat{Y}_t$ being defined by (11). In the proof of Lemma 4, particularly when proving that integrals J_2 and J_3 are of order $o\left(n^{-1/2}\right)$ almost surely, we need to assume that $F_n^* \Rightarrow F$ almost surely, where $\Rightarrow$ means the weak convergence. Bose [6] uses this fact without any proof or reference; Kreiss and Franke [12] have only proved that $F_n^* \Rightarrow F$ in probability. Here we prove the following lemma.

Lemma 7. Under assumptions A.1, A.3 and A.6

$$F_n^* \Rightarrow F \quad \text{and} \quad \int y^k \,\mathrm{d}F_n^*(y) \to \int y^k \,\mathrm{d}F(y) \quad \text{as} \quad n \to \infty$$

holds almost surely for k = 1,...,8.

Proof. For distributions α and β with finite moments of order $1 \le p < \infty$ let us introduce the metric

$$d_p(\alpha,\beta) = \inf\{E|X-Y|^p\}^{\frac{1}{p}}$$

where the infimum is taken over all pairs of random variables X, Y having marginal distributions α and β, respectively. Now, according to Lemma 8.3 in Bickel and Freedman [5], it is sufficient to prove that $d_8(F_n^*, F) \to 0$ a.s.

Let F_n be the empirical distribution function based on the random sample $Y_1, \ldots, \ldots, Y_n$ from F. Then, according to Lemma 8.1 in [5]

$$d_8(F_n^*, F) \le d_8(F_n^*, F_n) + d_8(F_n, F)$$

and according to Lemma 8.4 in the same paper

$$d_8(F_n, F) \to 0 \quad \text{a.s.}$$

Further, we have

$$d_8^8(F_n^*, F_n) \le \frac{1}{n}\sum_{t=1}^{n}(\hat{Y}_t - \overline{Y} - Y_t)^8 \le c\left(\frac{1}{n}\sum_{t=1}^{n}(\hat{Y}_t - Y_t)^8 + \overline{Y}^8\right),$$

where c is a constant which can change in the sequel. According to (1) and (11) we have

$$\frac{1}{n}\sum_{t=1}^{n}(\hat{Y}_t - Y_t)^8 = \frac{1}{n}\sum_{t=1}^{n}\left(\sum_{j=1}^{p_0}(\hat{b}_j - b_j)\,X_{t-j}\right)^8 \le \left(\sum_{j=1}^{p_0}(\hat{b}_j - b_j)^{8/7}\right)^7 \sum_{j=1}^{p_0}\left(\frac{1}{n}\sum_{t=1}^{n}X_{t-j}^8\right).$$

The last term tends to zero a.s. which follows from the strict consistency of $\hat{b}_j$ and the stacionarity and ergodicity of the sequence $\{X_t\}$ (see [11], Chap. 4).

Similarly,

$$\overline{Y}^8 = \left(\frac{1}{n}\sum_{t=1}^{n}\hat{Y}_t\right)^8 \le c\left(\frac{1}{n}\sum_{t=1}^{n}(\hat{Y}_t - Y_t)\right)^8 + c\left(\frac{1}{n}\sum_{t=1}^{n}Y_t\right)^8 \to 0 \quad \text{a.s.}$$

which follows from the same reasons as above and from the strong law of large numbers. □

Now, we can notice that the moments EZ_t^k are smooth functions of the parameters $b_j, j = 1, \ldots, p_0$ and the moments of Y_t up to order $2k, 1 \le k \le 4$ and that the similar relations hold between their bootstrap counterparts. Thus, from here and from Lemma 7 we can conclude that $E^* Z_t^{*k} \to EZ_t^k$ a.s. for $1 \le k \le 4$ (see also [14] and [13]) for similar considerations and calculations.)

(Received February 17, 1994.)

REFERENCES

[1] T. W. Anderson: The Statistical Analysis of Time Series. Mir, Moscow 1976 (Russian translation).

[2] G. J. Babu and K. Singh: On one term Edgeworth correction by Efron's bootstrap. Sankhyā Ser. A *46* (1984), 219–232.

[3] I. V. Basawa, A. K. Mallik, W. P. McCormick and R. L. Taylor: Bootstrapping explosive autoregressive processes. Ann. Statist. *17* (1989), 1479–1486.

[4] R. N. Bhattacharya and Ranga R. Rao: Normal Approximation and Asymptotic Expansions. Nauka, Moscow 1982 (Russian translation).

[5] P. J. Bickel and D. A. Freedman: Some asymptotic theory for the bootstrap. Ann. Statist. *9* (1981), 1196–1217.

[6] A. Bose: Edgeworth correction by bootstrap in autoregressions. Ann. Statist. *16* (1988), 1709–1722.

[7] P. J. Brockwell and R. A. Davis: Time Series: Theory and Methods. Springer–Verlag, New York 1987.

[8] D. Freedman: On bootstrapping two-stage least-squares estimates in stationary linear models. Ann. Statist. *12* (1984), 827–842.

[9] W. A. Fuller: Introduction to Statistical Time Series. Wiley, New York – Toronto 1976.

[10] F. Götze and C. Hipp: Asymptotic expansions for sums of weakly dependent random vectors. Z. Wahrsch. verw. Gebiete *64* (1983), 211–239.

[11] E. J. Hannan: Multivariate Time Series. Wiley, New York 1970.

[12] J.-P. Kreiss and J. Franke: Bootstrapping stationary autoregressive moving–average models. J. Time Ser. Anal. *13* (1992), 297–317.

[13] E. Paparoditis and B. Streitberg: Order identification statistics in stationary autoregressive moving-average models: vector autocorrelations and the bootstrap. J. Time Ser. Anal. *5* (1991), 415–434.

[14] Z. Prášková: Empirical Edgeworth expansion and bootstrap in AR(1) models. In: Trans. of the Eleventh Prague Conference (S. Kubík and J. Á. Víšek, eds.), Academia, Prague 1992, pp. 281–293.

Doc. RNDr. Zuzana Prášková, CSc., Matematicko-fyzikální fakulta Univerzity Karlovy (Faculty of Mathematics and Physics – Charles University), Sokolovská 83, 186 00 Praha 8. Czech Republic.