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A PENCIL APPROACH TO HIGH GAIN FEEDBACK AND GENERALIZED STATE SPACE SYSTEMS

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In this paper we study limits of generalized state space systems under high gain feedback modulo system equivalence. Different group actions on the space of system pencils are considered and related to the action of pencil equivalence. A recent result on the orbit closure problem for pencils is applied to obtain necessary conditions for a system to be a limit of a given system under high gain feedback. These conditions are shown to be sufficient for arbitrary state space systems. The result is used to investigate a high gain version of Rosenbrock's problem: invariant factor assignment in the limit via high gain state feedback.

1. INTRODUCTION

In the sixties the theory of matrix pencils [4], [14], [24] created by Weierstrass (1867) and Kronecker (1890) was the main mathematical source of inspiration for the emerging structure theory of linear state space systems, see [13], [21]. In the late seventies and eighties, Rosenbrock's description of linear systems by polynomial system matrices [21], [22] provided a bridge for the application of pencil ideas to *generalized state space systems*. Pencil ideas were used to work out an adequate equivalence concept for singular systems [20], [22], [23] and to obtain invariants and canonical forms for controllable systems under the action of the "state" feedback group [5], [15]. In this paper we apply pencil methods to classify limits of generalized state space systems under high gain feedback.

As explained in [7], a natural setting for studying limits of state space systems under high gain feedback is that of generalized state space systems. Neglecting the outputs, these systems are described by mixed linear algebraic and differential equations of the form

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0; \; E, A \in \mathbb{K}^{n\times n}, \; B \in \mathbb{K}^{n\times m}, \tag{1}$$

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where $\mathbb{K}$ is either the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers, and the pair (E, A) defines a regular pencil, i.e.

$$\det(sE - A) \not\equiv 0 \tag{2}$$

If $E = I$, we refer to the system (1) as a *state space system*.

Generalized state feedback $u = Fx$ transforms the triplet (E, A, B) into $(E, A + BF, B)$; but this triplet does not necessarily satisfy the regularity condition (2). Therefore it is convenient to study the effect of feedback transformations on the whole space of triplets $(E, A, B) \in \mathbb{K}^{n\times(2n+m)}$. If (F_ϵ) is a high gain feedback family (i.e. an unbounded family in $\mathbb{K}^{m\times n}$), the resulting closed loop triplets $\Sigma_\epsilon = (E, A + BF_\epsilon, B)$ will in general not converge as $\epsilon \to 0$. Generalizing an idea of Young, Kokotovic and Utkin [28], one may obtain a "limit" of this family of systems by replacing each system Σ_ϵ with an equivalent one in such a way that a limit exists as $\epsilon \to 0$. In [28], high gain feedback is applied to *state space systems* and only *scaling operations* are considered as equivalence transformations. The limits obtained are generalized state space systems. In this paper we apply high gain feedback to *generalized state space systems* and allow more general equivalence operations. To yield adequate equivalence concepts, these operations must preserve both the "finite" and the "infinite frequency behaviour" of a system. The most general operations of this kind are the transformations of *strong equivalence* as defined in [23]. Unfortunately these transformations depend on the individual system and do not form a transformation group. Therefore we also consider the stricter concept of *restricted system equivalence* [22] which is described by a $Gl_n(\mathbb{K}) \times Gl_n(\mathbb{K})$-action. Additionally we allow for linear coordinate transformations in the input space.

Two systems (E_i, A_i, B_i), $i = 1, 2$ are said to be *feedback equivalent* if one can be transformed into the other by state feedback and the above equivalence plus input transformations. In this paper we focus on the high gain feedback classification problem: Given a system $E\dot{x} = Ax + Bu$, which systems $\overline{E}\dot{x} = \overline{A}x + \overline{B}u$ can be obtained as limits of systems that are feedback equivalent to $E\dot{x} = Ax + Bu$?

There are two distinct versions of this problem (see (HGF1), (HGF2) in § 2) corresponding to the two different equivalence concepts considered in this paper. A third and closely related question is that of determining all the limits of *pencils* which are equivalent to a given system pencil $[sE - A \quad B]$ satisfying (2). While this problem has been solved completely [11], only partial results are available to date concerning the two versions of the high gain feedback problem. Our results include a complete solution of the state space high gain feedback classification problem, i.e. a complete description, up to feedback equivalence, of state space systems which can be approximated by applying high gain feedback to a given state space system.

This paper is partly a survey and partly a research paper. It develops a pencil approach to high gain feedback and gives a unified presentation with full proofs of recent results, which have been reported in a scattered way and without proofs in several conference proceedings [8], [9], [10]. On the other hand it also contains a number of new results. The paper is organized as follows. In the next section we analyze the action of various groups of pencil transformations on system pencils of generalized state space systems. We show that every operation of pencil equivalence which preserves the structure of a system pencil can be decomposed into the product of four transformations, vz. 1) an operation of state feedback, 2) an operation of strong equivalence, 3) an input transformation, and 4) another operation of state feedback. In Section 3 we derive from Kronecker's result a canonical form for generalized state space systems with respect to these transformations. For later use we also discuss a canonical form due to Loiseau, Öscaldiran, Malabre and Karcanias [17] with respect to the operations of state feedback, restricted system equivalence, and input transformation. Section 4 contains the main results of this paper and deals with orbit closure problems for pencils and, in particular, for system pencils. Necessary and sufficient conditions are specified which the Kronecker invariants of a pencil have to satisfy in order that it be representable as the limit of pencils belonging to a fixed equivalence class ("orbit") of pencils. However, these conditions are not sufficient if the limit is required to be a limit of system pencils. In Section 4 we show that these necessary conditions are sufficient in the case of state space systems. In Section 5 we present some system-theoretic applications. In particular, we discuss a high gain version of Rosenbrock's problem: Given a generalized state space system (E, A, B), which invariant factors can be obtained in the limit by applying high gain feedback to the system?

For the convenience of the reader, we have included a glossary of terms at the end of the paper.

2. SYSTEM TRANSFORMATIONS AND PENCIL EQUIVALENCE

The concept of high gain feedback limit depends on the underlying concept of system equivalence. In this paper we consider the two concepts of *strong equivalence* and *restricted system equivalence.* Two systems $E_i\dot{x} = A_ix + B_iu$ of the form (1) or the associated pencils $[sE_i - A_i \quad B_i]$, $i = 1, 2$ are said to be *strongly equivalent* [23] if one can be transformed into the other by a finite sequence of the following two kinds of operations:

i) *Operations of strong equivalence:*

$$[sE_2 - A_2 \quad B_2] = L[sE_1 - A_1 \quad B_1]\begin{bmatrix} R & X \\ 0 & I_m \end{bmatrix}, \text{ i.e.}$$

$$E_2 = LE_1R, \quad A_2 = LA_1R, \quad B_2 = L(B_1 - A_1X), \tag{3}$$

where

$$L,\ R \in Gl_n(\mathbb{K}), \quad X \in \mathbb{K}^{n\times m} \quad \text{and} \quad E_1X = 0. \tag{4}$$

ii) *Trivial augmentation/deflation:*

$$E_2 = \begin{bmatrix} E_1 & 0 \\ 0 & 0_{k\times k} \end{bmatrix}, \ A_2 = \begin{bmatrix} A_1 & 0 \\ 0 & I_k \end{bmatrix}, \ B_2 = \begin{bmatrix} B_1 \\ 0_{k\times m} \end{bmatrix} \quad \text{for some } k \in \mathbb{N}. \tag{5}$$

Two systems $E_i\dot{x} = A_i x + B_i u$ or the associated pencils $[sE_i - A_i \quad B_i]$, $i = 1, 2$ are *restricted system equivalent* if there exist $L,\ R \in Gl_n(\mathbb{K})$ such that

$$[sE_2 - A_2 \quad B_2] = L\,[sE_1 - A_1 \quad B_1]\begin{bmatrix} R & 0 \\ 0 & I_m \end{bmatrix}, \quad \text{i.e.}$$

$$E_2 = LE_1R,\ A_2 = LA_1R,\ B_2 = LB_1. \tag{6}$$

If $\det E \neq 0$ — in particular if the system is a state space system — the operations of strong equivalence coincide with the transformations of restricted system equivalence.

Note that systems of different dimensions may be strongly equivalent, whereas restricted system equivalence preserves the dimension of a system. It follows from [20, Thm. 7] that two systems of the same dimension are strongly equivalent if and only if they can be transformed into each other by operations of the form (3).

The definition of strong equivalence is of an algorithmic nature. A closed form expression which yields a common framework for both kinds of operations (3), (5) has been given in [20] ("complete system equivalence"). While the operations of strong equivalence (3) will be primarily used in this paper, the trivial deflations and augmentations (5) allow us to consider lower dimensional state space systems as feedback limits of higher dimensional state space systems.

In [23] arguments are given which show why strong equivalence is an adequate equivalence concept for singular systems. In fact, strong equivalence can be characterized by an isomorphy between the solution spaces of the corresponding system equations [23, Remarks 3.3, 3.4]. Moreover, two *state space systems* are strongly equivalent if and only if they are similar. Finally, an extended version of Kalman's realization theorem holds for this concept of equivalence: Controllable and observable generalized state space systems are strongly equivalent if and only if they have the same transfer matrix [23].

In (3), left multiplication by L corresponds to row operations on the system of equations and right multiplication by R corresponds to a change of state space coordinates, but what system-theoretic sense does the modification of B have? This question is best answered in Willems' behavioural framework [25]. The behaviour of a system $E\dot{x} = Ax + Bu$ is the set of pairs of C^∞ functions $x : \mathbb{R} \to \mathbb{K}^n$, $u : \mathbb{R} \to \mathbb{K}^m$ such that $E\dot{x}(t) = Ax(t) + Bu(t)$ for all $t \in \mathbb{R}$. Consider the transformed system

$$E\dot{x} = Ax + (B - AX)u$$

or, equivalently,

$$E(\dot{x} - X\dot{u}) = A(x - Xu) + Bu,$$

where $EX = 0$. The behaviour of this system is isomorphic (as a $\mathbb{K}[\frac{d}{dt}]$-module) to the behaviour of the original system $E\dot{x} = Ax + Bu$ via the isomorphism $(x, u) \to (x - Xu, u)$ which preserves the control but modifies the state by adding to it a linear combination of control variables.

The operations of restricted system equivalence (6) form a transformation group. On the contrary, the more general operations of strong equivalence (3) are defined relative to a given system (1) (because of the constraint $EX = 0$); they do not define transformations on the set of all systems of the form (1). In order to describe strong equivalence in terms of the action of a transformation group, we suspend the condition $EX = 0$ in the first part of the analysis. This is done by viewing systems and system transformations within the larger context of pencils and pencil equivalence.

For arbitrary integers $p, q \geq 1$, a $p \times q$ matrix pencil is, by definition, a pair (M, N) of matrices $M, N \in \mathbb{K}^{p \times q}$. These pairs are denoted by $sM + N$. Two $p \times q$ pencils are said to be *equivalent* [4] if they can be transformed into each other by the following action of the group $Gl_p(\mathbb{K}) \times Gl_q(\mathbb{K})$ on the space of $p \times q$ pencils

$$(P, Q) \cdot (sM + N) = (sPMQ^{-1} + PNQ^{-1}), \quad P \in Gl_p(\mathbb{K}),\ Q \in Gl_q(\mathbb{K}). \tag{7}$$

Every system (1) defines an $n \times (m+n)$ pencil $[sE - A \quad B]$; we refer to pencils of this form as *system pencils*. System pencils which satisfy the regularity condition (2) are called *regular*. By $\mathcal{P}_{n,m}(\mathbb{K}) = \mathbb{K}^{2\,n \times (n+m)}$ we denote the vector space of $n \times (m+n)$ pencils over $\mathbb{K}$ endowed with the usual product topology. The closed subset of all system pencils is denoted by $\mathcal{S}_{n,m}(\mathbb{K})$. The subset $\mathcal{S}_{n,m}^{reg}(\mathbb{K})$ of regular system pencils is dense in $\mathcal{S}_{n,m}(\mathbb{K})$.

Applying any pencil transformation (7) given by

$$P \in Gl_n(\mathbb{K}), \quad Q^{-1} = \begin{bmatrix} R & X \\ F & W \end{bmatrix} \in Gl_{n+m}(\mathbb{K}) \tag{8}$$

to a system pencil $[sE - A \quad B] \in \mathcal{S}_{n,m}(\mathbb{K})$ yields an $n \times (n+m)$ pencil

$$(P, Q) \cdot [sE - A \quad B] = [sPER - P(AR - BF) \quad P(sEX + BW - AX)] \tag{9}$$

which is, in general, not a system pencil. Thus $\mathcal{S}_{n,m}(\mathbb{K})$ is not invariant under arbitrary pencil operations. In fact we have, for any $[sE - A \quad B] \in \mathcal{S}_{n,m}(\mathbb{K})$,

$$P[sE - A \quad B] \begin{bmatrix} R & X \\ F & W \end{bmatrix} \in \mathcal{S}_{n,m}(\mathbb{K}) \Longleftrightarrow EX = 0. \tag{10}$$

If $(P, Q) \cdot [sE - A \quad B] = [s\tilde{E} - \tilde{A} \quad \tilde{B}] \in \mathcal{S}_{n,m}(\mathbb{K})$ we obtain from (9) and (10) that

$$\operatorname{rk} E = \operatorname{rk} PE[R \quad X] = \operatorname{rk}[PER \quad 0] = \operatorname{rk} \tilde{E}, \tag{11}$$

i.e. the rank of E does not change under pencil transformations which preserve the system structure of the pencil. Given any pencil $sM + N \in \mathcal{P}_{n,m}(\mathbb{K})$ there exists $Q \in Gl_{n+m}(\mathbb{K})$ such that MQ^{-1} is of the form $[E \quad 0]$, $E \in \mathbb{K}^{n \times n}$. Thus every pencil orbit in $\mathcal{P}_{n,m}(\mathbb{K})$

$$\mathcal{O}(sM + N) = \{P(sM + N)Q^{-1};\ P \in Gl_n(\mathbb{K}), Q \in Gl_{n+m}(\mathbb{K})\} \tag{12}$$

contains a system pencil $[sE - A \quad B]$. However, it does not necessarily contain a regular system pencil. In the following lemma we see that pencils whose orbits contain regular system pencils are those which are *regularizable* in the sense of Özcaldiran and Lewis [19]:

Lemma 2.1. A system orbit $\mathcal{O}([sE-A \quad B])$ contains a regular system pencil if and only if it is regularizable, i.e. there exists $F \in \mathbb{K}^{m\times n}$ such that $\det(sE-(A-BF)) \not\equiv 0$.

Proof. The 'if' part is immediate since every feedback transformation is a transformation of pencil equivalence. Now assume that there is a pair (P,Q) as in (8) such that

$$\det(sPER - P(AR-BF)) \not\equiv 0,$$

see (9). Then $\det(sER-(AR-BF)) \not\equiv 0$ and, by continuity of the determinant and density of nonsingular matrices in the space $\mathbb{K}^{n\times n}$ there exists a nonsingular matrix $\tilde{R}$ close to R such that

$$\det(sE\tilde{R}-(A\tilde{R}-BF)) = \det(sE-(A-BF\tilde{R}^{-1}))\det\tilde{R} \not\equiv 0.$$

Hence (E,A,B) is regularizable by state feedback. □

Regularizability has been characterized in terms of $(A,E,\mathrm{Im}\,B)$-invariant subspaces in [19]. In Section 3 we will express this property in terms of pencil invariants and thus obtain a characterization of those pencil orbits $\mathcal{O}(sM+N)$ which contain regular system pencils.

Remark 2.2. If $\mathcal{O}([sE-A \quad B])$ contains a regular system pencil it follows from similar arguments as in the previous proof that $\mathcal{O}([sE-A \quad B]) \cap \mathcal{S}^{\mathrm{reg}}_{n,m}(\mathbb{K})$ is dense in $\mathcal{O}([sE-A \quad B]) \cap \mathcal{S}_{n,m}(\mathbb{K})$:

$$\mathrm{cl}\,(\mathcal{O}([sE-A \quad B]) \cap \mathcal{S}^{\mathrm{reg}}_{n,m}(\mathbb{K})) = \mathrm{cl}\,(\mathcal{O}([sE-A \quad B]) \cap \mathcal{S}_{n,m}(\mathbb{K})). \tag{13}$$

The group $Gl_n(\mathbb{K}) \times Gl_{n+m}(\mathbb{K})$ of pencil operations (P,Q) on $\mathcal{P}_{n,m}(\mathbb{K})$ contains the following important subgroups of system transformations of which the first three leave $\mathcal{S}^{\mathrm{reg}}_{n,m}(\mathbb{K})$ invariant whereas the remaining ones ((17), (18)) preserve the system structure but may destroy the regularity condition (2):

1. Similarity transformations:
$$\left(P, \begin{bmatrix} P & 0 \\ 0 & I_m \end{bmatrix}\right), \quad P \in Gl_n(\mathbb{K}). \tag{14}$$

2. Input transformations:
$$\left(I_n, \begin{bmatrix} I_n & 0 \\ 0 & W \end{bmatrix}\right), \quad W \in Gl_m(\mathbb{K}). \tag{15}$$

3. Transformations of restricted system equivalence:
$$\left(P, \begin{bmatrix} R & 0 \\ 0 & I_m \end{bmatrix}\right), \quad P,R \in Gl_n(\mathbb{K}). \tag{16}$$

4. State feedback transformations:

$$\left(I_n, \begin{bmatrix} I_n & 0 \\ F & I_m \end{bmatrix}\right), \quad F \in \mathbb{K}^{m\times n}. \tag{17}$$

5. Generalized feedback transformations:

$$\left(P, \begin{bmatrix} R & 0 \\ F & W \end{bmatrix}\right), \quad P \in Gl_n(\mathbb{K}), \begin{bmatrix} R & 0 \\ F & W \end{bmatrix} \in Gl_{n+m}(\mathbb{K}). \tag{18}$$

The corresponding transformation groups are, respectively, $Gl_n(\mathbb{K})$, $Gl_m(\mathbb{K})$, $Gl_n(\mathbb{K}) \times Gl_n(\mathbb{K})$, $\mathbb{K}^{m\times n}$ and the *generalized feedback group*

$$\mathcal{G}_{n,m}(\mathbb{K}) = Gl_n(\mathbb{K}) \times \mathcal{F}_{n,m}(\mathbb{K}), \tag{19}$$

where $\mathcal{F}_{n,m}(\mathbb{K})$ is the *full feedback group*

$$\mathcal{F}_{n,m}(\mathbb{K}) = \left\{ \begin{bmatrix} R & 0 \\ F & W \end{bmatrix} : R \in Gl_n(\mathbb{K}),\ W \in Gl_m(\mathbb{K}), F \in \mathbb{K}^{m\times n} \right\}. \tag{20}$$

If the generalized feedback transformations are restricted to the set of state space systems, we have the full feedback group action on the set of state space systems:

$$\begin{bmatrix} R & 0 \\ F & W \end{bmatrix} \cdot [sI - A \quad B] = R[sI - A \quad B] \begin{bmatrix} R & 0 \\ F & W \end{bmatrix}^{-1}. \tag{21}$$

Pencil equivalence (7) is given by the action of a reductive group $(Gl_p(\mathbb{K}) \times Gl_q(\mathbb{K}))$ on a vector space whereas the generalized feedback transformations (18) and the full feedback transformations (21) are actions of non-reductive groups ($\mathcal{G}_{n,m}(\mathbb{K})$ and $\mathcal{F}_{n,m}(\mathbb{K})$ respectively) on vector spaces.

Definition 2.3. Two systems (E_i, A_i, B_i), $i = 1, 2$ are said to be *feedback equivalent in the first (resp. second) sense* if $[sE_2 - A_2 \quad B_2]$ is obtained from the pencil $[sE_1 - A_1 \quad B_1]$ via some combination of operations of state feedback (17), input change of coordinates (15), and strong equivalence (3) (resp. restricted system equivalence (6)).

Clearly, systems which are equivalent in the second sense are also equivalent in the first sense.

Using the transformations introduced above, we now turn to the main topic of this paper and consider limits of a given system under high gain feedback (in the sense indicated in the introduction). Depending on the equivalence concept used, two different notions are obtained, one (HFG1) is based on strong equivalence and the other (HFG2) is based on restricted system equivalence.

HGF1: The system $\overline{E}\dot{x} = \overline{A}x + \overline{B}u$ is said to be a *high gain feedback limit in the first sense* of the system $E\dot{x} = Ax + Bu$ (for short, HGF1-limit of (E, A, B)) if the pencil $[s\overline{E} - \overline{A} \quad \overline{B}]$ is a limit of a sequence of regular system pencils $[sE_k - A_k \quad B_k]$, where the system pencils $[sE_k - A_k \quad B_k]$ are feedback equivalent in the first sense to $[sE - A \quad B]$.

HGF2: The system $\overline{E}\dot{x} = \overline{A}x + \overline{B}u$ is said to be a *high gain feedback limit in the second sense* of the system $E\dot{x} = Ax + Bu$ (for short, HGF2-limit of (E, A, B)) if the pencil $[s\overline{E} - \overline{A} \quad \overline{B}]$ is a limit of a sequence of regular system pencils $[sE_k - A_k \quad B_k]$, where the system pencils $[sE_k - A_k \quad B_k]$ are feedback equivalent in the second sense to $[sE - A \quad B]$.

In either case, we refer to a high gain feedback limit as *nontrivial* if it is not feedback equivalent (in the first, resp. second sense) to the original system.

Note that the set of transformations applied in HGF2 is a subset of the set of transformations applied in HGF1; comparing the definitions of strong equivalence and restricted system equivalence, we see that the two concepts coincide when the matrix E is nonsingular.

Both concepts of limits under high gain feedback are defined via special transformations of pencil equivalence (7). We will now show that *arbitrary* pencil transformations may be used in the definition of HGF1-limits, i.e. $\overline{E}\dot{x} = \overline{A}x + \overline{B}u$ is a HGF1-limit of the system $E\dot{x} = Ax + Bu$ if it is the limit of a sequence of regular system pencils which are pencil equivalent to $E\dot{x} = Ax + Bu$. It is not a priori clear that a pencil equivalence operation (9) which transforms a system pencil into another one can be decomposed into a product of system transformations of the form (15), (17), (3). (Note, in particular, that in (8) the matrices R and W need not be nonsingular.) The following lemma resolves this issue.

Lemma 2.4. Every transformation $T \in Gl_{n+m}(\mathbb{K})$ can be written as a product of the form

$$T = \begin{bmatrix} I_n & 0 \\ F_1 & I_m \end{bmatrix} \begin{bmatrix} R & X \\ 0 & W \end{bmatrix} \begin{bmatrix} I_n & 0 \\ F_2 & I_m \end{bmatrix} = \begin{bmatrix} R + XF_2 & X \\ F_1R + F_1XF_2 + WF_2 & F_1X + W \end{bmatrix}, \tag{22}$$

where $F_1, F_2 \in \mathbb{K}^{m\times n}$, $X \in \mathbb{K}^{n\times m}$, $R \in Gl_n(\mathbb{K})$ and $W \in Gl_m(\mathbb{K})$.

Proof. Let $T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in Gl_{n+m}(\mathbb{K})$. We need to find F_1, R, W, X and F_2 such that R, W are nonsingular and

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} R + XF_2 & X \\ F_1R + F_1XF_2 + WF_2 & F_1X + W \end{bmatrix}. \tag{23}$$

Set $X = T_{12}$ and choose F_2 such that $R := T_{11} - T_{12}F_2$ is nonsingular. Since $\mathrm{rk}[T_{11} \quad T_{12}] = n$, this is possible. Now define $F_1 = (T_{21} - T_{22}F_2)(T_{11} - T_{12}F_2)^{-1}$ and $W = T_{22} - F_1T_{12}$. Then

$$F_1R + F_1XF_2 + WF_2 = F_1T_{11} + T_{22}F_2 - F_1T_{12}F_2 = T_{21}$$

so that (23) is satisfied. It only remains to show that $\det W \neq 0$. But this follows from

$$\begin{aligned} \det W &= \det(T_{22} - (T_{21} - T_{22}F_2)(T_{11} - T_{12}F_2)^{-1}T_{12}) \\ &= \det \begin{bmatrix} T_{11} - T_{12}F_2 & T_{12} \\ T_{21} - T_{22}F_2 & T_{22} \end{bmatrix} (\det(T_{11} - T_{12}F_2))^{-1} \end{aligned}$$

$$= \det \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} (\det(T_{11} - T_{12}F_2))^{-1} \neq 0.$$

□

Remark 2.5. i) The factorization of $T \in Gl_{n+m}(\mathbb{K})$ in (22) is not unique. But once $F_2 \in \mathbb{K}^{m\times n}$ is chosen such that $\det(T_{11} - T_{12}F_2) \neq 0$ the remaining submatrices F_1, R, W, X are uniquely determined.

ii) By transposition of (22) we see that every transformation $T \in Gl_{n+m}(\mathbb{K})$ can be written as a product of the form

$$T = \begin{bmatrix} I_n & X_1 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} R & 0 \\ F & W \end{bmatrix} \begin{bmatrix} I_n & X_2 \\ 0 & I_m \end{bmatrix}, \tag{24}$$

where again $R \in Gl_n(\mathbb{K})$ and $W \in Gl_m(\mathbb{K})$.

As a consequence of the above lemma we obtain

Proposition 2.6. Any two system pencils $[sE_i - A_i \quad B_i]$ which are pencil equivalent can be transformed into each other by applying successively the following operations: i) State feedback (17), ii) strong equivalence (3), iii) input transformation (15), iv) state feedback (17). In particular, they are feedback equivalent in the first sense.

Proof. If

$$(P, Q) \cdot [sE_1 - A_1 \quad B_1] = [sE_2 - A_2 \quad B_2], \tag{25}$$

$$P \in Gl_n(\mathbb{K}), \quad Q^{-1} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in Gl_{n+m}(\mathbb{K})$$

then $E_1T_{12} = 0$. Decomposing $T = Q^{-1}$ according to Lemma 2.4 we obtain

$$T = \begin{bmatrix} I_n & 0 \\ F_1 & I_m \end{bmatrix} \begin{bmatrix} R & XW^{-1} \\ 0 & I_m \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & W \end{bmatrix} \begin{bmatrix} I_n & 0 \\ F_2 & I_m \end{bmatrix}, \tag{26}$$

where $R \in Gl_n(\mathbb{K})$, $W \in Gl_m(\mathbb{K})$ and $E_1XW^{-1} = 0$. This concludes the proof, since the pair $\left(P, \begin{bmatrix} R & XW^{-1} \\ 0 & I_m \end{bmatrix}\right)$ defines an operation of strong equivalence for $[sE_1 - (A_1 - B_1F_1) \quad B_1]$. □

For state space systems, pencil equivalence simply reduces to equivalence modulo the conventional action of the full feedback group $\mathcal{F}_{n,m}(\mathbb{K})$ (21). In contrast with general pencil transformations (see Proposition 2.6), the operations of the full feedback group on state space systems can be decomposed into successive application of only *three* elementary operations (14), (17), (15). In particular, state feedback does not have to be used twice.

Corollary 2.7. Two state space systems $[sI - A_i \quad B_i]$, $i = 1,2$ are pencil equivalent if and only if they lie in the same orbit of the full feedback group $\mathcal{F}_{n,m}(\mathbb{K})$. More precisely, each one can be transformed into the other by successive application of the following three operations: i) Similarity transformation (14), ii) state feedback (17), iii) input transformation (15).

Proof. If (25) holds with $E_1 = E_2 = I_n$, then necessarily $T_{12} = 0$ and Q^{-1} has the form

$$Q^{-1} = \begin{bmatrix} P^{-1} & 0 \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} P^{-1} & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} I_n & 0 \\ T_{21} & I_m \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & T_{22} \end{bmatrix}.$$

Thus

$$[sI - A_2 \ B_2] = P[sI - A_1 \ B_1] \begin{bmatrix} P^{-1} & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} I_n & 0 \\ T_{21} & I_m \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & T_{22} \end{bmatrix}$$

and this concludes the proof. □

For our purposes, the most important consequence of Proposition 2.6 is that the first high gain feedback concept HGF1 can now be expressed in terms of pencil transformations:

HGF1′: The system $\overline{E}\dot{x} = \overline{A}x + \overline{B}u$ is a high gain feedback limit of the system $E\dot{x} = Ax + Bu$ if and only if

$$[s\overline{E} - \overline{A} \ \ \overline{B}] \in \mathrm{cl}\left(\mathcal{O}([sE - A \ \ B]) \cap \mathcal{S}^{\mathrm{reg}}_{n,m}(\mathbb{K})\right) = \mathrm{cl}\left(\mathcal{O}([sE - A \ \ B]) \cap \mathcal{S}_{n,m}(\mathbb{K})\right). \tag{27}$$

(The set equality follows from Remark 2.2.) This characterization establishes a connection between high gain feedback limits and orbit closures of system pencils under pencil equivalence. Note, however, that a characterization of the orbit closure $\mathrm{cl}\,(\mathcal{O}([sE - A \quad B]))$ does not determine the set of all HGF1-limits of $E\dot{x} = Ax + Bu$ since, in general, equality does not hold in the following inclusion

$$\mathrm{cl}\,(\mathcal{O}([sE - A \quad B]) \cap \mathcal{S}_{n,m}(\mathbb{K})) \subset \mathrm{cl}\,(\mathcal{O}([sE - A \quad B])) \cap \mathcal{S}_{n,m}(\mathbb{K}).$$

The following characterization of HGF2-limits follows immediately from the definition.

HGF2′: The system $\overline{E}\dot{x} = \overline{A}x + \overline{B}u$ is a high gain feedback limit of the system $E\dot{x} = Ax + Bu$ if and only if

$$[s\overline{E} - \overline{A} \quad \overline{B}] \in \mathrm{cl}\,(\mathcal{G}_{n,m}(\mathbb{K}) \cdot [sE - A \quad B]). \tag{28}$$

If we restrict our attention to state space systems ($E = \overline{E} = I$), the two concepts HGF1 and HGF2 coincide and we obtain the following characterization: A system $\dot{x} = \overline{A}x + \overline{B}u$ is a high gain feedback limit of the system $\dot{x} = Ax + Bu$ if and only if

$$[sI - \overline{A} \quad \overline{B}] \in \mathrm{cl}\,(\mathcal{F}_{n,m}(\mathbb{K}) \cdot [sI - A \quad B]). \tag{29}$$

3. CANONICAL FORMS OF MATRIX PENCILS AND SYSTEMS

In this section we describe Kronecker's canonical form for pencil equivalence [14] and a canonical form for the action of the generalized feedback group established by Loiseau, Özcaldiran, Malabre, and Karcanias [17]. A simple modification of Kronecker's canonical form provides a canonical form for arbitrary regularizable systems with respect to transformations of strong equivalence, state feedback, and change of coordinates in the input space.

Kronecker's canonical form is described in terms of the following invariants: column indices, row indices, and elementary divisors, see [4]. The column indices $(c_1, \dots, c_k)$ of a pencil $sM + N \in \mathcal{P}_{n,m}(\mathbb{K})$ are defined as follows. Choose a minimal degree solution $X_1(s) \in (\mathbb{K}[s])^{n+m}$ to the equation $(sM + N)X(s) = 0$. For each $i \geq 2$, choose a minimal degree solution $X_i(s) \in (\mathbb{K}[s])^{n+m}$ to the equation $(sM + N)X(s) = 0$ which is not contained in the $\mathbb{K}[s]$-span of $\{X_1(s), \dots, X_{i-1}(s)\}$. This process ends after a finite number k of steps ($k \leq m$). We define $c_i = \deg X_{k-i+1}(s), i = 1, .., k$. Obviously, $c_1 \geq .. \geq c_k$. The row indices $(r_1, \dots, r_\ell)$ of a pencil $sM + N$ are defined similarly as minimal degrees of solutions of $X(s)(sM + N) = 0$.

We also associate to each pencil $sM+N$ of rank ρ a set of polynomials $\{D_j(sM + tN)\}_{j=1}^{\rho}$ in $\mathbb{K}[s,t]$. Let $D_j(sM + tN)$ be the (normalized) greatest common divisor of the set of all minors of order j of the associated homogeneous pencil $sM + tN$, $j \leq \rho$. The *invariant factors* of the pencil $sM+N$ are, by definition, the homogeneous polynomials

$$\left\{ f_j(s,t) = \frac{D_{\rho-j+1}(sM + tN)}{D_{\rho-j}(sM + tN)} \right\}_{j=1}^{\rho}, \qquad (D_0(sM + tN) = 1). \tag{30}$$

Splitting the invariant factors into powers of homogeneous polynomials irreducible over $\mathbb{K}$ we obtain the *elementary divisors* of the pencil $sM + N$. Elementary divisors of the form t^j are called "infinite" and the other elementary divisors are called "finite".

Theorem 3.1. (Kronecker 1890 [14]) Two $n \times (n + m)$ pencils are equivalent if and only if they have the same row indices, column indices, and elementary divisors.

Because we shall see (Proposition 3.2) that regular system pencils have *no row indices*, we restrict our attention to $n \times (n + m)$ matrix pencils which have this property. The canonical form of such pencils has the following structure (see [4]):

$$\begin{bmatrix} 0 & \cdots & 0 & S_{c_1} & & & & 0 \\ \vdots & & \vdots & & S_{c_2} & & & \\ \vdots & & \vdots & & & \ddots & & \\ \vdots & & \vdots & & & & S_{c_h} & \\ 0 & \cdots & 0 & & & & & R \end{bmatrix}, \tag{31}$$

where the first $k-h$ columns are zero, $h = \max\{i : c_i > 0\}$, each block S_{c_i}, $i = 1, \ldots, h$ is a $c_i \times (c_i+1)$ rectangular block of the form

$$S_{c_i} = \begin{bmatrix} s & 1 & & \\ & \ddots & \ddots & \\ & & s & 1 \end{bmatrix}_{c_i \times (c_i+1)} \tag{32}$$

and the $r \times r$ regular block R ($r = n - \Sigma_{i=1}^{k} c_i$) is written in "Jordan form":

$$R = \begin{bmatrix} sI - J & 0 \\ 0 & sN - I \end{bmatrix}. \tag{33}$$

J is a matrix in (real or complex) Jordan form and N is a nilpotent matrix in Jordan form. Each nonzero column index c_i defines the number of rows in a rectangular block $S_{c_i}$and each zero column index corresponds to a zero column. Hence the number of column indices k is equal to the number of zero columns plus the number of rectangular blocks. It follows that $k = m$. The matrix J is determined by the finite elementary divisors and the matrix N is determined by the infinite elementary divisors.

In the following propositions we apply Kronecker's results to regular system pencils of generalized state space systems. Since our interest is in pencils which represent systems, we modify the pencil canonical form (31) as follows to obtain a "system canonical form". Move the last column of each block S_{c_i} to the right side of the matrix and move the zero columns to the right side of the matrix. This gives us the following canonical form:

$$\left[\begin{array}{cc|c} sI - A_1 & 0 & B_1 \\ 0 & R & 0 \end{array}\right], \tag{34}$$

where the pair (A_1, B_1) defines a state space system in Brunovsky canonical form (see [1]) and R is the regular block given by (33).

The next proposition, which also appears in [9], [17], characterizes regularizability (see Lemma 2.1 for the definition of regularizability).

Proposition 3.2. The pencil $[sE - A \quad B]$ is regularizable if and only if it has no row indices. In particular, every regular system pencil has no row indices.

Proof. If $[sE - A \quad B]$ is regularizable, then there is a pencil transformation $Q \in GL_{n+m}(\mathbb{K})$ such that the pencil $[sE - A \quad B]\, Q = [sE_1 - A_1 \quad B_1]$ is a regular system pencil. Suppose $[sE - A \quad B]$ has a row index. By Theorem 3.1, the pencil $[sE_1 - A_1 \quad B_1]$ has a row index, i.e. there is a polynomial vector $f(s) \in (\mathbb{K}[s])^n$, $f(s) \neq 0$ such that $f(s)\,[sE_1 - A_1 \quad B_1] = 0$. Then $f(s)\,(sE_1 - A_1) = 0$ and so $\det(sE_1 - A_1) \equiv 0$. This contradicts the statement that $[sE_1 - A_1 \quad B_1]$ is a regular system pencil.
Conversely, if the pencil $[sE - A \quad B]$ has no row indices, then it is equivalent to a pencil of the form (34). Since pencils of this form are regular system pencils, it follows from Lemma 2.1 that $[sE - A \quad B]$ is regularizable. □

In [1], Brunovsky introduced a canonical form for controllable state space systems under the action of the full feedback group. Here we extend the Brunovsky canonical form to arbitrary generalized state space systems.

Proposition 3.3. Every regularizable system $E\dot{x} = Ax + Bu$ is equivalent under transformations of strong equivalence, change of input coordinates, and state feedback to exactly one system (up to reordering of blocks) of the following form:

$$\dot{x}_1 = A_1x_1 + B_1u \tag{35}$$

$$\dot{x}_2 = A_2x_2 \tag{36}$$

$$E_3\dot{x}_3 = x_3, \tag{37}$$

where the subsystem (35) is a controllable state space system in Brunovsky canonical form, A_2 is a matrix in Jordan form, and E_3 is a nilpotent matrix in Jordan form. Furthermore, if E is nonsingular, then the canonical form consists only of the parts (35) and (36).

Proof. Expressing R in (34) by (33) and setting $A_2 = J$ and $E_3 = N$, the canonical form (35), (36), (37) follows directly from (34). If we write (35)–(37) in the form $\tilde{E}\dot{x} = \tilde{A}x + \tilde{B}u$ we obtain from (11) that $\operatorname{rk} E = \operatorname{rk} \tilde{E}$. Hence the singular part of $\tilde{E}$ must be absent if $\det E \neq 0$. □

Remark 3.4. Restricting Proposition 3.3 to the set of state space systems, one obtains a canonical form for arbitrary (possibly uncontrollable) state space systems under the action of the full feedback group (see Corollary 2.7). Note that in this canonical form the controllable and the uncontrollable parts of the system are decoupled.

Next we describe the system canonical form (35), (36), (37) for controllable generalized state space systems. For definitions and discussions concerning controllability of singular systems, see [27], [23], [2], and [16]. We will use the concept of controllability introduced in [23] which may be characterized as follows: $E\dot{x} = Ax + Bu$ is controllable if and only if

$$\operatorname{rk}[\alpha E - A \quad B] = n \quad \text{for all } \alpha \in \mathbb{C} \text{ and } \operatorname{Im} E + A \operatorname{Ker} E + \operatorname{Im} B = \mathbb{K}^n. \tag{38}$$

Note that controllability is preserved under operations of strong equivalence (3) and generalized feedback transformations.

Corollary 3.5. If an n-dimensional system $E\dot{x} = Ax + Bu$ is controllable, then it can be transformed to a controllable state space system of dimension less than or equal to n by state feedback (17), change of input coordinates (15), strong equivalence operations (3), and trivial deflations (5).

Proof. Let $E\dot{x} = Ax + Bu$ be a controllable system. Theorem 1 of [15] states that $\operatorname{rk} E = \Sigma_{i=1}^{m} c_i$. It follows that no "$s$" can appear in the $r \times r$ regular block of the canonical form (34). Hence the regular block is simply the $r \times r$ identity matrix I_r. Therefore the pencil $[sE - A \quad B]$ is equivalent to a pencil of the form:

$$\begin{bmatrix} sI - A_1 & 0 & B_1 \\ 0 & I_r & 0 \end{bmatrix}, \tag{39}$$

where the system $\dot{x} = A_1x + B_1u$ is controllable. This pencil can be trivially deflated to:

$$[sI - A_1 \quad B_1]. \tag{40}$$

This concludes the proof by Proposition 2.6. □

Up to this point, we have focused on transformation groups which include strong equivalence operations. We now move on to the smaller group of generalized feedback transformations (18) relevant to the HGF2 problem. Loiseau, Özcaldiran, Malabre, and Karcanias [17] describe a canonical form for the action of the generalized feedback group $\mathcal{G}_{n,m}(\mathbb{K})$. Their canonical form is applicable to arbitrary implicit systems $E\dot{x} = Ax + Bu$ with rectangular pencil $sE - A$; we present the theorem here only for the case of regularizable systems (i. e. $sE - A$ is a square pencil and the pencil $[sE - A \quad B]$ has no row indices). First we introduce some notation. For each integer $n_i \geq 2$, let

$$U_{n_i} = \begin{bmatrix} s & & \\ 1 & \ddots & \\ & \ddots & s \\ & & 1 \end{bmatrix}_{n_i \times (n_i - 1)} \quad \text{and} \quad e_{n_i} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n_i \times 1} . \tag{41}$$

Corresponding to each pencil of the form

$$U = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ U_{n_1} & & \\ & \ddots & \\ & & U_{n_{k_U}} \end{bmatrix}_{(j_U + \sum_{i=1}^{k_U} n_i) \times (\sum_{i=1}^{k_U} n_i - k_U)} , \quad (j_U \text{ zero rows}) \tag{42}$$

let

$$B_U = \begin{bmatrix} I_{j_U} & & & \\ & e_{n_1} & & \\ & & \ddots & \\ & & & e_{n_{k_U}} \end{bmatrix}_{(j_U + \sum_{i=1}^{k_U} n_i) \times (j_U + k_U)} . \tag{43}$$

Theorem 3.6. [17] Under the action of the generalized feedback group $\mathcal{G}_{n,m}(\mathbb{K})$, each regularizable system pencil $[sE - A \quad B]$ may be transformed to exactly one pencil (up to reordering of blocks) of the form:

$$\begin{bmatrix} sI - A_c & 0 & 0 & B_c & 0 \\ 0 & P & 0 & 0 & 0 \\ 0 & 0 & U & 0 & B_U \end{bmatrix}, \tag{44}$$

where (A_c, B_c) defines a controllable state space system in Brunovsky canonical form, P is a pencil in the canonical form (31), and (U, B_U) is a pair of the form (42), (43). If the pencil P has j_P zero columns and k_P blocks of the form (32) and the pencil U has j_U zero rows and k_U blocks of the form (41), then $j_P + k_P = j_U + k_U$.

To clarify the relationship between pencil orbits $\mathcal{O}([sE-A \quad B])$ and orbits under generalized feedback transformations $\mathcal{G}_{n,m}(\mathbb{K})\cdot[sE-A \quad B]$, recall that the regularizable system pencils have the following invariants with respect to pencil equivalence: column indices, finite elementary divisors, and infinite elementary divisors. Each nonzero column index corresponds to a block of the form (32) and each zero column index corresponds to a zero column. The finite elementary divisors correspond to the Jordan blocks in the matrix J of (33) given by the list of eigenvalues and block sizes $\{(\alpha_i, r_i)\}_{i=1}^{k_J}$, and the infinite elementary divisors correspond to the Jordan blocks of the nilpotent matrix N in (33) given by the list of block sizes $(n_1, \ldots, n_{n-e})$ where $e = \operatorname{rk} E$. In the following we refer to the block sizes $(n_1, \ldots, n_{n-e})$ as *nilpotency indices*.

In the canonical form for generalized feedback transformations (44), all blocks corresponding to the finite elementary divisors appear in the pencil P. The column indices correspond to blocks in both the pencil $[sI + A_c \quad B_c]$ and the pencil P. The column indices corresponding to blocks in the pencil $[sI + A_c \quad B_c]$ we call *regular* and those corresponding to blocks in the pencil P we call *singular*; thus the column indices are partitioned into two lists, the regular column indices $(c_1, \ldots, c_\mu)$ (listed in decreasing order) and the singular column indices $(c_{\mu+1}, \ldots, c_m)$ (listed in decreasing order). We denote the partitioned list by $(c_1, \ldots, c_\mu; c_{\mu+1}, \ldots, c_m)$. Similarly, the blocks defined by the infinite elementary divisors appear in both the pencil P and the pencil $[U \quad B_U]$, partitioning the list of nilpotency indices into *regular nilpotency indices* $(u_1, \ldots, u_\nu)$ from the pencil P and *singular nilpotency indices* $(u_{\nu+1}, \ldots, u_{n-e})$ from the pencil $[U \quad B_U]$ with $u_1 \geq \ldots \geq u_\nu$ and $u_{\nu+1} \geq \ldots \geq u_{n-e}$. Again we denote the partitioned list by $(u_1, \ldots, u_\nu; u_{\nu+1}, \ldots, u_{n-e})$. Because $j_P + k_P = j_U + k_U$, we have $m - \mu = n - e - \nu$, i.e. the number of singular column indices is the same as the number of singular nilpotency indices. If $[sE - A \quad B]$ is a pencil in the canonical form (44), then the horizontal rectangular blocks and the zero columns of the $n \times n$ pencil $sE-A$ correspond to singular column indices and the vertical rectangular blocks and the zero rows of $sE - A$ correspond to the singular nilpotency indices.

It follows from Theorem 3.6 that the $\mathcal{G}_{n,m}(\mathbb{K})$-orbits of regularizable pencils are parameterized by the invariants

$$((c_1, \ldots, c_\mu; c_{\mu+1}, \ldots, c_m), \{(\alpha_i, r_i)\}_{i=1}^{f}, (u_1, \ldots, u_\nu; u_{\nu+1}, \ldots, u_{n-e})). \tag{45}$$

The canonical form (44) which has only regular indices is the system canonical form (34) described in Section 2. Let $[sE - A \quad B]$ be a regularizable system pencil with column indices $(d_1, \ldots, d_m)$, finite elementary divisor blocks given by $\{(\alpha_i, r_i)\}_{i=1}^{f}$, and nilpotency indices $(n_1, \ldots, n_{n-e})$. Comparing these invariants with the generalized feedback invariants, we see that the set $\mathcal{O}([sE - A \quad B]) \cap \mathcal{S}_{n,m}(\mathbb{K})$ of system pencils equivalent to $[sE - A \quad B]$ is a finite union of $\mathcal{G}_{n,m}(\mathbb{K})$-orbits, specifically the union of orbits with invariants (45) where $(c_1, \ldots, c_\mu; c_{\mu+1} \ldots, c_m)$ and $(u_1, \ldots, u_\nu; u_{\nu+1}, \ldots, u_{n-e})$ are rearrangements of $(d_1, \ldots, d_m)$ and $(n_1, \ldots, n_{n-e})$, respectively. Note that, if E is nonsingular, there are no nilpotency indices and so all column indices are regular. Hence $\mathcal{O}([sE - A \quad B]) \cap \mathcal{S}_{n,m}(\mathbb{K})$ is a $\mathcal{G}_{n,m}(\mathbb{K})$ orbit.

4. ORBIT CLOSURES AND LIMITS UNDER HIGH GAIN FEEDBACK

At the end of Section 2 we observed that a determination of limits under high gain feedback requires computation of $\mathrm{cl}\,(\mathcal{O}([sE - A \quad B]) \cap \mathcal{S}_{n,m}(\mathbb{K}))$ for HGF1 and of $\mathrm{cl}\,(\mathcal{G}_{n,m}(\mathbb{K}) \cdot [sE - A \quad B])$ for HGF2. The conclusion of Section 3 brings the two problems together; there we established that the set $\mathcal{O}([sE - A \quad B]) \cap \mathcal{S}_{n,m}(\mathbb{K})$ of equivalent system pencils is a finite union of $\mathcal{G}_{n,m}(\mathbb{K})$-orbits, i. e. there is a finite set of system pencils $\{[sE_i - A_i \quad B_i]\}_{i=1}^{t}$ (specifically those of the form (44) with the same pencil invariants as $[sE - A \quad B]$) such that

$$\mathcal{O}([sE - A \quad B]) \cap \mathcal{S}_{n,m}(\mathbb{K}) = \bigcup_{i=1}^{t} \mathcal{G}_{n,m}(\mathbb{K}) \cdot [sE_i - A_i \quad B_i]$$

Thus the intersection of the $Gl_n(\mathbb{K}) \times Gl_{n+m}(\mathbb{K})$-orbit of $[sE - A \quad B] \in \mathcal{S}_{n,m}(\mathbb{K})$ with the linear subspace $\mathcal{S}_{n,m}(\mathbb{K}) \subset \mathcal{P}_{n,m}(\mathbb{K})$ is a union of $\mathcal{G}_{n,m}(\mathbb{K})$-orbits and

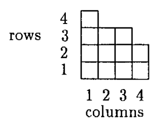
$$\mathrm{cl}\,((Gl_n(\mathbb{K}) \times Gl_{n+m}(\mathbb{K}) \cdot [sE - A \quad B]) \cap \mathcal{S}_{n,m}(\mathbb{K})) = \bigcup_{i=1}^{t} \mathrm{cl}\,(\mathcal{G}_{n,m}(\mathbb{K}) \cdot [sE_i - A_i \quad B_i]).$$

Hence a complete description of HGF2-limits would produce a complete description of HGF1-limits. Unfortunately, a complete description of the closures of $\mathcal{G}_{n,m}(\mathbb{K})$-orbits is not yet available. As noted previously, $\mathcal{G}_{n,m}(\mathbb{K})$ is a non-reductive group whereas $Gl_n(\mathbb{K}) \times Gl_{n+m}(\mathbb{K})$ is reductive. Since the representation theory of non-reductive groups is known to be much more complex than that of reductive groups, one would expect descriptions of orbit closures to be more difficult to determine for the action of $\mathcal{G}_{n,m}(\mathbb{K})$ than for the action of $Gl_n(\mathbb{K}) \times Gl_{n+m}(\mathbb{K})$. Indeed, a complete characterization of the closures of $Gl_n(\mathbb{K}) \times Gl_{n+m}(\mathbb{K})$-orbits containing regular system pencils will be given in Theorem 4.1. As mentioned at the end of the previous section this does not completely solve the high gain feedback problem in the sense of HGF1. On the other hand there exist only partial results for closures of $\mathcal{G}_{n,m}(\mathbb{K})$-orbits. For instance, in [6] these closures have been determined for *reachable* input pairs (see the glossary for a definition), and some sufficient conditions for orbit closure appear in [12]. For arbitrary state space systems we will give a complete description at the end of this section (Theorem 4.6). In spite of these results the complete description of the sets of all HGFi-limits, $i = 1, 2$ (for arbitrary regular system pencils) still remains an unresolved problem.

The first theorem of this section gives a complete characterization of the *pencils* in $\mathrm{cl}\,(\mathcal{O}([sE - A \quad B]))$; a consequence of this characterization is a set of conditions necessary for a system to be a limit of another system under high gain feedback (in either sense). We begin by introducing some notation. The *dual* column indices of an arbitrary $n \times (n+m)$ pencil are defined as follows:

$$c'_j = |\{i : c_i \geq j\}|, \quad j = 1, \ldots, n, \tag{46}$$

where $(c_1, \ldots, c_m)$ is the list of column indices of the pencil and $|S|$ denotes the cardinality of the set S. Note that the dual column indices do not depend upon the ordering of the list $(c_1, \ldots, c_m)$. A useful illustration of the dual indices is via Young diagrams which consist of squares arranged as follows: row 1 has c_1 squares, row 2 has c_2 squares, etc. Then the dual indices (c'_i) are the lengths of the columns. For example, the dual indices to $(4, 4, 3, 1)$ are $(4, 3, 3, 2, 0, \ldots, 0)$.



The dual column indices are "controllability indices" of the system (E, A, B) in the following sense (which generalizes the characterization of controllability indices for state space systems, see [26, § 5.7]): Define the subspaces $\mathcal{V}_i$ and $\mathcal{R}_i$ of $\mathbb{K}^n$ by:

$$\mathcal{V}_0 = \mathbb{K}^n, \quad \mathcal{V}_i = A^{-1}(\operatorname{Im} B + E\mathcal{V}_{i-1}), \quad i \geq 1; \qquad \mathcal{V}^* = \bigcap_{i \in \mathbb{N}} V_i$$

$$\mathcal{R}_0 = 0, \quad \mathcal{R}_i = E^{-1}(\operatorname{Im} B + A\mathcal{R}_{i-1}), \quad i \geq 1.$$

Then it follows from [17, Prop. 2.2] that

$$\dim\, \mathcal{R}_j \cap \mathcal{V}^* = \sum_{i=1}^{j} c'_i, \qquad j = 1, \ldots, n.$$

The following theorem gives necessary and sufficient conditions for an arbitrary regularizable system pencil $s\overline{M} + \overline{N} \in \mathcal{S}_{n,m}(\mathbb{K})$ to lie in the closure of the orbit $\mathcal{O}(sM + N)$ of another $n \times (n+m)$ pencil with no row indices. We write the invariants for $s\overline{M} + \overline{N}$ with an upper bar; e.g. the column indices of $s\overline{M} + \overline{N}$ are denoted $\bar{c}_i$. The proof of the theorem appears in [11].

Theorem 4.1. Let $s\overline{M} + \overline{N}$, $sM + N \in \mathcal{P}_{n,m}(\mathbb{K})$ be pencils with column indices $(\bar{c}_1, \ldots, \bar{c}_m)$, $(c_1, \ldots, c_m)$ respectively, and with no row indices. Then $s\overline{M} + \overline{N}$ is in the closure of the orbit $\mathcal{O}(sM + N)$ if and only if the following two conditions hold:

$$\sum_{i=1}^{j} \bar{c}'_i \leq \sum_{i=1}^{j} c'_i, \quad j = 1, \ldots, n \tag{47}$$

$$D_j(sM + tN) \mid D_j(s\overline{M} + t\overline{N}), \quad j = 1, \ldots, n. \tag{48}$$

The general orbit closure problem for arbitrary singular matrix pencils (possibly having row indices) is still open.

Example 4.2. The following regular system pencil, representing a state space system, has column indices $(1,1)$ and divisor polynomials $D_3 = s-1$, $D_2 = D_1 = 1$:

$$\left[\begin{array}{ccc|cc} s & 0 & 0 & 1 & 0 \\ 0 & s & 0 & 0 & 1 \\ 0 & 0 & s-1 & 0 & 0 \end{array}\right]. \tag{49}$$

The list of dual column indices is $(2,0,0)$. From Theorem 4.1 it follows that any regular system pencil in the closure of the orbit of the pencil (49) is equivalent to (49) or to one of the following pencils (in canonical form (34)):

(i) Pencils in the orbit closure with column indices $(2,0)$ (i. e. dual column indices $(1,1,0)$):

$$\left[\begin{array}{ccc|cc} s & 1 & 0 & 0 & 0 \\ 0 & s & 0 & 1 & 0 \\ 0 & 0 & s-1 & 0 & 0 \end{array}\right]. \tag{50}$$

(ii) Pencils in the orbit closure with column indices $(1,0)$:

$$\left[\begin{array}{ccc|cc} s & 0 & 0 & 1 & 0 \\ 0 & s-1 & 1 & 0 & 0 \\ 0 & 0 & s-1 & 0 & 0 \end{array}\right] \quad \text{or} \quad \left[\begin{array}{ccc|cc} s & 0 & 0 & 1 & 0 \\ 0 & s-1 & 0 & 0 & 0 \\ 0 & 0 & \alpha s+\beta & 0 & 0 \end{array}\right], \quad (\alpha,\beta) \neq (0,0). \tag{51}$$

(iii) Pencils in the orbit closure with column indices $(0,0)$:

$$\left[\begin{array}{ccc|cc} s-1 & 1 & 0 & 0 & 0 \\ 0 & s-1 & 1 & 0 & 0 \\ 0 & 0 & s-1 & 0 & 0 \end{array}\right]$$

$$\text{or} \quad \left[\begin{array}{ccc|cc} s-1 & 1 & 0 & 0 & 0 \\ 0 & s-1 & 0 & 0 & 0 \\ 0 & 0 & \alpha s+\beta & 0 & 0 \end{array}\right], \qquad (\alpha,\beta) \neq (0,0)$$

$$\text{or} \quad \left[\begin{array}{ccc|cc} s-1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 s+\beta_1 & 1 & 0 & 0 \\ 0 & 0 & \alpha_1 s+\beta_1 & 0 & 0 \end{array}\right], \qquad \alpha_1 \neq 0$$

$$\text{or} \quad \left[\begin{array}{ccc|cc} s-1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 s+\beta_1 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 s+\beta_2 & 0 & 0 \end{array}\right], \qquad (\alpha_i,\beta_i) \neq (0,0)$$

$$\text{or} \quad \left[\begin{array}{ccc|cc} s-1 & 0 & 0 & 0 & 0 \\ 0 & 1 & s & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]. \tag{52}$$

As an immediate consequence of Theorem 4.1 we obtain the following necessary condition for a system to occur as the limit of another system under high gain feedback.

Corollary 4.3. If a generalized state space system $\overline{E}\dot{x} = \overline{A}x + \overline{B}u$ is a limit of the system $E\dot{x} = Ax + Bu$ under high gain feedback (in the sense of HGF1 or HGF2), then the pencils $[s\overline{E} - \overline{A} \quad \overline{B}]$ and $[sE - A \quad B]$ satisfy the conditions (47), (48).

Remark 4.4. In the context of the HGF2 problem, we have as invariants regular and singular column indices $(c_1, \dots, c_\mu; c_{\mu+1}, \dots, c_m)$, regular and singular nilpotency indices $(u_1, \dots, u_\nu; u_{\nu+1}, \dots, u_{n-e})$, and finite elementary divisors. The inequalities of (47) are conditions on the column indices written in decreasing order; the condition (48) may be rewritten in terms of the nilpotency indices and finite elementary divisors as follows. Because $D_j([sE - tA \quad tB])$ is a homogeneous polynomial in the variables s, t, it factors over $\mathbb{C}$ as

$$D_j([sE - tA \quad tB] = t^{r_j} \prod_{i=1}^{h_j} (s - \alpha_i t) =: t^{r_j} g_j(s,t)$$

The condition (48) is equivalent to:

$$r_j \leq \overline{r}_j \qquad \text{and} \qquad g_j(s,t) \mid \overline{g}_j(s,t)\,, \qquad j = 1, \dots, n. \tag{53}$$

Let $q = n - \operatorname{rk} E$, $\overline{q} = n - \operatorname{rk} \overline{E}$. From Section 3 of [7], we have

$$\sum_{i=1}^{j} n_i = \sum_{i=1}^{q} n_i - r_{n-j}\,, \quad j = 1, \dots, n$$

and

$$\sum_{i=1}^{j} \overline{n}_i = \sum_{i=1}^{\overline{q}} \overline{n}_i - \overline{r}_{n-j}\,, \quad j = 1, \dots, n,$$

where $(n_i)_{i=1}^{q}$ (resp. $(\overline{n}_i)_{i=1}^{q}$) are the nilpotency indices written in decreasing order, $n_i = 0$ for $q < i \leq n$ and $\overline{n}_i = 0$ for $\overline{q} < i \leq n$. Because $q \leq \overline{q}$, the condition $\overline{r}_j \leq r_j$ may be written as

$$\sum_{i=j}^{q} n_i \leq \sum_{i=j}^{\overline{q}} \overline{n}_i\,, \qquad j = 1, \dots, q \tag{54}$$

For the second part of (53), we note that $g_j(s,t) | \overline{g}_j(s,t)$ if and only if $g_j(s,1) | \overline{g}_j(s,1)$. Using the fact that each $j \times j$ minor of $[sE - tA \quad tB]$ is homogeneous and hence can be factored as $t^r \prod_{i=1}^{h} (s - \alpha_i t)$, we see that $g_j(s,1)$ is the greatest common divisor of the $j \times j$ minors of the polynomial matrix $[sE - A \quad B]$. Therefore the second part of (53) is equivalent to (54) along with the following condition on the finite elementary divisors:

$$D_j([sE - A \quad B]) \mid D_j([s\overline{E} - \overline{A} \quad \overline{B}]), \quad j = 1, \dots, n. \tag{55}$$

Note that, in the conditions (47), (54), and (55) no distinction is made between regular and singular indices.

In general, the converse to Corollary 4.3 does not hold. Suppose there are system pencils $[sE_k - A_k \quad B_k]$ equivalent to $[sE - A \quad B]$ such that

$$\lim_{k\to\infty} [sE_k - A_k \quad B_k] = [s\overline{E} - \overline{A} \quad \overline{B}]$$

If E is nonsingular, we have $B_k = L_k B W_k$ for some $L_k \in Gl_n(\mathbb{K})$, $W_k \in Gl_m(\mathbb{K})$ so that the rank of $\overline{B}$ cannot be greater than the rank of B. On the other hand it is not hard to find examples of pencils $[s\overline{E}-\overline{A} \quad \overline{B}]$ and $[sI-A \quad B]$ which satisfy conditions (47), (48) with $\operatorname{rk}\overline{B} > \operatorname{rk} B$. Hence conditions (47) and (48) are not sufficient for either concept of high gain feedback limits. (The two concepts coincide since $E = I$ is nonsingular.) The next example illustrates that an increase in the rank of B is not the only way in which a pencil limit can fail to be a limit of equivalent *system pencils*.

Example 4.5. Consider the following system pencils $\overline{P}$ and P which have invariants $((2,1),(t))$ and $(3,1)$ respectively:

$$\bar{P} = \left[\begin{array}{cccc|cc} s & 1 & 0 & 0 & 0 & 0 \\ 0 & s & 0 & 0 & 1 & 0 \\ 0 & 0 & s & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \qquad P = \left[\begin{array}{cccc|cc} s & 1 & 0 & 0 & 0 & 0 \\ 0 & s & 1 & 0 & 0 & 0 \\ 0 & 0 & s & 0 & 1 & 0 \\ 0 & 0 & 0 & s & 0 & 1 \end{array}\right].$$

Again the concepts HGF1 and HGF2 coincide since P represents a state space system. The pencils $\overline{P}$ and P satisfy the conditions (47) and (48) of Theorem 4.1, so $\overline{P}$ is a limit of pencils equivalent to P. Limits of P under high gain feedback are pencils which lie in the closure of the $\mathcal{G}_{n,m}(\mathbb{K})$-orbit of P. In terms of the $\mathcal{G}_{n,m}(\mathbb{K})$-invariants described at the end of Section 3, P has only regular column indices (invariants $(3,1;\ )$) and the column index list of $\overline{P}$ is $(2,1)$. Since both of the systems represented by P and $\overline{P}$ are reachable (69), we may apply the characterization in [6] of closures of $\mathcal{G}_{n,m}(\mathbb{K})$-orbits within the set of reachable systems. For the case $\operatorname{rk}\overline{E} = \operatorname{rk} E - 1$ and P with only regular column indices, the characterization is as follows. We define the following partial ordering on lists of nonnegative integers: If $a = (a_1, \ldots, a_m)$ (not necessarily in increasing or decreasing order), let

$$r_{i,j}(a) = (i-1)|\{k : a_k \geq i, 1 \leq k \leq m\}| + |\{k : a_k \geq i, 1 \leq k \leq j\}| + \sum_{1\leq g\leq m, a_g \leq i-1} a_g.$$

We say $\overline{a} = (\overline{a}_1, \ldots, \overline{a}_m) \preceq (a_1, \ldots, a_m) = a$ if $r_{i,j}(\overline{a}) \leq r_{i,j}(a)$ for all i, j. A reachable system with column indices $(\overline{c}_1, \ldots, \overline{c}_{\overline{k}}; \overline{c}_{\overline{k}+1}, \ldots, \overline{c}_m)$ is in the closure of the $\mathcal{G}_{n,m}(\mathbb{K})$-orbit of a reachable system with column indices $(c_1, \ldots, c_m;\ )$ if and only if there is an integer h, $1 \leq h \leq m$, such that $(\overline{c}_1, \ldots, \overline{c}_{\overline{k}}, \overline{c}_{\overline{k}+1}, \ldots, \overline{c}_m) \preceq (c_1, \ldots, \hat{c}_h, \ldots, c_m, c_h - 1)$where $\hat{}$ denotes omission of this index. In this example, if $\overline{P}$ lies in the closure of the $\mathcal{G}_{n,m}(\mathbb{K})$-orbit of P, then

$$(2,1) \preceq (1,2) \quad \text{or} \quad (2,1) \preceq (3,0).$$

Neither of these inequalities hold ($3 = r_{2,1}(2,1) \not\preceq r_{2,1}(1,2) = 2$ and $3 = r_{2,1}(2,1) \not\preceq r_{2,1}(3,0) = 2$); it follows that the pencil $\overline{P}$ cannot be in the closure of the $\mathcal{G}_{n,m}$-orbit of P.

In Corollary 2.7 we showed that the restriction of pencil equivalence to the set of state space system pencils $[sI - A \quad B]$ coincides with equivalence under the action (21) of the full feedback group $\mathcal{F}_{n,m}(\mathbb{K})$. It follows directly from Theorem 4.1 that the conditions (47), (48) are necessary for orbit closure of state space systems under the action of the full feedback group. Now we establish that these conditions are, in fact, sufficient.

Theorem 4.6. A state space system $\dot{x} = \overline{A}x + \overline{B}u$ is a limit of the state space system $\dot{x} = Ax + Bu$ under the action (21) of the full feedback group $\mathcal{F}_{n,m}(\mathbb{K})$ if and only if the pencil $[sI - \overline{A} \quad \overline{B}]$ is in the closure of $\mathcal{O}([sI - A \quad B])$, i.e. if and only if the pencils $[sI - \overline{A} \quad \overline{B}]$ and $[sI - A \quad B]$ satisfy the conditions (47), (48).

Proof. Because elements of the full feedback group $\mathcal{F}_{n,m}(\mathbb{K})$ are pencil transformations, we know that $[sI - \overline{A} \quad \overline{B}]$ is in the closure of $\mathcal{O}([sI - A \quad B])$ whenever $\dot{x} = \overline{A}x + \overline{B}u$ is a limit of a system $\dot{x} = Ax + Bu$ under the full feedback group action.
Conversely, suppose $[sI - \overline{A} \quad \overline{B}]$ is in the closure of $\mathcal{O}([sI - A \quad B])$, i.e. for some $L_k \in Gl_n(\mathbb{K})$, $\begin{bmatrix} R_k & X_k \\ F_k & W_k \end{bmatrix} \in Gl_{n+m}(\mathbb{K})$,

$$\lim_{k\to\infty} L_k[sI - A \quad B]\begin{bmatrix} R_k & X_k \\ F_k & W_k \end{bmatrix} = [sI - \overline{A} \quad \overline{B}] \tag{56}$$

i.e.

$$\lim_{k\to\infty} L_k R_k = I, \quad \lim_{k\to\infty} L_k (AR_k - BF_k) = \overline{A} \tag{57}$$

$$\lim_{k\to\infty} L_k X_k = 0, \quad \lim_{k\to\infty} L_k BW_k - L_k AX_k = \overline{B}. \tag{58}$$

Set $M_k = L_k(AR_k - BF_k) - \overline{A}$; then $\lim_{k\to\infty} M_k = 0$. From (57) and the continuity of the determinant function it follows that there is an integer N such that R_k is invertible for $k > N$. From (58) we have $\lim_{k\to\infty} L_k R_k R_k^{-1} X_k = 0$, hence $\lim_{k\to\infty} R_k^{-1} X_k = 0$. Substituting

$$L_k AX_k = L_k AR_k R_k^{-1} X_k = (M_k + L_k BF_k + \bar{A}) R_k^{-1} X_k$$

into (58), we obtain

$$\begin{aligned} \overline{B} &= \lim_{k\to\infty} L_k BW_k - (M_k + L_k BF_k + \bar{A}) R_k^{-1} X_k \\ &= \lim_{k\to\infty} L_k B(W_k - F_k R_k^{-1} X_k) - (M_k + \bar{A}) R_k^{-1} X_k. \end{aligned}$$

Because $\lim_{k\to\infty} M_k = 0$ and $\lim_{k\to\infty} R_k^{-1} X_k = 0$, it follows that

$$\overline{B} = \lim_{k\to\infty} L_k B(W_k - F_k R_k^{-1} X_k). \tag{59}$$

If the matrix $W_k - F_k R_k^{-1} X_k$ is not invertible, there is an $m \times m$ matrix $\tilde{W}_k$ arbitrarily close to W_k such that $\tilde{W}_k - F_k R_k^{-1} X_k$ is invertible; hence, without loss of generality, we may assume that $V_k = W_k - F_k R_k^{-1} X_k$ is invertible. It follows from (57), (58), and (59) that

$$\lim_{k\to\infty} L_k[sI - A \quad B] \begin{bmatrix} R_k & 0 \\ F_k & V_k \end{bmatrix} = [sI - \overline{A} \quad \overline{B}]. \tag{60}$$

Because $\lim_{k\to\infty} L_k R_k = I$, we may replace L_k in the equation (60) by $R_k^{-1} = (L_k R_k)^{-1} L_k$ (the matrix $L_k R_k$ is invertible for large k). Therefore

$$\lim_{k\to\infty} R_k^{-1}[sI - A \quad B] \begin{bmatrix} R_k & 0 \\ F_k & W_k \end{bmatrix} = [sI - \overline{A} \quad \overline{B}].$$

Altogether we have shown that every pencil $[sI - \overline{A} \quad \overline{B}]$ representing a state space system which is in the closure of $\mathcal{O}([sI - A \quad B])$ belongs to the closure of $\mathcal{F}_{n,m}(\mathbb{K}) \cdot [sI - A \quad B]$. The theorem then follows from Theorem 4.1. □

Theorem 4.6 extends the main result of [18] which characterized orbit closure only within the set of controllable state space systems.

5. SYSTEM-THEORETIC APPLICATIONS

In this section we discuss some consequences of the results of § 4. We consider only systems which are regularizable. In the two concepts of "limits under high gain feedback" we have included limits which may be realized without applying feedback transformations; we identify cases in which feedback transformations are necessarily applied. We then discuss how many nonequivalent systems may occur as high gain feedback limits of a given one and we identify those systems which never occur as a nontrivial high gain feedback limit of any system and those systems which have no nontrivial high gain feedback limits. The last part of this section is devoted to a discussion of the "high gain Rosenbrock problem". We compare the set of lists of invariant factors which can be achieved by applying state feedback to a given system (E, A, B) with the set of invariant factor lists which are producible in the limit by high gain state feedback. If controllability and the rank of E are to be preserved, we will see that the two sets coincide, i. e. nothing is gained with respect to the assignability of invariant factors by allowing high gain feedback limits. On the other hand, if we admit noncontrollable high gain feedback limits of a given controllable state space systems, *any* set of invariant factors can be achieved. We begin by summarizing the system-theoretic results of the previous section:

1. If $\overline{E}\dot{x} = \overline{A}x + \overline{B}u$ is a limit of $E\dot{x} = Ax + Bu$ under high gain feedback (in either sense), then the pencils $[s\overline{E} - \overline{A} \quad \overline{B}]$ and $[sE - A \quad B]$ satisfy conditions (47), (48).
2. For state space systems, HGF1 and HGF2 coincide and we have: $\dot{x} = \overline{A}x + \overline{B}u$ is a limit of $\dot{x} = Ax + Bu$ under high gain feedback if and only if the pencils $[sI - \overline{A} \quad \overline{B}]$ and $[sI - A \quad B]$ satisfy conditions (47), (48).

A system $\overline{E}\dot{x} = \overline{A}x + \overline{B}u$ which is a limit under high gain feedback of another system $E\dot{x} = Ax + Bu$ (in the sense of HGF1 or HGF2) may have actually arisen as a limit in which feedback transformations were absent. In the following propositions, we identify some limits which can be achieved only when feedback transformations are applied.

Lemma 5.1. Suppose a regular system pencil $[s\overline{E} - \overline{A} \quad \overline{B}]$ is a limit of system pencils which are equivalent to a regular system pencil $[sE - A \quad B]$ under input transformations and operations of strong equivalence alone. Then:

$$\det(s\overline{E} - \overline{A}) \text{ is a nonzero scalar multiple of } \det(sE - A) \tag{61}$$

and

$$D_j(sE - tA) \mid D_j(s\overline{E} - t\overline{A}), \qquad 1 \leq j \leq n. \tag{62}$$

In particular, rk $\overline{E}$ = rk E and hence a singular system cannot be a limit of a state space system under transformations of input and strong equivalence alone.

P r o o f. By assumption, we have

$$[s\overline{E} - \overline{A} \quad \overline{B}] = \lim_{k \to \infty} L_k [sE - A \quad B] \begin{bmatrix} R_k & X_k \\ 0 & W_k \end{bmatrix},$$

where $EX_k = 0$. It follows that

$$s\overline{E} - \overline{A} = \lim_{k \to \infty} L_k (sE - A) R_k.$$

The conclusion (61) follows from [7, Lemma 3.3], and (62) follows from [7, Theorem 3.7]. □

Proposition 5.2. Let $\overline{E}\dot{x} = \overline{A}x + \overline{B}u$ and $E\dot{x} = Ax + Bu$ be systems given by nonequivalent pencils in system canonical form (34) which have the same regular blocks $(\overline{R} = R)$. If the system $\overline{E}\dot{x} = \overline{A}x + \overline{B}u$ is a limit of $E\dot{x} = Ax + Bu$ under high gain feedback (in either sense), then the limit could not have been achieved by using only operations of strong equivalence and input transformations.

P r o o f. Let $(\overline{c}_1, \ldots, \overline{c}_m)$ and $(c_1, \ldots, c_m)$ be the column indices of $[s\overline{E} - \overline{A} \quad \overline{B}]$ and $[sE - A \quad B]$ respectively. Because $\overline{R} = R$, we have

$$n - r = \sum_{i=1}^{m} \overline{c}_i = \sum_{i=1}^{m} c_i,$$

where $\overline{R} = R$ is an $r \times r$ pencil. Since the system $\overline{E}\dot{x} = \overline{A}x + \overline{B}u$ is a limit of $E\dot{x} = Ax + Bu$ under high gain feedback, it follows from Corollary 4.3 that conditions (47) and (48) hold. Because the sequences of column indices for the two pencils are partitions of the same integer, the condition (47) is equivalent to

$$\sum_{i=1}^{j} \overline{c}_i \geq \sum_{i=1}^{j} c_i, \quad j = 1, \ldots, m. \tag{63}$$

(This is a standard combinatorics result; see for instance [3].) Since the pencils $[s\overline{E}-\overline{A} \quad \overline{B}]$ and $[sE-A \quad B]$ are not equivalent, at least one of these inequalities is a strict inequality; i.e. there is an h, $1 \le h \le m$, such that

$$\sum_{i=1}^{h} \overline{c}_i > \sum_{i=1}^{h} c_i. \tag{64}$$

In system canonical form (34), the *nonzero* column indices $(\overline{c}_1, \ldots, \overline{c}_{\overline{p}})$ (respectively $(c_1, \ldots, c_p)$) are the block sizes of the Jordan blocks $sI + N_{\overline{c}_i}$ (respectively $sI + N_{c_i}$) of the pencil $sI - \overline{A}_1$ (respectively $sI - A_1$). Let $r_1, \ldots, r_k$ be the block sizes of the 'nilpotent' Jordan blocks $sI + N_{r_i}$ in $R = \overline{R}$; then the sizes of the Jordan blocks of this form in $sE - A$ (resp. $s\overline{E} - \overline{A}$) are $b_1 \ge \ldots \ge b_w$ ($\overline{b}_1 \ge \ldots \ge \overline{b}_{\overline{w}}$) where $(b_1, \ldots, b_w)$ is a rearrangement of the list $(c_1, \ldots, c_p, r_1, \ldots, r_k)$ and $(\overline{b}_1, \ldots, \overline{b}_{\overline{w}})$ is a rearrangement of the list $(\overline{c}_1, \ldots, \overline{c}_{\overline{p}}, r_1, \ldots, r_k)$ (both in decreasing order). For ease of exposition, set $b_i = 0$ for $w < i \le n$ and $\overline{b}_i = 0$ for $\overline{w} < i \le n$. Let h_0 be the minimal integer h such that (64) holds, so that $c_i = \overline{c}_i$ for $i < h_0$ and $c_{h_0} < \overline{c}_{h_0}$. Let z be the smallest integer such that $b_z \ne \overline{b}_z$. Simple combinatorial reasoning shows that $b_z < \overline{b}_z$ and so:

$$\sum_{i=1}^{z} \overline{b}_i > \sum_{i=1}^{z} b_i. \tag{65}$$

Let d_j be the largest power of s dividing $D_j(sE+tA)$ and define $\overline{d}_j$ similarly. From the discussion of block sizes and invariant factors in [7, Sec. 3], we see that

$$\sum_{i=1}^{j} b_i = d_n - d_{n-j}, \quad j = 1, \ldots, n-1$$

and

$$\sum_{i=1}^{j} \overline{b}_i = \overline{d}_n - \overline{d}_{n-j}, \quad j = 1, \ldots, n-1$$

Suppose $\overline{E}\dot{x} = \overline{A}x + \overline{B}u$ is a limit of systems which are equivalent to $E\dot{x} = Ax + Bu$ under input transformations and operations of strong equivalence alone. By Lemma 5.1, we have $\overline{d}_n = d_n$ and $d_j \le \overline{d}_j$, $1 \le j \le n-1$. It follows that

$$\sum_{i=1}^{z} \overline{b}_i \le \sum_{i=1}^{z} b_i$$

contradicting (65). □

In the following propositions, we discuss the number of non-feedback equivalent systems which may occur as high gain feedback limits of a given system $E\dot{x} = Ax + Bu$ (see Definition 2.3).

Proposition 5.3. For HGF1 or HGF2, there are only finitely many high gain feedback limits $\overline{E}\dot{x} = \overline{A}x + \overline{B}u$ of $E\dot{x} = Ax + Bu$ which are non-feedback equivalent and satisfy

$$\sum_{i=1}^{m} \overline{c}_i = \sum_{i=1}^{m} c_i, \tag{66}$$

where $(\bar{c}_1, \ldots, \bar{c}_m)$ and $(c_1, \ldots, c_m)$ are the column indices of $[s\overline{E}-\overline{A} \quad \overline{B}]$ and $[sE-A \quad B]$, respectively.

Proof. It suffices to establish the proposition for the HGF2 case. Let $\overline{E}\dot{x} = \overline{A}x+\overline{B}u$ be an HGF2 limit of $E\dot{x} = Ax+Bu$ such that the condition (66) is satisfied. It follows from Corollary 4.3 that conditions (47) and (48) are also satisfied. Given n and m, there are only finitely many possible column index lists for $[s\overline{E} - \overline{A} \quad \overline{B}]$. We may assume that both systems are in the canonical form (34) with regular blocks $\overline{R}$ and R respectively. From condition (66) it follows that $\overline{R}$ and R are both $r \times r$ blocks ($r = n - \sum_{i=1}^{m} c_i$). Let R_1 and R_2 (respectively $\overline{R}_1$ and $\overline{R}_2$) be the $r \times r$ matrices such that $R = sR_1 + R_2$ (respectively $\overline{R} = s\overline{R}_1 + \overline{R}_2$). It follows from condition (48) that

$$D_j(sR_1 + tR_2) \mid D_j(s\overline{R}_1 + t\overline{R}_2), \qquad j = 1, \ldots, r$$

(see [4], vol. 2). In particular, $D_r(sR_1 + tR_2) = D_r(s\overline{R}_1 + t\overline{R}_2)$. Because $D_j(s\overline{R}_1 + t\overline{R}_2)$ divides $D_r(s\overline{R}_1 + t\overline{R}_2)$ for $j = 1, \ldots, r$, there are only finitely many possible lists $(D_1(s\overline{R}_1 + t\overline{R}_2), \ldots, D_r(s\overline{R}_1 + t\overline{R}_2))$. It follows that, under the condition (66), there are only finitely many possible lists of invariants for the pencil $[s\overline{E} - \overline{A} \quad \overline{B}]$; the conclusion follows from Kronecker's classification (Theorem 3.1) and the fact that each class of equivalent pencils contains only finitely many HGF2 equivalence classes (see the last paragraph of § 3). □

Proposition 5.4. A system $E\dot{x} = Ax + Bu$ has infinitely many non-feedback equivalent limits under high gain feedback (in either sense) if and only if the pencil $[sE - A \quad B]$ has at least one nonzero column index.

Proof. Suppose all column indices of $[sE - A \quad B]$ are zero. If $\overline{E}\dot{x} = \overline{A}x + \overline{B}u$ is a high gain feedback limit of $E\dot{x} = Ax + Bu$ (in either sense), then by Corollary 4.3, the corresponding pencils satisfy (47); hence $[s\overline{E} - \overline{A} \quad \overline{B}]$ has no nonzero column indices. In particular, (66) is satisfied and we conclude from Proposition 5.3 that $E\dot{x} = Ax + Bu$ has only finitely many non-feedback equivalent limits under high gain feedback (in the sense of HGF1 or of HGF2).
Suppose the pencil $[sE - A \quad B]$ has a nonzero column index, i. e. $c_1 \neq 0$. It suffices to produce infinitely many HGF2 limits which are nonequivalent in both senses. Without loss of generality, we may assume that the pencil $[sE - A \quad B]$ is in the canonical form (44).If c_1 is a regular column index, then the pencil $[sE - A \quad B]$ has a "controllable state space block" $[sI - A_{c_1} \quad B_{c_1}]$, where A_{c_1} is a nilpotent $c_1 \times c_1$ Jordan block and $B_{c_1} = \begin{bmatrix} 0 & \cdots & 1 \end{bmatrix}^{\mathrm{T}}$. It suffices to show that there are infinitely many nonequivalent high gain feedback limits of $[sI - A_{c_1} \quad B_{c_1}]$ in canonical form, since every generalized feedback transformation of $[sI - A_{c_1} \quad B_{c_1}]$ can be trivially extended to a generalized feedback transformation of the whole pencil $[sE - A \quad B]$. We see from Theorem 4.6 that all of the following pencils are high gain feedback limits (HGF1 and HGF2) of the pencil $[sI - A_{c_1} \quad B_{c_1}]$:

$$\begin{bmatrix} sI + A_{c_1-1} & 0 & B_{c_1-1} \\ 0 & s-\alpha & 0 \end{bmatrix}, \qquad \alpha \in \mathbb{K}.$$

Because there are infinitely many choices for α, giving nonequivalent pencils, the conclusion follows.
If c_1 is a singular column index, then the pencil $[sE - A \quad B]$ has a $c_1 \times (c_1 + 1)$ block in P of the form S_{c_1} (defined by (32)). Applying HGF2 transformations only to this block, it follows from Lemma 5.4 of [11] that all of the following pencils are in the orbit closure of S_{c_1}:

$$\begin{bmatrix} S_{c_1-1} & 0 \\ 0 & s-\alpha \end{bmatrix}, \qquad \alpha \in \mathbb{K}.$$

Any pencil equivalence transformation for the block S_{c_1} can be trivially extended to a transformation of restricted equivalence for the whole pencil $[sE - A \quad B]$. Since different α's lead to nonequivalent pencils, the conclusion follows. □

Proposition 5.5. There is a state space system $\dot{x} = A_0x + B_0u$ such that every system $\overline{E}\dot{x} = \overline{A}x + \overline{B}u$ is a limit of $\dot{x} = A_0x + B_0u$ under high gain feedback (in either sense). Moreover, up to transformation by elements of $\mathcal{G}_{n,m}(\mathbb{K})$, the system $\dot{x} = A_0x + B_0u$ is the only such system. Consequently, the system $\dot{x} = A_0x + B_0u$ is the only system (up to transformation by elements of $\mathcal{G}_{n,m}(\mathbb{K})$) which is a nontrivial high gain feedback limit of no other system.

Proof. It suffices to consider only the HGF2 case since all HGF2 limits are HGF1 limits as well. Let $\dot{x} = A_0x + B_0u$ be the controllable state space system in Brunovsky canonical form with associated pencil $[sI - A_0 \quad B_0]$ having the following invariants:

$$D_j([sI - tA_0 \quad tB_0]) = 1 \qquad \text{for all } j = 1, \ldots, n$$

$$c_i = \left\{ \begin{array}{lll} \left[\frac{n}{m}\right] + 1 & \text{if} & 1 \le i \le n - \left[\frac{n}{m}\right] m \\ \left[\frac{n}{m}\right] & \text{if} & n - \left[\frac{n}{m}\right] m + 1 \le i \le m \end{array} \right\}$$

i.e.

$$\begin{aligned} c'_i &= m \qquad \text{for } 1 \le i \le \left[\frac{n}{m}\right] \\ c'_{\left[\frac{n}{m}\right]+1} &= \left\{ \begin{array}{ll} 0 & \text{if } m \mid n \\ n - \left[\frac{n}{m}\right] m & \text{otherwise} \end{array} \right\} \\ c'_i &= 0 \qquad \text{for } \left[\frac{n}{m}\right] + 2 \le i \le n. \end{aligned}$$

Among all lists of invariants of $n \times (n+m)$ pencils with no row indices, the above list of invariants is the (unique) largest one with respect to the partial ordering given by conditions (47) and (48). If $\dot{x} = A_0x + B_0u$ is a high gain feedback limit of another system $E\dot{x} = Ax + Bu$, then E is nonsingular and so the system is equivalent to a state space system. We conclude from Theorem 4.6 that the two systems are equivalent. It also follows from Theorem 4.6 that every state space system pencil $[sI - A \quad B]$ lies in the closure of $\mathcal{G}_{n,m}(\mathbb{K}) \cdot [sI - A_0 \quad B_0]$. It follows from the characterization of $\mathcal{G}_{n,m}(\mathbb{K})$-orbit closure given in [6] that every controllable system $E\dot{x} = Ax + Bu$ is in the $\mathcal{G}_{n,m}(\mathbb{K})$-orbit closure of some state space system. Let $\overline{E}\dot{x} =$

$\overline{A}x + \overline{B}u$ be an arbitrary system. If $\mathcal{R}_{n,m}(\mathbb{K})$ is the subgroup of $\mathcal{G}_{n,m}(\mathbb{K})$ consisting of transformations of restricted system equivalence, it follows from [7, Prop. 4.6] that there is a regular system pencil $[sE - A \quad B]$ representing a controllable system such that $[s\overline{E} - \overline{A} \quad \overline{B}]$ is in the closure of $\mathcal{R}_{n,m}(\mathbb{K}) \cdot [sE - A \quad B]$. Since $\mathcal{R}_{n,m}(\mathbb{K})$ is a subgroup of $\mathcal{G}_{n,m}(\mathbb{K})$, it follows that $[s\overline{E} - \overline{A} \quad \overline{B}] \in \mathrm{cl}\,(\mathcal{G}_{n,m}(\mathbb{K}) \cdot [sE - A \quad B])$. By transitivity, we have

$$[s\overline{E} - \overline{A} \quad \overline{B}] \in \mathrm{cl}\,(\mathcal{G}_{n,m}(\mathbb{K}) \cdot [sI - A_0 \quad B_0]).$$

This concludes the proof. □

A consequence of Theorem 5.5 is that every system occurs as a high gain feedback limit of some state space system. Moreover, with the exception of the system $\dot{x} = A_0x + B_0u$ described above, every system occurs as a high gain feedback limit of a system which is not equivalent to it.

Proposition 5.6. Let $E\dot{x} = Ax$ be a system satisfying $\det(sE - A) \not\equiv 0$ such that $sE - A$ is equivalent to a diagonal pencil. Then the system $E\dot{x} = Ax$ has no nontrivial HGF1 limits.

Proof. The pencil $[sE - A \quad 0]$ with $sE - A$ diagonal has all column indices equal to zero and all elementary divisors of the pencil have degree 1. If $\overline{E}\dot{x} = \overline{A}x + \overline{B}u$ is a high gain feedback limit of $E\dot{x} = Ax$, then it follows from Corollary 4.3 that the pencils $[s\overline{E} - \overline{A} \quad \overline{B}]$ and $[sE - A \quad 0]$ must satisfy conditions (47) and (48). Hence the pencil $[s\overline{E} - \overline{A} \quad \overline{B}]$ has all column indices equal to zero and the two pencils have the same elementary divisors. It follows from Kronecker's Theorem and Proposition 2.6 that the pencils are equivalent in the sense of HGF1. □

We conclude the paper with a brief discussion of what we call the *high gain feedback version of Rosenbrock's problem*. While a complete high gain counterpart of Rosenbrock's Theorem is not yet available, we will present results concerning two extreme special cases.

The pencils in canonical form (44) which represent *controllable* systems are those with no regular nilpotency indices, no finite elementary divisors, and the singular nilpotency indices are all equal to 1. Using Theorem 4.1 and the generalization of Rosenbrock's Theorem which appears in [15], we obtain the following result for controllable systems with fixed degree of singularity (i. e. $\mathrm{rk}\,E$ fixed).

Proposition 5.7. Let $[s\overline{E} - \overline{A} \quad \overline{B}]$ and $[sE - A \quad B]$ be pencils representing controllable systems with $\mathrm{rk}\,E = \mathrm{rk}\,\overline{E}$ such that $[s\overline{E} - \overline{A} \quad \overline{B}]$ is in the closure of $\mathcal{O}([sE - A \quad B])$. For any feedback transformation $\overline{F} \in \mathbb{K}^{m\times n}$ such that $\det(s\overline{E} - \overline{A} + \overline{B}\,\overline{F}) \not\equiv 0$, there is a feedback transformation $F \in \mathbb{K}^{m\times n}$ such that the pencils $sE - A + BF$ and $s\overline{E} - \overline{A} + \overline{B}\,\overline{F}$ have the same invariant factors.

Proof. By the remarks above, the pencil invariants of the two pencils $[s\overline{E} - \overline{A} \quad \overline{B}]$ and $[sE - A \quad B]$ are as follows: $\overline{k}$ (respectively k) infinite elementary divisors, all of degree 1, and column indices $(\overline{c}_1, \ldots, \overline{c}_m)$ (respectively $(c_1, \ldots, c_m)$). Since the pencil $[s\overline{E}-\overline{A} \quad \overline{B}]$ is in the closure of $\mathcal{O}([sE-A \quad B])$, it follows from Theorem 4.1 that $k \leq \overline{k}$ and

$$\sum_{i=1}^{j} \overline{c}_i' \leq \sum_{i=1}^{j} c_i', \qquad j=1,\ldots,n. \tag{67}$$

Since by assumption $\sum_{i=1}^{m} c_i = \operatorname{rk} E = \operatorname{rk} \overline{E} = \sum_{i=1}^{m} \overline{c}_i$, (67) is equivalent to

$$\sum_{i=1}^{j} \overline{c}_i \geq \sum_{i=1}^{j} c_i, \qquad j=1,\ldots,m. \tag{68}$$

Let $\overline{F} \in \mathbb{K}^{m\times n}$ and let $\overline{p}_1(s),\ldots,\overline{p}_m(s),1,\ldots,1$ be the n invariant factors of the pencil $s\overline{E}-\overline{A}+\overline{B}\,\overline{F}$. From the main theorem in [15] and (68), it follows that

$$\sum_{i=1}^{j} \deg\ \overline{p}_i(s) \geq \sum_{i=1}^{j} \overline{c}_i \geq \sum_{i=1}^{j} c_i, \quad j=1,\ldots,m.$$

Again applying the main theorem of [15] it follows that the invariant polynomials

$$\overline{p}_1(s),\ldots,\overline{p}_m(s),1,\ldots,1$$

are the invariant factors of the pencil $sE-A+BF$ for some feedback transformation F. □

Roughly speaking, the previous proposition can be summarized as follows: Nothing new is gained as long as neither controllability is sacrificed nor $\operatorname{rk} E$ is strictly decreased.

On the other extreme, if we do not require any controllability property of the high gain feedback limit we will see that *all* lists of invariant factors may occur.

Proposition 5.8. Let $p_1(s),\ldots,p_n(s)$ be monic polynomials in $\mathbb{K}[s]$ such that $p_{i+1}(s)$ divides $p_i(s)$, $i=1,\ldots,n-1$, and $\sum_{i=1}^{n} \deg\ p_i(s) = n$. If $\dot{x} = Ax+Bu$ is a controllable state space system, then there is a high gain feedback limit $\dot{x} = \overline{A}x+\overline{B}u$ of $\dot{x} = Ax+Bu$ (in the sense of HGF1 or HGF2) such that the invariant factors of the pencil $sI-\overline{A}$ are $p_1(s),\ldots,p_n(s)$.

Proof. Let $\overline{A}$ be an $n\times n$ matrix such that the invariant factors of $sI-\overline{A}$ are $p_1(s),\ldots,p_n(s)$. Then the $n\times(n+m)$ pencil $[sI-\overline{A} \quad 0]$ has only zero column indices and invariant factors $f_j = p_j$, $j=1,\ldots,n$. Because $\dot{x} = Ax+Bu$ is controllable, we have $D_j([sI-A \quad B]) = 1$ for all $j=1,\ldots,n$. It follows from Theorem 4.6 that $\dot{x} = \overline{A}x$ is a limit of $\dot{x} = Ax+Bu$ under high gain feedback. (Recall that, in the case of state space systems, the high gain feedback concepts HGF1 and HGF2 coincide). □

The previous two propositions show an interesting trade-off between the gain in assignability of invariant factors and the loss in controllability. If controllability is to be preserved in the limit no new lists of invariant factors can be assigned in the high gain feedback limit (compared to ordinary state feedback). If, however, we are willing to sacrifice controllability completely, then all possible lists of invariant factors can be produced in the limit.

APPENDIX. GLOSSARY

$$\mathcal{P}_{n,m}(\mathbb{K}) = \mathbb{K}^{n\times 2(n+m)} = \text{the set of } n\times(n+m) \text{ pencils } sM+N \text{ over } \mathbb{K}$$

$$\mathcal{O}(sM+N) = \{P(sM+N)Q : P\in Gl_n(\mathbb{K}),\ Q\in Gl_{n+m}(\mathbb{K})\} \text{ where } sM+N\in\mathcal{P}_{n,m}(\mathbb{K})$$

$$\mathcal{S}_{n,m}(\mathbb{K}) = \{[sE-A \quad B] : E,A\in\mathbb{K}^{n\times n}, B\in\mathbb{K}^{n\times m}\}$$

$$\mathcal{S}^{\text{reg}}_{n,m}(\mathbb{K}) = \{[sE-A \quad B]\in\mathcal{S}_{n,m}(\mathbb{K}) :\ \det(sE-A)\not\equiv 0\}.$$

Strong equivalence: $[sE-A \quad B]\sim[sLER-LAR \quad L(B-AX)]$ where $L,R\in Gl_n(\mathbb{K}),\ X\in\mathbb{K}^{n\times m},\ EX=0.$

Restricted system equivalence: $[sE-A \quad B]\sim[sLER-LAR \quad LB]$ where $L,R\in Gl_n(\mathbb{K})$.

Full feedback group:

$$\mathcal{F}_{n,m}(\mathbb{K}) = \left\{\begin{bmatrix} R & 0\\ F & W\end{bmatrix} : R\in Gl_n(\mathbb{K}),\ W\in Gl_m(\mathbb{K}), F\in\mathbb{K}^{m\times n}\right\}.$$

Action of $\mathcal{F}_{n,m}(\mathbb{K})$ on the set of state space systems:

$$\begin{bmatrix} R & 0\\ F & W\end{bmatrix}\cdot[sI-A \quad B] = R[sI-A \quad B]\begin{bmatrix} R & 0\\ F & W\end{bmatrix}^{-1}.$$

Generalized feedback group: $\mathcal{G}_{n,m}(\mathbb{K}) = Gl_n(\mathbb{K})\times\mathcal{F}_{n,m}(\mathbb{K})$.
Action of $\mathcal{G}_{n,m}(\mathbb{K})$ on $\mathcal{S}_{n,m}(\mathbb{K})$:

$$\left(P,\begin{bmatrix} R & 0\\ F & W\end{bmatrix}\right)\cdot[sE-A \quad B] = P[sE-A \quad B]\begin{bmatrix} R & 0\\ F & W\end{bmatrix}^{-1}.$$

Limits under high gain feedback ($\text{cl}(\cdot)$ denotes topological closure):

HGF1: The system $\overline{E}\dot{x} = \overline{A}x + \overline{B}u$ is a *high gain feedback limit* of the system $E\dot{x} = Ax + Bu$ *in the sense* HGF1 if

$$[s\overline{E}-\overline{A} \quad \overline{B}]\in\text{cl}\,(\mathcal{O}([sE-A \quad B])\cap\mathcal{S}_{n,m}(\mathbb{K}))$$

HGF2: The system $\overline{E}\dot{x} = \overline{A}x + \overline{B}u$ is a *high gain feedback limit* of the system $E\dot{x} = Ax + Bu$ *in the sense* HGF2 if

$$[s\overline{E}-\overline{A} \quad \overline{B}]\in\text{cl}\,(\mathcal{G}_{n,m}(\mathbb{K})\cdot[sE-A \quad B]).$$

A system $E\dot{x} = Ax + Bu$ is *controllable* if

$$\text{rk}[\alpha E-A \quad B] = n \text{ for all } \alpha\in\mathbb{C} \text{ and } \operatorname{Im}E + A\operatorname{Ker}E + \operatorname{Im}B = \mathbb{K}^n.$$

A system $E\dot{x} = Ax + Bu$ is *reachable* if

$$\operatorname{rk}[\alpha E + \beta A \quad B] = n \text{ for all } (\alpha, \beta) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{69}$$

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