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TWO SPECIAL MODELS OF AR(n) PROCESSES WITH TIME-DEPENDENT RANDOM PARAMETERS

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Two special models of AR(n) series with MA(1) random parameters are investigated. Conditions for their second-order stationarity and explicit forms for their covariance functions are derived. In the case of nonzero covariance function spectral density and the best linear prediction are computed.

1. INTRODUCTION

Random coefficient autoregresive series are natural variations of classical models with fixed or non-random time trending parameters. In view of practise they are suitable for modelling the time series data e. g. in hydrology, meteorology or other situations in which the underlying mechanism described by the parameters may be expected to change in a nondeterministic fashion. The theory of these series has been developed almost twenty years ago by Conlisk [4], [5], Anděl [2], Nicholls and Quinn [9] and others. There were many questions of interest – stationarity, stability, estimates of parameters, testing of hypotheses etc. Later there was a perceptible effort of some authors to generalize the basic model with independent random coefficients assuming some type of dependence among them – see e. g. Brandt [3], Koubková [6]–[8], Tjøstheim [10] or Weiss [11]. But soon it turned out that computations in such models are rather complicated. The example of this fact is this paper, the main question of which is that of the second-order stationarity of finite AR(n) processes with MA(1) random parameters.

Let us remind the model solved by Anděl [2]. He introduced the scalar AR(n) series with random parameters as a series $X_1, \dots X_N$ for which

$$X_t = b_1(t)\, X_{t-1} + \dots + b_n(t)\, X_{t-n} + a^{-1} Y_t, \quad t = n+1, \dots, N, \tag{1}$$

where

(i) $X_1, \dots, X_n$ are random variables with zero mean and variance matrix $\boldsymbol{C} = (\sigma_{ij})$;

(ii) $Y_{n+1}, \dots, Y_N$ are independent random variables with zero mean and unit variance independent of $X_1, \dots, X_n$;

(iii) a is a positive number;

(iv) $\boldsymbol{b}(t) = (b_1(t), \dots, b_n(t))'$, $t = n+1, \dots, N$, are vectors of random parameters independent of $X_1, \dots, X_n$ and $Y_{n+1}, \dots, Y_N$;

(v) $\boldsymbol{b}(t)$, $t = n+1, \dots, N$, are independent samples from a distribution with mean $(\beta_1, \dots, \beta_n)'$ and variance matrix $\Delta = (\delta_{ij})$.

Assuming (i) – (v) he derived conditions for second-order stationarity of the series (1). Nicholls and Quinn [9] generalized Anděl's model to the multivariate case.

Now suppose that the random parameters are not independent in time. The simplest kind of their dependence is that of MA(1). So we shall replace the assumption (v) by

$$\boldsymbol{b}(t) = \boldsymbol{K} + \boldsymbol{A}\boldsymbol{Z}(t) + \boldsymbol{B}\boldsymbol{Z}(t-1) \tag{v'}$$

where $\boldsymbol{K}, \boldsymbol{A}, \boldsymbol{B}$ are real matrices and $\boldsymbol{Z}(n), \dots, \boldsymbol{Z}(N)$ are independent random vectors with zero mean and the same variance matrix $\boldsymbol{D}$ (independent of $X_1 \dots, X_n$ and $Y_{n+1}, \dots, Y_N$).

The simplest special cases of such a model, i.e. AR(1) processes with MA(1) parameters, were solved by Koubková [6], [8] and Tjøstheim [10]. It is quite interesting to compare the results of them. The basic model is the same in all cases. The only difference is that in [6], [8] there are no assumptions concerning the distribution of the process $\{Z_t\}$ while in [10] $\{Z_t\}$ is assumed to be Gaussian white noise. Furthermore, in [6] and [8] the second-order stationarity of the finite series $\{X_t\}$ is investigated. It has been proved that stationarity of $\{X_t\}$ depends both on the form of the variance of $\{X_t\}$ and the forms of the third and the fourth moments of random parameters. In [10] the asymptotical second-order stationarity is the question of interest. For this type of stationarity all roots of some specified polynom are required to be inside the unit circle – without any assumption concerning the variance of $\{X_t\}$. It can be easy to see that in [6] and [8] $\{Z_t\}$ cannot be normally distributed and thus the conditions for the second-order stationarity derived in [6] and [8] do not imply the asymptotical one in [10] (this implication evidently holds without the assumption of normality in [10]). The correlation structure is always given by the same second-order difference equation with generally different initial conditions and it is similar to that of the classical AR(2) or ARMA(2, 1) series.

Two special models investigated in this paper are generalizations of the model of Koubková [6] for the case of the higher-order autoregression with $n \geq 2$.

2. MODELS AND RESULTS

Assume that $X_1, \dots, X_N$ is defined by (1) and that (i) – (iv), (v') and (vi) are satisfied where

$$\boldsymbol{A} = \begin{bmatrix} \alpha_1 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_2 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_n \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \beta_1 & 0 & 0 & \cdots & 0 \\ 0 & \beta_2 & 0 & \cdots & 0 \\ 0 & 0 & \beta_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_n \end{bmatrix}, \quad \boldsymbol{K} = \boldsymbol{0} \tag{vi}$$

Then introduce

Model I. $\boldsymbol{Z}(t) = (Z_1(t), \dots, Z_n(t))'$, $t = n, \dots, N$, where $Z_i(t)$ are independent random variables with zero mean and variance $\delta_i^2 > 0$;

Model II. $\boldsymbol{Z}(t) = (Z(t), \dots, Z(t))'$, $t = n, \dots, N$, where $Z(t)$ is a random variable with zero mean and variance $\delta^2 > 0$.

Note that under the assumptions of Model I

$$b_i(t) = \alpha_i Z_i(t) + \beta_i Z_i(t-1), \quad t = n+1, \dots, N, \tag{2}$$

i.e. $\{b_i(t)\}$, $i = 1, \dots, n$, are independent scalar MA(1) processes. In Model II

$$b_i(t) = \alpha_i Z(t) + \beta_i Z(t-1), \quad t = n+1, \dots, N, \tag{3}$$

i.e. $\{b_i(t)\}$, $i = 1, \dots, n$, are scalar MA(1) processes generated by the same system $\{Z(t)\}$ and thus they cannot be independent.

Theorem 1. Under the assumptions of Model I, the series $X_1, \dots, X_N$ defined by (1) is stationary iff the following three conditions are fulfilled:

$$\alpha_1 \beta_1 = 0, \tag{4}$$

$$\sum_{i=1}^{n} (\alpha_i^2 + \beta_i^2)\, \delta_i^2 < 1, \tag{5}$$

$$\sigma_{ij} = \begin{cases} a^{-2}[1 - \sum_{i=1}^{n} (\alpha_i^2 + \beta_i^2)\, \upsilon_i^2]^{-1} = \sigma^2 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j. \end{cases} \tag{6}$$

Its covariance function is given by

$$R(t) = \begin{cases} \sigma^2 & \text{for} \quad t = 0 \\ 0 & \text{for} \quad t \neq 0. \end{cases} \tag{7}$$

Remarks.

(i) Under the assumption (4) process $\{b_1(t)\}$ is white noise.

(ii) Under the assumption (4), one of the matrices $\boldsymbol{A}$, $\boldsymbol{B}$ is necessarily singular.

(iii) The covariance function (7) is the same as that of white noise.

Theorem 2. Under the assumptions of Model II, the series $X_1, \dots, X_N$ defined by (1) is stationary iff

$$\sum_{i=1}^{n} (\alpha_i^2 + \beta_i^2)\, \delta^2 < 1, \tag{8}$$

$$\sigma_{ij} = \begin{cases} a^{-2}[1 - \sum_{i=1}^{n} (\alpha_i^2 + \beta_i^2)\, \delta^2]^{-1} = \sigma^2 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j, \end{cases} \tag{9}$$

and also one of the following three conditions is satisfied:

$$\beta_1 = 0 \text{ and } \alpha_1\beta_2 + \ldots + \alpha_{n-1}\beta_n = 0, \tag{10}$$

$$\beta_1 \neq 0 \text{ and } \alpha_1 = \ldots = \alpha_n = 0, \tag{11}$$

$$\beta_1 \neq 0,\ \alpha_1 = \ldots = \alpha_{n-1} = 0,\ \alpha_n \neq 0,\ EZ^3(t) = 0 \text{ and } EZ^4(t) = \delta^4 \text{ for all } t = n, \ldots, N. \tag{12}$$

Its covariance function is

$$R(t) = \begin{cases} \sigma^2 & \text{for} \quad t = 0 \\ 0 & \text{for} \quad t \neq 0 \end{cases} \tag{13}$$

if (10) or (11) hold or

$$R(t) = \begin{cases} (\alpha_n\beta_1\delta^2)^{|k|}\sigma^2 & \text{for } t = k(n+1) \text{ where } k \text{ is an integer} \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

if (12) holds.

Remarks.

(i) Under the assumption (10) the process $\{b_1(t)\}$ is white noise.

(ii) Under the assumptions (11) the parameters $\{b(t)\}$ are independent in time.

(iii) Under the assumption (12) $\{b_n(t)\}$ can be the only MA(1) process, the others are white noises.

(iv) If both matrices $\boldsymbol{A}$, $\boldsymbol{B}$ are regular then $X_1, \ldots, X_N$ cannot be stationary.

Theorem 3. The covariance function (13) of the series $X_1, \ldots, X_N$ is the same as that of white noise. The covariance function (14) of the series $X_1, \ldots, X_N$ is the same as that of a classical AR$(n+1)$ process $V_1, \ldots, V_N$ generated by

$$V_t = \alpha_n\beta_1\delta^2 V_{t-n-1} + c^{-1}Y_t, \quad t = n+2, \ldots, N, \tag{15}$$

where $V_1, \ldots, V_{n+1}$ are random variables with zero mean and variance matrix $\boldsymbol{E} = \sigma^2\boldsymbol{I}$ which are independent of $Y_{n+2}, \ldots, Y_N$ and

$$c^{-1} = a^{-1}\left[\frac{1-(\alpha_n\beta_1\delta^2)^2}{1-(\beta_1^2+\ldots+\beta_n^2+\alpha_n^2)\,\delta^2}\right]^{\frac{1}{2}}.$$

Corollary 4. Under the assumptions (8), (9), (12) the spectral density of the series $X_1, \ldots, X_N$ exists and it is equal to

$$f(\lambda) = \frac{\sigma^2}{2\pi}\cdot\frac{1-(\alpha_n\beta_1\delta^2)^2}{1-2\alpha_n\beta_1\delta^2\cos((n+1)\,\lambda)+(\alpha_n\beta_1\delta^2)^2}, \tag{16}$$

$\lambda \in \langle -\pi, \pi \rangle$.

Corollary 5. Let (8), (9) and (12) hold. Denote $\boldsymbol{G} = \mathrm{Var}(X_1, \ldots, X_N)$ where $N \geq n+1$. Then elements $h_{s,t}$ of the matrix $\boldsymbol{H} = \boldsymbol{G}^{-1}$ are:

(i) for $N = n+1$:

$$h_{s,t} = \begin{cases} \sigma^{-2} & \text{for} \quad s = t \\ 0 & \text{for} \quad s \neq t; \end{cases}$$

(ii) for $N = n+1+k$ where $k = 1, \ldots, n$:

$$h_{s,t} = \begin{cases} \frac{1}{\sigma^2[1-(\alpha_n\beta_1\delta^2)^2]} & \text{for} \quad s = t = 1, \ldots, k, N-k+1, \ldots, N, \\ \sigma^{-2} & \text{for} \quad s = t = k+1, \ldots, N-k, \end{cases}$$

$$h_{s,t+n+1} = h_{s+n+1,t} = \frac{-\alpha_n\beta_1\delta^2}{\sigma^2[1-(\alpha_n\beta_1\delta^2)^2]} \quad \text{for } s = t = 1, \ldots, N-n-1,$$

$h_{s,t} = 0$ otherwise;

(iii) for $N = 2(n+1)$:

$$h_{s,t} = \frac{1}{\sigma^2[1-(\alpha_n\beta_1\delta^2)^2]} \quad \text{for} \quad s = t,$$

$$h_{s,t+n+1} = h_{s+n+1,t} = \frac{-\alpha_n\beta_1\delta^2}{\sigma^2[1-(\alpha_n\beta_1\delta^2)^2]} \quad \text{for} \quad s = t = 1, \ldots, N-n-1,$$

$h_{s,t} = 0$ otherwise;

(iv) for $N > 2(n+1)$:

$$h_{s,t} = \begin{cases} \frac{1}{\sigma^2[1-(\alpha_n\beta_1\delta^2)^2]} & \text{for} \quad s = t = 1, \ldots, n+1, N-n, \ldots, N, \\ \frac{1+(\alpha_n\beta_1\delta^2)^2}{\sigma^2[1-(\alpha_n\beta_1\delta^2)^2]} & \text{for} \quad s = t = n+2, \ldots, N-n-1, \end{cases}$$

$$h_{s,t+n+1} = h_{s+n+1,t} = \frac{-\alpha_n\beta_1\delta^2}{\sigma^2[1-(\alpha_n\beta_1\delta^2)^2]} \quad \text{for } s = t = 1, \ldots, N-n-1,$$

$h_{s,t} = 0$ otherwise.

Corollary 6. Under the assumptions (8), (9), (12) the best linear prediction of the variable X_{N+t} based on $X_1, \ldots, X_N$ is

$$\widehat{X}_{N+t} = \begin{cases} (\alpha_n\beta_1\delta^2)^k X_N & \text{for } t = k(n+1) \text{ where } k = 0, 1, \ldots, \\ (\alpha_n\beta_1\delta^2)^{k+1} X_{N-j} & \text{for } t = (k+1)(n+1) - j \\ & \quad \text{and } k = 0, 1, \ldots;\ j = 1, \ldots, n. \end{cases} \tag{17}$$

The residual variance is

$$\Delta^2 = E(X_{N+t} - \widehat{X}_{N+t})^2 = \begin{cases} \sigma^2[1-(\alpha_n\beta_1\delta^2)^{2k}] & \text{for } t = k(n+1), \\ \sigma^2[1-(\alpha_n\beta_1\delta^2)^{2(k+1)}] & \text{otherwise.} \end{cases} \tag{18}$$

3. PROOFS OF MODEL I

Proof of Theorem 1. We first note that the process (1) has zero mean because

$$EX_{n+1} = Eb_1(n+1)\,EX_n + \ldots + Eb_n(n+1)\,EX_1 + a^{-1}EY_{n+1} = 0$$

and

$$EX_t = \alpha_1\beta_1\delta_1^2 EX_{t-2} \qquad \text{for } t = n+2, \ldots, N.$$

Its covariance function $R(s,t)$ is then

$$R(s,t) = EX_sX_t.$$

The necessity of condition (6) follows from the requirements

$$\begin{aligned} &\sigma_{12} = \sigma_{23} = \ldots = \sigma_{n-1,n} = R(n+1,n), \\ &\sigma_{13} = \sigma_{24} = \ldots = \sigma_{n-2,n} = R(n+1,n-1), \\ &\vdots \\ &\sigma_{1n} = R(n+1,2), \\ &\sigma_{11} = \ldots = \sigma_{nn} = \sigma^2 = R(n+1,n+1), \end{aligned}$$

where it is easy to compute

$$R(n+1,n) = R(n+1,n-1) = \ldots = R(n+1,2) = 0$$

and

$$R(n+1,n+1) = \sigma^2 \sum_{i=1}^{n} (\alpha_i^2 + \beta_i^2)\delta_i^2 + a^{-2}.$$

(5) is a necessary and sufficient condition for $\sigma^2 > 0$. The necessity of (4) we get from

$$0 = \sigma_{13} = R(n+2,n) = \alpha_1\beta_1\delta_1^2\sigma^2.$$

The sufficiency of (4), (6) will be proved by the straightforward computation of the covariance function $R(s+t,s)$ for $s+t \geq n+2$. It is not difficult to see that

$$R(s+t,s) = Eb_1(s+t)\,b_1(s+t-1)\,X_{s+t-2}X_s = \alpha_1\beta_1\delta_1^2 R(s+t-2,s) = 0 \qquad \text{for } t \geq 2.$$

Computation of the value $R(s+1,s)$ for $s \geq n+1$ and $R(s,s)$ for $s \geq n+2$ is technically more complicated. To compute $R(s+1,s)$ we first obtain that for $s \geq n+1$

$$\begin{aligned} &Eb_1(s+1)\,b_i(s)\,b_j(s)\,X_{s-i}X_{s-j} = 0 \qquad \text{for } i \geq 2 \text{ and } j \geq i, \\ &Eb_1(s+1)\,b_1^2(s)\,X_{s-1}^2 = 0 \end{aligned}$$

and

$$Eb_1(s+1)\,b_1(s)\,b_j(s)\,X_{s-1}X_{s-j} = 0 \qquad \text{for } j \geq 2.$$

Using these formulas and independence among the systems of variables $\{Z_s\}$ and $\{Y_s\}$ we obtain that

$$R(s+1,s)=0.$$

Analogously, for derivation $R(s,s)$ we show

$$\beta_1^2 EZ_1^2(s)\,X_s^2=\beta_1^2\delta_1^2\sigma^2 \qquad \text{for } s\ge n+1$$

and

$$Eb_1^2(s)\,X_{s-1}^2=(\alpha_1^2+\beta_1^2)\,\delta_1^2\sigma^2 \quad \text{for } s\ge n+2.$$

The first expression can be proved by induction using (4) and (6), the second one is a consequence of the former. Then we get

$$R(s,s)=EX_s^2=Eb_1^2(s)\,X_{s-1}^2+\sum_{i=2}^{n}Eb_i^2(s)\,EX_{s-i}^2+a^{-2}=\sum_{i=1}^{n}\sigma^2(\alpha_i^2+\beta_i^2)\,\delta_i^2+a^{-2}=\sigma^2.$$

□

4. PROOFS OF MODEL II

We first give two auxiliary lemmas.

Lemma 7. Let the variables $Z(n),\dots,Z(N)$ have the same finite moments $EZ^3(t)$ and $EZ^4(t)$ for all t. Then

(i) $Eb_{i_k}(j+k)\,b_{i_{k-1}}(j+k-1)\dots b_{i_1}(j+1)=\begin{cases}0 & \text{for } k \text{ odd}\\ \alpha_{i_1}\beta_{i_2}\dots\alpha_{i_{k-1}}\beta_{i_k}\delta^k & \text{for } k \text{ even,}\end{cases}$
where $j\ge n$ and $i_1,\dots,i_k\in\{1,\dots,n\}$ are different numbers;

(ii) $Eb_i(n+3)\,b_j(n+3)\,b_k(n+2)=\beta_i\beta_j\alpha_k EZ^3(t)$;

(iii) $Eb_i^2(n+3)\,b_j^2(n+2)=\alpha_i^2(\alpha_j^2+\beta_j^2)\,\delta^4+\beta_i^2\alpha_j^2EZ^4(t)+\beta_i^2\beta_j^2\delta^4$;

(iv) $Eb_i(n+3)\,b_j(n+3)\,b_k(n+2)\,b_m(n+1)=\beta_k\alpha_m(\alpha_i\alpha_j+\beta_i\beta_j)\,\delta^4$;

(v) $Eb_i^2(n+3)\,b_j(n+2)\,b_k(n+2)\,b_m(n+1)=\alpha_m[(\alpha_i^2+\beta_i^2)\,\beta_j\beta_k+\beta_i^2(\alpha_j\beta_k+\beta_j\alpha_k)]\delta^2EZ^3(t)$;

(vi) $Eb_i(n+3)\,b_j(n+3)\,b_k(n+2)\,b_m^2(n+1)$
$=[\beta_k\alpha_m^2(\alpha_i\alpha_j+\beta_i\beta_j)+\beta_i\beta_j\alpha_k(\alpha_m^2+\beta_m^2)]\delta^2EZ^3(t)$;

(vii) $Eb_i^2(n+3)\,b_j^2(n+2)\,b_k^2(n+1)=2\beta_i^2\alpha_j\beta_j\alpha_k^2[EZ^3(t)]^2$
$+[\alpha_i^2\alpha_j^2(\alpha_k^2+\beta_k^2)+\beta_j^2\beta_k^2(\alpha_i^2+\beta_i^2)]\delta^6+[\beta_j^2\alpha_k^2(\alpha_i^2+\beta_i^2)+\beta_i^2\alpha_j^2(\alpha_k^2+\beta_k^2)]\delta^2EZ^4(t)$.

Proof. The first formula can be proved by induction, the others by strightforward computation. □

Lemma 8. Under the assumptions (8), (9) and (12) we have

(i) $EZ(t)\,X_tX_{t-i} = \alpha_n\delta^2 EX_{t-i}X_{t-n}$ for $t \geq n+1$ and $i = 1,\ldots,t-1$;

(ii) $EZ(t)X_t^2 = 2\alpha_n^2\beta_1\delta^4 EX_{t-n}X_{t-n-1}$ for $t \geq n+2$;

(iii) $Eb_i(t)b_j(t)\,X_{t-i}X_{t-j} = \beta_i\beta_j\delta^2 EX_{t-i}X_{t-j}$ for $2 \leq i < j$;

(iv) $Eb_1(t)\,b_i(t)\,X_{t-1}X_{t-i} = \alpha_n\beta_1^2\beta_i\delta^4 EX_{t-i}X_{t-n-2}$ for $t \geq n+3$ and $i \geq 3$;

(v) $Eb_1(t)\,b_2(t)\,X_{t-1}X_{t-2} = \alpha_n\beta_1\beta_2\delta^4[2\beta_1^2\alpha_n\delta^2 EX_{t-n-2}X_{t-n-3}$
$+\beta_2 EX_{t-3}X_{t-n-2} + \ldots + \beta_n EX_{t-n-1}X_{t-n-2}]$ for $t \geq n+4$;

(vi) $EZ^2(t)\,X_tX_{t-i} = \alpha_n\beta_1 EX_{t-i}X_{t-n-1}$ for $t \geq n+2$ and $i \geq 2$;

(vii) $EZ^2(t)\,X_tX_{t-1} = 2\alpha_n^2\beta_1^2\delta^6 EX_{t-n-1}X_{t-n-2}$
$+\alpha_n\beta_2\delta^4 EX_{t-2}X_{t-n-1} + \ldots + \alpha_n\beta_n\delta^4 EX_{t-n}X_{t-n-1}$ for $t \geq n+3$.

Proof. All the formulas can again be proved by straightforward computing. □

Proof of Theorem 2. In view of the mutual dependence among the processes $\{b_i(t)\}$, some parts of the proof are technically quite difficult. For the reason of readibility an understanding we shall describe only the idea without tedious details which can be found in Koubková [7]. We first prove that all X_t have zero mean. This fact follows by induction from

$$EX_t = \alpha_1\beta_1\delta^2 EX_{t-2} + \ldots + \alpha_n\beta_1\delta^2 EX_{t-n-1} \quad \text{for} \quad t \geq n+2$$
$$EX_t = 0 \quad \text{for} \quad t \leq n+1.$$

The necessity of (8) and (9) can be proved in the same way as (5) and (6). Next we shall prove the necessity of (10) or (11) or (12). Solve the system of equations

$$\begin{aligned} &R(n+2,n+2) = \sigma^2,\\ &R(n+3,n+3) = \sigma^2,\\ &R(n+2,n+1) = \sigma_{12} = 0,\\ &R(n+2,i+1) = R(n+1,i) = 0 \quad \text{for} \quad i = 1,\ldots,n-1 \end{aligned} \tag{19}$$

which is a necessary condition for the stationarity of $\{X_t\}$. Using Lemma 7, we obtain after tedious transformations the equivalent system

$$\beta_1^2(\alpha_1^2 + \ldots + \alpha_n^2)\,(EZ^4(t) - \delta^4) + 2\beta_1(\alpha_1\beta_2 + \ldots + \alpha_{n-1}\beta_n)\,EZ^3(t) = 0, \tag{20}$$

$$\begin{aligned} &2\alpha_1\beta_1^3(\alpha_1^2 + \ldots + \alpha_n^2)[EZ^3(t)]^2\\ &\quad+\beta_1^2(\alpha_1^2 + \ldots + \alpha_n^2)(EZ^4(t) - \delta^4)[1 + (\alpha_1^2 + \beta_1^2)\,\delta^2]\\ &\quad+2\beta_1(\alpha_1\beta_2 + \ldots + \alpha_{n-1}\beta_n)[1 + (\alpha_1^2 + \beta_1^2)\,\delta^2]EZ^3(t)\\ &\quad+2\beta_1^2[\alpha_1(\alpha_1\beta_2 + \beta_1\alpha_2) + \ldots + \alpha_{n-1}(\alpha_1\beta_n + \beta_1\alpha_n)]\delta^2 EZ^3(t)\\ &\quad+2\beta_1(\alpha_1^2 + \ldots + \alpha_n^2)\,(\alpha_1\alpha_2 + \beta_1\beta_2)\,\delta^2 EZ^3(t) \end{aligned} \tag{21}$$

$$+2(\alpha_1\alpha_2+\beta_1\beta_2)\,(\alpha_1\beta_2+\ldots\alpha_{n-1}\beta_n)\,\delta^4$$
$$+2\beta_1[\alpha_1(\alpha_1\alpha_3+\beta_1\beta_3)+\ldots+\alpha_{n-2}(\alpha_1\alpha_n+\beta_1\beta_n)]\delta^4=0,$$
$$\beta_1(\alpha_1^2+\ldots+\alpha_n^2)\,EZ^3(t)+(\alpha_1\beta_2+\ldots\alpha_{n-1}\beta_n)\,\delta^2=0, \quad (22)$$
$$\alpha_{n-i}\beta_1\delta^2\sigma^2=0 \quad \text{for } i=1,\ldots,n-1. \quad (23)$$

Now it is evident that (10) is one solution of (19). If $\beta_1\neq 0$ then (23) implies $\alpha_1=\ldots\alpha_{n-1}=0$ and the system of (21), (23) becomes

$$\beta_1^2\alpha_n^2(EZ^4(t)-\delta^4)=0,$$
$$\beta_1\alpha_n^2EZ^3(t)=0.$$

If $\alpha_n=0$ or $EZ^4(t)=\delta^4$ and $EZ^3(t)=0$ then (21) is evidently satisfied and thus (11) or (12) are solutions of (19). Hence the necessity is proved.

Computing the value $R(s+t,s)$ we can prove the sufficiency.

(i) Sufficiency of (8), (9), (10).
Compute $R(s+t,s)$ for $s+t\geq n+2$. If $t\geq 2$ then

$$R(s+t,s)=Eb_1(s+t)\,EX_{s+t-1}X_s+\ldots+Eb_n(s+t)\,EX_{s+t-n}X_s+a^{-1}EY_{s+t}X_s=0.$$

If $t=1$ or $t=0$ we can use induction. Evidently,

$$R(s,s)=\sigma^2 \quad \text{and} \quad R(s,s-1)=0 \quad \text{for } s\leq n+1.$$

For $s\geq n+2$ we get

$$EX_s^2=\sum_{i=1}^{n}Eb_i^2(s)\,EX_{s-i}^2+2\sum_{j=1}^{n}\sum_{i=1}^{j-1}Eb_i(s)\,b_j(s)\,X_{s-i}X_{s-j}+a^{-2}$$

from which it follows by the induction assumption

$$EX_s^2=\sum_{i=1}^{n}Eb_i^2(s)\,\sigma^2+a^{-2}=\sigma^2.$$

Analogously,

$$EX_{s+1}X_s=Eb_2(s+1)\,X_sX_{s-1}+Eb_n(s+1)\,X_sX_{s-n+1}$$

and using induction we have

$$EX_{s+1}X_s=(\alpha_1\beta_2+\ldots+\alpha_{n-1}\beta_n)\,\delta^2\sigma^2=0.$$

(ii) Sufficiency of (8), (9), (11).
Under the assumption (11) the random parameters are independent in time and we can use the result of Anděl [2].

(iii) Sufficiency of (8), (9), (12).
It is quite easy to prove that

$$EX_{s+t}X_s = \alpha_n\beta_1\delta^2 EX_{s+t-n-1}X_s \tag{24}$$

for $t \geq 2$ and $s+t \geq n+2$, especially

$$R(s+t,s) = \alpha_n\beta_1\delta^2 R(s+t-n-1,s)$$

for $t \geq n+1$ and all s. Furthermore, from (24) it can be proved by induction that

$$R(s+t,s) = 0 \quad \text{for } t = 1,\dots,n \quad \text{and all } s.$$

Now it remains only to compute the variance of X_s. We have

$$EX_s^2 = \sum_{i=1}^{n} Eb_i^2(s)\,X_{s-i}^2 + 2\sum_{j=1}^{n}\sum_{i=1}^{j-1} Eb_i(s)\,b_j(s)\,X_{s-i}X_{s-j} + a^{-2}.$$

Using induction and Lemma 8 (note that all the expressions of Lemma 8 are equal to zero now) we first prove that

$$Eb_1^2(s)\,X_{s-1}^2 = \beta_1^2\delta^2\sigma^2$$

and then

$$EX_s^2 = \beta_1^2\delta^2\sigma^2 + \sum_{i=2}^{n} Eb_i^2(s)\,\sigma^2 + a^{-2} = \sigma^2.$$

The form (14) of the covariance function can be obtained as a solution of the system

$$\begin{aligned}
&R(t) = \alpha_n\beta_1\delta^2 R(t-n-1) && \text{for } t \geq n+1,\\
&R(t) = 0 && \text{for } t = 1,\dots,n,\\
&R(0) = \sigma^2, &&\\
&R(t) = R(-t) && \text{for } t < 0.
\end{aligned}$$

□

Proof of Theorem 3. We get a function of the form (14) as a solution of the Yule–Walker equation system of the process (15). □

Corollaries 4, 5, 6 can be proved by the methods known from the classical autoregression theory (see Anděl [1] e. g.).

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REFERENCES

[1] J. Anděl: Statistische Analyse von Zeitreihen. Akademie Verlag, Berlin 1984.

[2] J. Anděl: Autoregressive series with random parameters. Math. Operationsforsch. Statist. *7* (1976), 735–741.

[3] A. Brandt: The stochastic equation $Y_n + 1 = A_n Y_n + B_n$ with stationary coefficients. Adv. in Appl. Prob. *18* (1986), 211–220.

[4] J. Conlisk: Stability in a random coefficient model. Internat. Econom. Rev. *15* (1974), 529–533.

[5] J. Conlisk: A further note on stability in a random coefficient model. Econom. Rev. *17* (1976), 759–764.

[6] A. Koubková: First-order autoregressive processes with time-dependent random parameters. Kybernetika *18* (1982), 408–414.

[7] A. Koubková: Časové řady s náhodnými parametry. PhD Thesis, Praha 1986 (in Czech).

[8] A. Koubková: Random coefficient AR(1) process. In: Trans. Tenth Prague Conf. on Inform. Theory, Stat. Dec. Functions and Random Proc., Academia, Prague 1988, pp. 51–58.

[9] D. P. Nicholls and B. G. Quinn: Random coefficient autoregressive models: An introduction. Springer–Verlag, New York - Heidelberg - Berlin 1982.

[10] D. Tjøstheim: Some doubly stochastic time series models. J. Time Ser. Anal. *7* (1986), 51–72.

[11] A. A. Weiss: The stability of the AR(1) process with an AR(1) coefficients. J. Time Ser. Anal. *6* (1986), 181–186.

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