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BAYESIAN ANALYSIS OF THE MODEL OF HIDDEN PERIODICITIES

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Consider a model of hidden periodicities $X_t = Y_t + \sum_{i=1}^k (a_i \cos\omega_i t + b_i \sin\omega_i t)$, $t = 1, \ldots, 2m+1$. It is assumed that Y_t are i.i.d. $N(0, \sigma^2)$ variables and that $\omega_i \in \{\lambda_1, \ldots, \lambda_m\}$ where $\lambda_r = 2\pi r/(2m+1)$. Let a_i, b_i and σ have a vague prior distribution and let the vector $(\omega_1, \ldots, \omega_k)'$ have a rectangular distribution. The posterior distribution of the parameters is derived and its asymptotic properties are investigated. The results can be used for estimating the number of periodical components k.

1. INTRODUCTION

Many observed time series are directly or indirectly influenced by periodically repeated events. One of the most popular models for describing such time series has been the model of hidden periodicities

$$X_t = Y_t + \sum_{i=1}^k (a_i \cos\omega_i t + b_i \sin\omega_i t), \quad t = 1, \ldots, N \tag{1}$$

where $X_1, \ldots, X_N$ is the observed time series, $\{Y_t\}$ are i.i.d. $N(0, \sigma^2)$ variables with $\sigma^2 > 0$ and $\omega_i \in (0, \pi]$ for $i = 1, \ldots, k$. We assume that $\omega_1, \ldots, \omega_k$ are different. A basic tool for investigating the model (1.1) is the periodogram

$$\begin{aligned} I(\lambda) &= \frac{1}{2\pi N} \left| \sum_{t=1}^N X_t e^{-it\lambda} \right|^2 = \\ &= \frac{1}{2\pi N} \left\{ \left(\sum_{t=1}^N X_t \cos\lambda t \right)^2 + \left(\sum_{t=1}^N X_t \sin\lambda t \right)^2 \right\}, \quad \lambda \in [0, \pi]. \end{aligned}$$

The periodogram was introduced by Schuster [30]. Later on, periodicities in disturbed series were investigated by Yule [38].

We shall consider only the case that the frequencies ω_i are not known. The first problem is to test if the periodic component in (1.1) is present or not. It means that we want to test $H_0 : \sum_{i=1}^k (a_i^2 + b_i^2) = 0$ against $H_1 : \sum_{i=1}^k (a_i^2 + b_i^2) > 0$. To test

H_0, we can apply the famous Fisher test of periodicity (see [13, 14, 2]). It is assumed that N is an odd number, $N = 2m + 1$. Define

$$\lambda_r = \frac{2\pi r}{N}, \qquad I_r = I(\lambda_r), \qquad r = 1, \dots, m.$$

Denote

$$I_{(1)} \geq I_{(2)} \geq \cdots \geq I_{(m)}$$

the ordered values of the periodogram. The Fisher test statistic is

$$F = \frac{I_{(1)}}{I_1 + I_2 + \cdots + I_m}. \tag{2}$$

If F exceeds the critical value we reject H_0.

Modifications of Fisher's test and other tests based on the periodogram were proposed in [19, 31, 22, 3, 6].

The Fisher test has good properties if $k = 1$. For $k > 1$ the power of the test can be rather low. In this case the Siegel test can be recommended (see [32]). A formula for asymptotic percentage points for Siegel's test was derived in [34].

Quinn [27] introduces a method for estimating the number of frequencies k. A test of periodicity in multiple time series can be found in [24].

A generalization of the Fisher test to the case that $\{Y_t\}$ in (1.1) are dependent variables was proposed by Whittle [36, 37]. However, the power of his test is rather low (see [26]) and so the problem was further investigated by Hannan [14, 15], Priestley [25, 26] and Cipra [8]. Not all proposed procedures are based on the periodogram directly. For example, a test suggested by Priestley [25, 26] is based on the correlogram approach. Kedem [21] also presents a method which does not use the periodogram.

Since the Fisher test looks for the frequencies ω_i only in the set

$$\Lambda = \{\lambda_1, \dots, \lambda_m\},$$

Cipra [7] suggested a modification for the case that ω_i can be between two Fisher's frequencies λ_j and λ_{j+1}. Tests of periodicity when some observations are missing can be found in Cipra [9].

Several papers are devoted to the problem of estimating frequencies ω_i and to the asymptotic properties of the corresponding estimates ([36, 35, 16, 17, 10, 18, 5, 23, 28, 29]). A review is given by Brillinger [4]. Statistical properties of the maximum of the periodogram are described in [1].

Our paper has two main parts. First, Section 2 contains a Bayesian analysis of the model (1.1). It is assumed that there are exactly k periodicities in (1.1) and all their frequencies ω_i are of the form $2\pi r_i/N$. The parameters a_i, b_i have a vague prior density. The assumption that $\omega_i = 2\pi r_i/N$ $(i = 1, \dots, k)$ is rather restrictive. On the other hand, in many cases when a routine statistical analysis of real data is carried out, the investigator confines himself to Fisher's or Siegel's tests and so, in fact, he also considers only f¿equencies $\omega_i = 2\pi r_i/N$. Then our approach can give some additional information.

The second main part of our paper (Section 4) deals with asymptotics. Here we assume that the length $N_j = 2m_j + 1$ ($m_j \to \infty$) is chosen in such a way that the frequencies $2\pi r_i/(2m_j+1)$ are all of the form $2\pi r/N$ for the initial N. This restriction has the following reason. If the frequency of a harmonic component falls approximately mid–way between the two periodogram ordinates, the height of the ordinate is reduced by a factor $4/\pi^2$ (see [36]; cf. [26]). This would be a source of difficulties if general sequences $\{N_j\}$ were allowed.

A simulation study shows that the derived results can also indicate the number k of the periodic components.

2. BAYESIAN APPROACH

Consider the model (1.1) and assume that $a_i^2 + b_i^2 > 0$ for $i = 1, \dots, k$. Define

$$\Lambda_k = \{(\lambda_{i_1}, \dots, \lambda_{i_k})' : \lambda_{i_1} < \dots < \lambda_{i_k} \text{ are elements of } \Lambda\}.$$

Assume that $1 \le k \le m$. Introduce the following notation:

$$\boldsymbol{a} = (a_1, \dots, a_k)', \quad \boldsymbol{b} = (b_1, \dots, b_k)', \quad \boldsymbol{\omega} = (\omega_1, \dots, \omega_k)',$$

$$\Omega = \{\omega_1, \dots, \omega_k\}, \quad \bar{X} = \frac{1}{N}\sum_{t=1}^{N} X_t, \quad Q = \sum_{t=1}^{N} X_t^2,$$

$$C(\lambda) = \sum_{t=1}^{N} X_t \cos \lambda t, \quad S(\lambda) = \sum_{t=1}^{N} X_t \sin \lambda t, \quad P(\lambda) = C^2(\lambda) + S^2(\lambda), \quad \lambda \in \Lambda.$$

Theorem 2.1. Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_k)' \in \Lambda_k$. Define

$$\hat{a}_i = \frac{2}{N} C(\omega_i), \qquad \hat{b}_i = \frac{2}{N} S(\omega_i). \tag{3}$$

Then the density of the vector $\boldsymbol{X} = (X_1, \dots, X_N)'$ is

$$f(\boldsymbol{x}|\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\omega}, \sigma) = (2\pi)^{-N/2} \sigma^{-N} \exp\left\{ -\frac{Z}{2\sigma^2} \right\} \tag{4}$$

where

$$Z = \frac{N}{2} \sum_{i=1}^{k} [(a_i - \hat{a}_i)^2 + (b_i - \hat{b}_i)^2] + Q - \frac{2}{N} \sum_{i=1}^{k} P(\omega_i). \tag{5}$$

P r o o f. It is clear that

$$f(\boldsymbol{x}|\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\omega}, \sigma) = (2\pi)^{-N/2} \sigma^{-N} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{t=1}^{N} \left[X_t - \sum_{i=1}^{k} (a_i \cos \omega_i t + b_i \sin \omega_i t) \right]^2 \right\}.$$

But for $\boldsymbol{\omega} \in \Lambda_k$ we have

$$\sum_{t=1}^{N} \cos^2 \omega_i t = \sum_{t=1}^{N} \sin^2 \omega_i t = \frac{N}{2}, \qquad \sum_{t=1}^{N} \cos \omega_i t \sin \omega_i t = 0 \tag{6}$$

and

$$\sum_{t=1}^{N} \cos \omega_i t \cos \omega_j t = \sum_{t=1}^{N} \sin \omega_i t \sin \omega_j t = 0 \qquad \text{for } i \neq j. \tag{7}$$

After some computation we get the assertion of the theorem. □

The well known problem in the Bayesian approach is the choice of a prior distribution. In our case we investigate the situation when $\boldsymbol{a}$, $\boldsymbol{b}$ and σ have prior density σ^{-1} for $\sigma > 0$ and zero otherwise. The density σ^{-1} for σ is quite common in the Bayesian analysis. It reflects the fact that $\sigma > 0$. It is supposed that $\ln \sigma$ has the improper rectangular density on the real line. The vague prior density has some advantages, e. g. it "approximates" any other reasonable prior density in such a way that the posterior probabilities do not differ too much (see [12, § 10.4, Theorem 1]). Moreover, the modus $\hat{\boldsymbol{a}}$, $\hat{\boldsymbol{b}}$ of the posterior distribution is identical with the maximum likelihood estimate (MLE) and the modus $\hat{\sigma}$ is nearly identical with the corresponding MLE. Thus our choice of the prior distribution gives also some information about the behaviour of MLE's.

In our paper the symbols like $c(\boldsymbol{x})$ denote constants which may generally depend on $\boldsymbol{x}$.

Theorem 2.2. Let $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{\omega}$, σ be independent vectors and variables. Let $\boldsymbol{a}$, $\boldsymbol{b}$, σ have prior density σ^{-1} for $\sigma > 0$ and zero for $\sigma \leq 0$. Let $\boldsymbol{\omega} \in \Lambda_k$ have the rectangular distribution on Λ_k. Then the modus $\hat{\boldsymbol{a}}$, $\hat{\boldsymbol{b}}$, $\hat{\boldsymbol{\omega}}$, $\hat{\sigma}$ of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{\omega}$, σ is given by the following rules:

(a) $\hat{\boldsymbol{\omega}}$ is the element of Λ_k which maximizes the sum $P(\omega_1) + \ldots + P(\omega_k)$.

(b) $\hat{\boldsymbol{a}} = (\hat{a}_1, \ldots, \hat{a}_k)'$, $\quad \hat{\boldsymbol{b}} = (\hat{b}_1, \ldots, \hat{b}_k)'$ where

$$\hat{a}_i = \frac{2}{N} C(\hat{\omega}_i), \qquad \hat{b}_i = \frac{2}{N} S(\hat{\omega}_i).$$

(c) $\hat{\sigma}^2 = \frac{1}{N+1} \left[Q - \frac{2}{N} \sum_{i=1}^{k} P(\hat{\omega}_i) \right]$.

Proof. From the Bayes theorem we get the joint posterior density

$$g(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\omega}, \sigma | \boldsymbol{x}) = c_0(\boldsymbol{x}) \sigma^{-N-1} \exp \left\{ -\frac{Z}{2\sigma^2} \right\} \tag{8}$$

for $\sigma > 0$ and for $\boldsymbol{\omega} \in \Lambda_k$; otherwise g vanishes. The variable Z is given in the formula (2.3). □

The modus of the posterior distribution can be used as an estimator of the unknown parameters.

Theorem 2.3. The marginal posterior density $h(\boldsymbol{\omega}|\boldsymbol{x})$ of the vector $\boldsymbol{\omega}$ with respect to the counting measure on Λ_k is

$$h(\boldsymbol{\omega}|\boldsymbol{x}) = c(\boldsymbol{x}) \left[1 - \frac{2}{N} \sum_{i=1}^{k} \frac{P(\omega_i)}{Q} \right]^{-\frac{N}{2}+k}$$

where the expression $c(\boldsymbol{x})$ is determined from the condition

$$\sum_{\boldsymbol{\omega} \in \Lambda_k} h(\boldsymbol{\omega}|\boldsymbol{x}) = 1.$$

Proof. We use the formula (2.6). First of all we calculate the marginal posterior density

$$g_1(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\omega}|\boldsymbol{x}) = \int_0^\infty g(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\omega}, \sigma|\boldsymbol{x}) \, \mathrm{d}\sigma = c_1(\boldsymbol{x}) Z^{-\frac{N}{2}}.$$

It gives

$$h(\boldsymbol{\omega}|\boldsymbol{x}) = \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} g_1(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\omega}|\boldsymbol{x}) \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}\boldsymbol{b} = c(\boldsymbol{x}) \left[1 - \frac{2}{N} \sum_{i=1}^{k} \frac{P(\omega_i)}{Q} \right]^{-\frac{N}{2}+k}.$$

□

3. A MODIFICATION

The computation of $h(\boldsymbol{\omega}|\boldsymbol{x})$ is difficult when N and k are large. Thus we propose a modification of the above results. Define

$$p(\omega_i) = \exp \left\{ \frac{P(\omega_i)}{Q} \right\}.$$

Instead of h introduce now the posterior density

$$v(\boldsymbol{\omega}|\boldsymbol{x}) = C_0(\boldsymbol{x}) p(\omega_1) p(\omega_2) \dots p(\omega_k), \qquad \boldsymbol{\omega} \in \Lambda_k$$

where $C_0(\boldsymbol{x})$ is determined from the condition

$$\sum_{\boldsymbol{\omega} \in \Lambda_k} v(\boldsymbol{\omega}|\boldsymbol{x}) = 1.$$

The function $v(\boldsymbol{\omega}|\boldsymbol{x})$ can be considered as an approximation of the function $h(\boldsymbol{\omega}|\boldsymbol{x})$. In Section 4 we prove that the asymptotic properties of the both functions h and v are the same.

But $P(\omega_i)/Q$ can be very large, in a limit case it can reach even the value $N/2$. Denote

$$P_{\max} = \max\{P(\omega_1), \dots, P(\omega_m)\}, \qquad q(\omega_i) = \exp \left\{ \frac{P(\omega_i) - P_{\max}}{Q} \right\}.$$

Then

$$v(\boldsymbol{\omega}|\boldsymbol{x}) = C(\boldsymbol{x})q(\omega_1)q(\omega_2)\dots q(\omega_k).$$

If we put

$$s_k = \sum_{\substack{\{\omega_1,\dots,\omega_k:\omega_1,\dots,\omega_k \\ \text{are different elements of } \Lambda\}}} q(\omega_1)q(\omega_2)\dots q(\omega_k)$$

then $C(\boldsymbol{x}) = k!/s_k$. The values s_k can be namely calculated using the tables published by David, Kendall [11]. If we use the same notation as that in the cited paper, viz.

$$(r) = \sum_{i=1}^{m} [q(\omega_i)]^r$$

then it holds

$$\begin{aligned}
s_2 &= -(2) + (1)^2, \\
s_3 &= 2(3) - 3(2)(1) + (1)^3, \\
s_4 &= -6(4) + 8(3)(1) + 3(2)^2 - 6(2)(1)^2, \\
s_5 &= 24(5) - 30(4)(1) - 20(3)(2) + 20(3)(1)^2 + 15(2)^2(1) - 10(2)(1)^3 + (1)^5, \\
s_6 &= -120(6) + 144(5)(1) + 90(4)(2) - 90(4)(1)^2 + 40(3)^2 - 120(3)(2)(1) + 40(3)(1)^3 \\
&\quad -15(2)^3 + 45(2)^2(1)^2 - 15(2)(1)^4 + (1)^6.
\end{aligned}$$

The tables by David and Kendall enable to extend these formulas to the expression for s_{12}.

4. ASYMPTOTICS

In this section we investigate the limit behaviour of the posterior probabilities and related variables for the case that $m \to \infty$. We assume that k is fixed and that $\Omega \subset \Lambda$ for all sufficiently large m. Generally, Ω may depend on m but we do not denote it explicitly. It would be also possible to consider a fixed set Ω where $\Omega \subset \Lambda$ for some $m = m_0$ and then to deal with a sequence $m_0 < m_1 < m_2 \dots$ such that $\Omega \subset \Lambda$ for every m_j, $j \ge 0$.

First, we remember a definition. A sequence of random variables $\{U_n, n \ge 1\}$ is said to converge completely to a constant c if

$$\sum_{n=1}^{\infty} P[|U_n - c| > \varepsilon] < \infty \qquad \text{for each } \varepsilon > 0. \tag{9}$$

This definition is due to Hsu and Robbins [20]. It is well known that the condition (4.1) ensures that $U_n \to c$ a.s. but the converse does not hold (see [33, p. 11, Theorem 2.1.1]).

Theorem 4.1. Let $\{a_{ni}, i \geq 1, n \geq 1\}$ denote a matrix of real numbers. Let $\xi_1, \xi_2, \ldots$ be i.i.d. random variables. Define

$$C_n = \sum_{i=1}^{n} a_{ni}^2, \qquad T_n = \sum_{i=1}^{n} a_{ni}\xi_i.$$

Let $E|\xi_1|^{2/\alpha} < \infty$ for some $0 < \alpha \leq 1$, $E\xi_1 = 0$,

$$|a_{ni}| \leq Kn^{-\alpha} \qquad \text{for } i \leq n \text{ and some } K < \infty$$

and

$$C_n = o\left(\frac{1}{\ln n}\right).$$

Then T_n converges completely to zero as $n \to \infty$.

P r o o f. See [33, p. 226, Theorem 4.1.3]. □

Lemma 4.2. Let $\{c_{ni}, i \geq 1, n \geq 1\}$ be a matrix of real numbers such that $|c_{ni}| \leq 1$ for $i \leq n$. Let $\xi_1, \xi_2, \ldots$ be i.i.d. random variables with $E\xi_1 = 0$, $E\xi_1^2 < \infty$. Define

$$T_n = \frac{1}{n}\sum_{i=1}^{n} c_{ni}\xi_i. \tag{10}$$

Then T_n converges completely to zero as $n \to \infty$.

P r o o f. The assertion follows from Theorem 4.1 when we put $\alpha = 1$ and $a_{ni} = c_{ni}/n$. □

Lemma 4.3. If $\{X_t\}$ is given by (1.1) and if $\boldsymbol{\omega} \in \Lambda_k$ for all sufficiently large m then

$$\frac{1}{N}Q \to R = \sigma^2 + \frac{1}{2}\sum_{i=1}^{k}(a_i^2 + b_i^2) \qquad \text{a.s. as } N \to \infty.$$

P r o o f. Using (2.4) and (2.5) we get

$$Q = \sum_{t=1}^{N} Y_t^2 + 2\sum_{t=1}^{N} Y_t \sum_{i=1}^{k}(a_i \cos\omega_i t + b_i \sin\omega_i t) + \frac{N}{2}\sum_{i=1}^{k}(a_i^2 + b_i^2).$$

Since we assume that $\{Y_t\}$ are i.i.d. $N(0, \sigma^2)$ variables, the remaining part of the proof follows from Lemma 4.2 and from the strong law of large numbers. □

Lemma 4.4. We have

$$Q \geq \frac{2}{N} \sum_{i=1}^{m} P(\lambda_i).$$

P r o o f. The inequality follows from the known formula

$$\sum_{t=1}^{N} (X_t - \bar{X})^2 = \frac{2}{N} \sum_{i=1}^{m} P(\lambda_i)$$

(see [2, p. 85, Theorem 7.4]) because

$$Q \geq \sum_{t=1}^{N} (X_t - \bar{X})^2.$$

□

Lemma 4.5. Let $N \to \infty$. If $\omega_i \in \Lambda$ then

$$\frac{1}{N^2} P(\omega_i) \to \frac{1}{4}(a_i^2 + b_i^2) \qquad \text{a. s.}$$

If $\lambda \in \Lambda$ and $\lambda \notin \Omega \subset \Lambda$ then

$$\frac{1}{N^2} P(\lambda) \to 0 \qquad \text{a. s.}$$

P r o o f. Using (2.4) and (2.5) we obtain

$$C(\lambda) = \begin{cases} \frac{1}{2} N a_i + \sum_{t=1}^{N} Y_t \cos \omega_i t & \text{for } \lambda = \omega_i \in \Omega \subset \Lambda, \\ \sum_{t=1}^{N} Y_t \cos \lambda t & \text{for } \lambda \in \Lambda,\ \lambda \notin \Omega \subset \Lambda, \end{cases}$$

$$S(\lambda) = \begin{cases} \frac{1}{2} N b_i + \sum_{t=1}^{N} Y_t \sin \omega_i t & \text{for } \lambda = \omega_i \in \Omega \subset \Lambda, \\ \sum_{t=1}^{N} Y_t \sin \lambda t & \text{for } \lambda \in \Lambda,\ \lambda \notin \Omega \subset \Lambda. \end{cases}$$

From here we get the assertion. □

Lemma 4.6. If $\Omega \subset \Lambda$ then for any $\varepsilon > 0$ and for all sufficiently large m we have

$$\sum_{\{i:\lambda_i \in \Lambda, \lambda_i \notin \Omega\}} \frac{1}{N^2} P(\lambda_i) < \frac{1}{2}\sigma^2 + \varepsilon \qquad \text{a. s.}$$

Proof. Lemma 4.4 gives

$$\frac{1}{N}Q \geq 2 \sum_{\{i:\lambda_i \in \Omega\}} \frac{1}{N^2} P(\lambda_i) + 2 \sum_{\{i:\lambda_i \in \Lambda, \lambda_i \notin \Omega\}} \frac{1}{N^2} P(\lambda_i)$$

Now, we use Lemma 4.3 and Lemma 4.5. □

Lemma 4.7. If $m \to \infty$ then

$$\max_{\{i:\lambda_i \in \Lambda, \lambda_i \notin \Omega\}} \frac{1}{N^2} P(\lambda_i) \to 0 \qquad \text{a. s.}$$

Proof. Since $N = 2m + 1$, we can denote the variables

$$\left\{ \frac{1}{N^2} P(\lambda_i),\ \lambda_i \in \Lambda, \lambda_i \notin \Omega \right\}$$

shortly by

$$\xi_{m,1}, \dots, \xi_{m,m-k}.$$

The variables $\{C(\lambda_i), S(\lambda_i),\ \lambda_i \in \Lambda, \lambda_i \notin \Omega\}$ have joint normal distribution and because of (2.4) and (2.5) they all are uncorrelated. Thus $\xi_{m,1}, \dots, \xi_{m,m-k}$ are independent. Lemma 4.6 gives that for all sufficiently large m we have

$$\xi_{m,1} + \ldots + \xi_{m,m-k} < \sigma^2 \qquad \text{a. s.}$$

Then for all $i = 1, \dots, m - k$ we obtain

$$P\left(\xi_{m,i} \geq \frac{\sigma^2}{m-k}\right) = 0$$

and thus

$$\max\{\xi_{m,1}, \dots, \xi_{m,m-k}\} < \frac{\sigma^2}{m-k} \qquad \text{a. s.} \tag{11}$$

Now, we let $m \to \infty$. □

Introduce variables $A_i = a_i^2 + b_i^2$, $i = 1, \dots, k$. Further define $A_i = 0$ for $i > k$ and $A = A_1 + \ldots + A_k$. In our Bayesian approach we assume that $a_1, b_1, \dots, a_k, b_k$ are independent and have the vague prior density which is equal to 1 on $\mathbb{R}^{2k}$. Then all the variables $A_1, \dots, A_k$ will be different. To simplify the next derivation we shall assume that the frequencies $\omega_1, \dots, \omega_k$ are ordered in such a way that $A_1 > A_2 > \ldots > A_k$.

Theorem 4.8. Let $\{\gamma_1, \dots, \gamma_\ell\} \subset \Lambda$ where $1 \leq \ell \leq m$. Put $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_\ell)'$. Define

$$p(\gamma_i) = \exp\left\{\frac{P(\gamma_i)}{Q}\right\}$$

and

$$h(\boldsymbol{\gamma}|\boldsymbol{x}) = c(\boldsymbol{x}) \left[1 - \frac{2}{N} \sum_{i=1}^{\ell} \frac{P(\gamma_i)}{Q} \right]^{-\frac{N}{2}+k}, \qquad v(\boldsymbol{\gamma}|\boldsymbol{x}) = C(\boldsymbol{x}) p(\gamma_1) p(\gamma_2) \dots p(\gamma_\ell)$$

where $c(\boldsymbol{x})$ and $C(\boldsymbol{x})$ are positive constants determined from the conditions

$$\sum_{\boldsymbol{\gamma} \in \Lambda_\ell} h(\boldsymbol{\gamma}|\boldsymbol{x}) = 1 \qquad \text{and} \qquad \sum_{\boldsymbol{\gamma} \in \Lambda_\ell} v(\boldsymbol{\gamma}|\boldsymbol{x}) = 1,$$

respectively. Assume that $m \to \infty$.

Let $\ell \le k$. If $\{\gamma_1, \dots, \gamma_\ell\} = \{\omega_1, \dots, \omega_\ell\}$ then $h(\boldsymbol{\gamma}|\boldsymbol{x}) \to 1$ a. s. and $v(\boldsymbol{\gamma}|\boldsymbol{x}) \to 1$ a. s.; if $\{\gamma_1, \dots, \gamma_\ell\} \neq \{\omega_1, \dots, \omega_\ell\}$ then $h(\boldsymbol{\gamma}|\boldsymbol{x}) \to 0$ a. s. and $v(\boldsymbol{\gamma}|\boldsymbol{x}) \to 0$ a. s.

Let $\ell > k$. Then $h(\boldsymbol{\gamma}|\boldsymbol{x}) \to 0$ a. s. and $v(\boldsymbol{\gamma}|\boldsymbol{x}) \to 0$ a. s. for arbitrary $\{\gamma_1, \dots, \gamma_\ell\} \subset \Lambda$.

P r o o f . In the first part of the proof we deal with the assertions concerning the function h. Assume $\ell \le k$ and $\{\gamma_1, \dots, \gamma_\ell\} = \{\omega_1, \dots, \omega_\ell\}$. Then we have

$$h(\omega_1, \dots, \omega_\ell|\boldsymbol{x}) = \frac{1}{1+D}$$

where

$$D = \sum_{\substack{\{\gamma_1, \dots, \gamma_\ell : P(\gamma_1) > \dots > P(\gamma_\ell),\\ \{\gamma_1, \dots, \gamma_\ell\} \neq \{\omega_1, \dots, \omega_\ell\}\}}} \left[\frac{1 - \frac{2}{N} \sum_{i=1}^{\ell} \frac{P(\omega_i)}{Q}}{1 - \frac{2}{N} \sum_{i=1}^{\ell} \frac{P(\gamma_i)}{Q}} \right]^{\frac{N}{2}-k}.$$

It follows from Lemmas 4.3, 4.5 and 4.7 that

$$\frac{2}{N} \sum_{i=1}^{\ell} \frac{P(\omega_i)}{Q} \to \frac{A_1 + \dots + A_\ell}{2\sigma^2 + A} \qquad \text{a. s.}$$

and

$$\frac{2}{N} \sum_{i=1}^{\ell} \frac{P(\gamma_i)}{Q}$$

tends a. s. to a non-negative limit which does not exceed

$$\frac{A_1 + \dots + A_{\ell-1} + A_{\ell+1}}{2\sigma^2 + A}.$$

Thus for any arbitrary small $\varepsilon > 0$ and for all sufficiently large m we have a. s. that

$$0 \le \frac{1 - \frac{2}{N} \sum_{i=1}^{\ell} \frac{P(\omega_i)}{Q}}{1 - \frac{2}{N} \sum_{i=1}^{\ell} \frac{P(\gamma_i)}{Q}} \le \frac{2\sigma^2 + A - A_1 - \dots - A_\ell}{2\sigma^2 + A - A_1 - \dots - A_{\ell-1} - A_{\ell+1}} + \varepsilon = \alpha.$$

We can choose such a small ε that $\alpha < 1$. Then for all sufficiently large m we have

$$D \le \left[\binom{m}{\ell} - 1\right] \alpha^{\frac{N}{2}-k} \to 0.$$

Now, consider the case $\ell < k$ when $\{\gamma_1, \ldots, \gamma_\ell\} \ne \{\omega_1, \ldots, \omega_\ell\}$. We get

$$h(\gamma|\boldsymbol{x}) = \frac{1}{1+D}$$

where

$$D = \sum_{\substack{\{\beta_1,\ldots,\beta_\ell : P(\beta_1) > \ldots > P(\beta_\ell), \\ \boldsymbol{\beta} \ne \boldsymbol{\gamma}\}}} \left\{ \frac{1 - \frac{2}{N}\sum\limits_{i=1}^{\ell} \frac{P(\gamma_i)}{Q}}{1 - \frac{2}{N}\sum\limits_{i=1}^{\ell} \frac{P(\beta_i)}{Q}} \right\}^{\frac{N}{2}-k}.$$

For sufficiently large N we obtain

$$D \ge Z^{\frac{N}{2}-k}$$

where

$$Z = \frac{1 - \frac{2}{N}\sum\limits_{i=1}^{\ell-1} \frac{P(\omega_i)}{Q} - \frac{2}{N}\frac{P(\omega_{\ell+1})}{Q}}{1 - \frac{2}{N}\sum\limits_{i=1}^{\ell} \frac{P(\omega_i)}{Q}}.$$

If $N \to \infty$ then

$$Z \to \frac{2\sigma^2 + A - A_1 - \ldots - A_{\ell-1} - A_{\ell+1}}{2\sigma^2 + A - A_1 - \ldots - A_\ell} \ge 1 \qquad \text{a.s.}$$

and thus $D \to \infty$ a.s.

The case $\ell = k$, $\{\gamma_1, \ldots, \gamma_k\} \ne \{\omega_1, \ldots, \omega_k\}$ can be treated analogously using lemmas 4.3 and 4.5.

Now, assume $\ell > k$. For large m the maximum of $h(\boldsymbol{\gamma}|\boldsymbol{x})$ is reached at a point $\boldsymbol{\gamma} = (\omega_1, \ldots, \omega_k, \beta_{k+1}, \ldots, \beta_\ell)'$ where β_i are some frequencies from Λ. Then

$$h(\omega_1, \ldots, \omega_k, \beta_{k+1}, \ldots, \beta_\ell) = \frac{1}{1+D}$$

where

$$\begin{aligned} D &= \sum_{\substack{\{\gamma_1,\ldots,\gamma_\ell : P(\gamma_1) > \ldots > P(\gamma_\ell), \\ \{\gamma_1,\ldots,\gamma_\ell\} \ne \{\omega_1,\ldots,\omega_k,\beta_{k+1},\ldots,\beta_\ell\}\}}} \left\{ \frac{1 - \frac{2}{N}\sum\limits_{i=1}^{k} \frac{P(\omega_i)}{Q} - \frac{2}{N}\sum\limits_{i=k+1}^{\ell} \frac{P(\beta_i)}{Q}}{1 - \frac{2}{N}\sum\limits_{i=1}^{\ell} \frac{P(\gamma_i)}{Q}} \right\}^{\frac{N}{2}-k} \\ &\ge \sum_{\substack{\{\gamma_{k+1},\ldots,\gamma_\ell : P(\gamma_{k+1}) > \ldots > P(\gamma_\ell), \gamma_{k+1} \notin \Omega, \ldots, \gamma_\ell \notin \Omega, \\ \{\gamma_{k+1} \cdots \gamma_\ell\} \ne \{\beta_{k+1},\ldots,\beta_\ell\}\}}} \left\{ \frac{1 - \frac{2}{N}\sum\limits_{i=1}^{k} \frac{P(\omega_i)}{Q} - \frac{2}{N}\sum\limits_{i=k+1}^{\ell} \frac{P(\beta_i)}{Q}}{1 - \frac{2}{N}\sum\limits_{i=1}^{k} \frac{P(\omega_i)}{Q} - \frac{2}{N}\sum\limits_{i=k+1}^{\ell} \frac{P(\gamma_i)}{Q}} \right\}^{\frac{N}{2}-k} \end{aligned}$$

$$\geq \left[\binom{m-k}{\ell-k}-1\right]\left\{1-\frac{\frac{2}{N}\sum\limits_{i=k+1}^{\ell}\frac{P(\beta_i)}{Q}}{1-\frac{2}{N}\sum\limits_{i=1}^{k}\frac{P(\omega_i)}{Q}}\right\}^{\frac{N}{2}-k}.$$

From Lemmas 4.3 and 4.5 we get

$$1-\frac{2}{N}\sum_{i=1}^{k}\frac{P(\omega_i)}{Q}\to 1-\frac{A}{2R}>0$$

and (4.3) yields

$$\frac{2}{N}\sum_{i=k+1}^{\ell}\frac{P(\beta_i)}{Q}\leq 2(\ell-k)\frac{\sigma^2}{m-k}\frac{N}{Q}\sim\frac{2(\ell-k)\sigma^2}{R}\frac{1}{m-k}.$$

Thus $D\to\infty$.

Now, we prove our assertions for $v(\boldsymbol{\gamma}|\boldsymbol{x})$. Let $\ell\leq k$. Then we can write $v(\omega_1,\dots,\omega_\ell|\boldsymbol{x})=1/D$ where

$$D=\sum_{\{\gamma_1,\dots,\gamma_\ell:P(\gamma_1)>\dots>P(\gamma_\ell)\}}\exp\left\{\frac{P(\gamma_1)-P(\omega_1)}{Q}\right\}\dots\exp\left\{\frac{P(\gamma_\ell)-P(\omega_\ell)}{Q}\right\}.$$

For sufficiently large m we have $P(\omega_1)>\dots>P(\omega_\ell)>\dots>P(\omega_k)$. Then

$$D=1+\sum_{\substack{\{\gamma_1,\dots,\gamma_\ell:P(\gamma_1)>\dots>P(\gamma_\ell),\\ \{\gamma_1,\dots,\gamma_\ell\}\neq\{\omega_1,\dots,\omega_\ell\}\}}}\exp\left\{\frac{P(\gamma_1)-P(\omega_1)}{Q}\right\}\dots\exp\left\{\frac{P(\gamma_\ell)-P(\omega_\ell)}{Q}\right\}\leq 1+D_1$$

where

$$D_1=\sum_{\substack{\{\gamma_1,\dots,\gamma_\ell:P(\gamma_1)>\dots>P(\gamma_\ell),\\ \{\gamma_1,\dots,\gamma_\ell\}\neq\{\omega_1,\dots,\omega_\ell\}\}}}\exp\left\{\frac{P(\gamma_\ell)-P(\omega_\ell)}{Q}\right\}.$$

The sum D_1 has $\binom{m}{\ell}-1$ terms. It follows from lemmas 4.3 and 4.5 that for sufficiently large m we have

$$D_1\leq\left[\binom{m}{\ell}-1\right]\exp\{-c(A_\ell-A_{\ell+1})N\}$$

where c is a positive constant. Thus $D_1\to 0$ a.s. as $m\to\infty$ and the assertion is proved.

Let $\ell\leq k$ and $\{\gamma_1,\dots,\gamma_\ell\}\neq\{\omega_1,\dots,\omega_\ell\}$. Assume first that $\ell<k$. Then for large N

$$v(\gamma_1,\dots,\gamma_\ell|\boldsymbol{x})=1/D$$

where

$$D=\sum_{\substack{\{\beta_1,\dots,\beta_\ell:\\ P(\beta_1)>\dots>P(\beta_\ell),\boldsymbol{\beta}\neq\boldsymbol{\gamma}\}}}\exp\left\{\frac{P(\beta_1)-P(\gamma_1)}{Q}\right\}\dots\exp\left\{\frac{P(\beta_\ell)-P(\gamma_\ell)}{Q}\right\}$$

$$\geq \sum_{\substack{\{\beta_1,\dots,\beta_\ell:\\ P(\beta_1)>\dots>P(\beta_\ell),\boldsymbol{\beta}\neq\boldsymbol{\gamma}\}}} \exp\left\{\frac{P(\beta_1)-P(\omega_1)}{Q}\right\}\dots\exp\left\{\frac{P(\beta_{\ell-1})-P(\omega_{\ell-1})}{Q}\right\}$$
$$\times\exp\left\{\frac{P(\beta_\ell)-P(\omega_{\ell+1})}{Q}\right\}$$
$$\geq \exp\left\{\frac{P(\omega_\ell)-P(\omega_{\ell+1})}{Q}\right\}\to\infty \qquad \text{a. s.}$$

If $\ell = k$ then analogously $v(\gamma_1,\dots,\gamma_\ell|\boldsymbol{x}) = 1/D$ where

$$D \geq \exp\left\{\frac{P(\omega_\ell)-P(\beta)}{Q}\right\}, \qquad \beta\in\Lambda,\ \beta\notin\Omega.$$

Again we can see that $D\to\infty$ a. s.

Let $\ell > k$. For sufficiently large m the maximum of $v(\boldsymbol{\gamma}|\boldsymbol{x})$ is reached at a point $\boldsymbol{\gamma} = (\omega_1,\dots,\omega_k,\beta_{k+1},\dots,\beta_\ell)'$. Then $v(\omega_1,\dots,\omega_k,\beta_{k+1},\dots,\beta_\ell|\boldsymbol{x}) = 1/D$ where

$$D = \sum_{\substack{\{\gamma_1,\dots,\gamma_\ell:\\ P(\gamma_1)>\dots>P(\gamma_\ell)\}}} \frac{p(\gamma_1)\dots p(\gamma_k)p(\gamma_{k+1})\dots p(\gamma_\ell)}{p(\omega_1)\dots p(\omega_k)p(\beta_{k+1})\dots p(\beta_\ell)}$$
$$\geq \sum_{\substack{\{\gamma_{k+1},\dots,\gamma_\ell:P(\gamma_{k+1})>\dots>P(\gamma_\ell),\\ \gamma_{k+1}\notin\Omega,\dots,\gamma_\ell\notin\Omega\}}} \frac{p(\gamma_{k+1})\dots p(\gamma_\ell)}{p(\beta_{k+1})\dots p(\beta_\ell)} \geq \binom{m-k}{\ell-k}\exp\left\{-N\frac{1}{N}\sum_{i=k+1}^{\ell}\frac{p(\beta_i)}{Q}\right\}$$
$$\geq \binom{m-k}{\ell-k}\exp\left\{-\frac{(\ell-k)\sigma^2}{m-k}\,\frac{N}{Q}\right\}\sim\binom{m-k}{\ell-k}\exp\left\{\frac{(\ell-k)\sigma^2}{R}\,\frac{1}{m-k}\right\}\to\infty.$$

□

Using Theorem 4.8 we can estimate, at least asymptotically, the number k of the periodic components. The next theorem shows that the periodogram itself cannot give such an estimate.

Theorem 4.9. Let $A_1 > A_2 > \dots > A_k > 0$ be fixed numbers. Define

$$F_\ell = \frac{I_{(1)}+\dots+I_{(\ell)}}{I_1+\dots+I_m}.$$

Then for arbitrary ℓ we have $F_\ell \to 1$ a. s. as $m\to\infty$.

Proof. The assertion can be proved in a similar way as Theorem 4.8. □

5. RESULTS OF SIMULATION

A process

$$X_t = Y_t + \sum_{i=1}^{3}(a_i\cos\omega_i t + b_i\sin\omega_i t), \qquad t = 1,\dots,N$$

where $Y_t \sim N(0,2)$ was simulated with parameters introduced in Table 5.1.

Table 5.2 shows results of analysis of one realization for several values of the length N. Remember that ω_i are Fisher's frequencies such that

$$\omega_i = \frac{2\pi r_i}{N}, \qquad i = 1,2,3.$$

Table 5.1. Parameters used in a simulation

i	ω_i	a_i	b_i	$A_i = a_i^2 + b_i^2$
1	1.197	0.8	0.9	1.45
2	1.496	0.6	1.0	1.36
3	2.394	0.5	0.7	0.74

Table 5.2. Results of a simulation

$N = 21$

Fisher test: $F = 0.28,\ P = 0.49$

i	r_i	$\hat{r}_i$	$v(\hat{\omega}_1,\dots,\hat{\omega}_i\|\boldsymbol{x})$
1	4	8	0.47
2	5	2	0.11
3	8	9	0.02
4	–	5	0.00

$N = 63$

Fisher test: $F = 0.17,\ P = 0.11$

i	r_i	$\hat{r}_i$	$v(\hat{\omega}_1,\dots,\hat{\omega}_i\|\boldsymbol{x})$
1	12	24	0.68
2	15	15	0.20
3	24	25	0.02
4	–	12	0.00

$N = 189$

Fisher test: $F = 0.15,\ P = 0.00$

i	r_i	$\hat{r}_i$	$v(\hat{\omega}_1,\dots,\hat{\omega}_i\|\boldsymbol{x})$
1	36	45	0.99
2	45	72	0.57
3	72	36	0.44
4	–	6	0.00

$N = 567$

Fisher test: $F = 0.12,\ P = 0.00$

i	r_i	$\hat{r}_i$	$v(\hat{\omega}_1,\dots,\hat{\omega}_i\|\boldsymbol{x})$
1	108	135	0.98
2	135	108	0.90
3	216	216	0.50
4	–	62	0.00

Since ω_1, ω_2 and ω_3 we keep fixed, the values r_1, r_2 and r_3 are different for different values of N. The estimates $\hat{r}_i$ introduced in Table 5.2 are defined in such a way that $P(\hat{r}_1) > P(\hat{r}_2) > \dots$. For information the value F of the Fisher test given in (1.2) and its significance P are also introduced.

The difference between A_1 and A_2 is small and so even for $N = 567$ the frequency ω_2 was found more significant than ω_1. For $\ell \le k$ the convergence $v(\boldsymbol{\omega}|\boldsymbol{x}) \to 1$ is not very fast, especially for the values of ℓ which are near to k. But for $\ell > k$ the convergence $v(\boldsymbol{\omega}|\boldsymbol{x}) \to 0$ seems to be quite good. Similar numerical evidence was obtained also from other simulations which are not reported here. Thus $v(\boldsymbol{\omega}|\boldsymbol{x})$ could be used for detection of the numbers of periodicities; if $\max_{\boldsymbol{\omega}} v(\omega_1,\dots,\omega_\ell|\boldsymbol{x})$ is small then the number of periodicities is smaller than ℓ.

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