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SELF BOUNDED CONTROLLED INVARIANTS FOR SINGULAR SYSTEMS

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A number of recent results in the geometric framework have been obtained for nonsingular systems, using the notion of Self Bounded Controlled Invariant Subspace, and of Self Hidden Conditioned Invariant Subspace. The aim of this note is to extend the above mentioned notion of Self Bounded Controlled Invariant Subspace to singular systems, to investigate its dynamical properties and to study its possible applications to noninteracting control problems.

1. INTRODUCTION

The geometric theory for singular systems has been greatly developed in the last years thanks to the work of several authors [3, 5, 6, 4], and various applications of geometric notions to synthesis and design problems have been developed (see e.g. the collection of papers in [7]). From a methodological viewpoint, a key step in this process consists in extending, from the classical case, the definitions of the basic geometric objects, like the Controlled Invariant Subspace and the Conditioned Invariant Subspace, and in describing suitable algorithms for their construction.

For nonsingular systems, a number of recent results in the geometric framework have been obtained using the notion of Self Bounded Controlled Invariant Subspace, and of Self Hidden Conditioned Invariant Subspace introduced in [1]. In particular such concepts have been proved to be useful in the solution of noninteracting control problems with stability requirements [2]. The aim of this note is to extend the above mentioned notion of Self Bounded Controlled Invariant Subspace to the framework of singular systems, to investigate its dynamical properties and to study its possible applications to noninteracting control problems.

2. SELF BOUNDED CONTROLLED INVARIANT SUBSPACES

Let us consider a linear, time invariant, discrete-time singular system S described by the equations:

$$\begin{cases} E\,x(t+1) = A\,x(t) + B\,u(t) \\ y(t) = C\,x(t), \end{cases} \tag{2.1}$$

where $x(\cdot) \in \mathbb{R}^n =: \mathbf{X}$ is the state, $u(\cdot) \in \mathbb{R}^p =: \mathbf{U}$ is the input, $y(\cdot) \in \mathbb{R}^q =: \mathbf{Y}$ is the output. The matrix E is supposed to be square and singular. The system is assumed to be solvable and conditionable, i.e. such that $\operatorname{rank}[sE - A] = n$ a.e.

Given a system S of the form (2.1), we recall that a subspace $\mathcal{L}$ of the state space $\mathbf{X}$ is said *Controlled Invariant* if it satisfies:

$$A\mathcal{L} \subseteq E\mathcal{L} + \operatorname{Im}(B). \tag{2.2}$$

In the following, for any subspace $\mathcal{N}$ of $\mathbf{X}$ we will denote by $\mathrm{CI}(N)$ the class of all the Controlled Invariant Subspaces of $\mathbf{X}$ that are contained in $\mathcal{N}$. It is known that $\mathrm{CI}(N)$ is closed with respect to subspace addition and that it contains a maximum element, denoted by $\mathcal{V}^*(\mathcal{N})$. An algorithm for constructing such maximum element is described in [4].

If $\mathcal{L}$ is controlled invariant, there exists a linear map $F : \mathbf{X} \mapsto \mathbf{U}$ such that:

$$(A + BF)\mathcal{L} \subset E\mathcal{L}. \tag{2.3}$$

Any F with the above properties is called a *friend* of $\mathcal{L}$.

We can introduce now the following Definition.

Definition 2.1. Given a system S of the form (2.1), and a subspace $\mathcal{N}$ of the state space $\mathbf{X}$, a subspace $\mathcal{L} \subseteq \mathcal{N}$ is called *Self Bounded Controlled Invariant with respect to $\mathcal{N}$* if the following two conditions hold:

i) $\mathcal{L}$ belongs to $\mathrm{CI}(\mathcal{N})$, i.e. $A\mathcal{L} \subseteq \mathrm{E}\mathcal{L} + \operatorname{Im}(B)$

ii) $E\mathcal{L} \supset E\mathcal{V}^*(\mathcal{N}) \cap \operatorname{Im}(B)$,

where $\mathcal{V}^*(\mathcal{N})$ is the maximum element in $\mathrm{CI}(\mathcal{N})$.

As in [8], the following characterization of the notion of Self Bounded Controlled Invariant with respect to $\mathcal{N}$ can be given:

Proposition 2.1. A subspace $\mathcal{L} \subseteq \mathcal{N}$ is *Self Bounded Controlled Invariant with respect to $\mathcal{N}$* if and only if:

i) $E\mathcal{L} \supset E\mathcal{V}^*(\mathcal{N}) \cap \operatorname{Im}(B)$ and

ii) For all matrices K such that $(A+BK)\,\mathcal{V}^*(\mathcal{N}) \subset E\,\mathcal{V}^*(\mathcal{N})$, one has $(A+BK)\,\mathcal{L} \subset E\mathcal{L}$.

Proof. Let K be such that

$$(A + BK)\mathcal{V}^*(\mathcal{N}) \subset E\mathcal{V}^*(\mathcal{N})$$

and let K' be such that

$$(A + BK')\,\mathcal{L} \subset E\mathcal{L}.$$

Thus, $\forall x \in \mathcal{L}$ we have: $(A+BK)\,x-(A+BK')\,x = B(K-K')\,x \in E\mathcal{V}^*(\mathcal{N}) \cap \operatorname{Im}(B)$. By $E\mathcal{V}^*(\mathcal{N}) \cap \operatorname{Im}(B) \subset E\mathcal{L}$ and $(A+BK')\,x \in E\mathcal{L}$, we have that $(A+BK)\,x$ belongs to $E\mathcal{L}$. This proves one of the implications, the other being obvious. □

In the following, we will denote by $\mathrm{SBCI}(\mathcal{N})$ the class of all the Self Bounded Controlled Invariant Subspaces w.r.t. $\mathcal{N}$. The following basic theorem, that recalls the corresponding results holding for regular systems, can be stated:

Theorem 2.1. Let us assume that $\mathcal{V}^*(\mathcal{N}) \cap \mathrm{Ker} E = \{0\}$, and let $\mathcal{L} \in \mathrm{SBCI}(\mathcal{N})$. Then, if $x_0 \in \mathcal{L}$ and $u(t)$ is an input such that the corresponding trajectories $x(t)$ is contained in $\mathcal{V}^*(\mathcal{N})$, we have that $x(t)$ is contained in $\mathcal{L}$.

Proof. Let $K : \mathcal{L} \to \mathcal{U}$ be a feedback such that $(A + BK)\mathcal{L} \subseteq E\mathcal{L}$. Then, we have:

$$Ex(1) = Ax(0) + BKx(0) - BKx(0) + Bu(0) = \tag{2.4}$$

$$= (A + BK)\,x(0) + B(u(0) - Kx(0)). \tag{2.5}$$

The two terms $Ex(1)$ and $(A + BF)\,x(0)$ both belong to $E\mathcal{V}^*(\mathcal{N})$, hence the term $B(u(0) - Fx(0))$ belongs to $E\mathcal{V}^*(\mathcal{N}) \cap \mathrm{Im}(B) \subset E\mathcal{L}$. Therefore $Ex(1) \in E\mathcal{L}$, and, by $\mathcal{V}^*(\mathcal{N}) \cap \mathrm{Ker} E = \{0\}$, $x(1)$ belongs to $\mathcal{L}$. Iterating this argument, we get the thesis. □

3. PROPERTIES OF THE CLASS SBCI($\mathcal{N}$)

It is easily realized that the class $\mathrm{SBCI}(\mathcal{N})$ is closed with respect to subspace addition. Moreover, the following Proposition holds:

Proposition 3.1. If the condition $\mathcal{V}^*(\mathcal{N}) \cap \mathrm{Ker} E = \{0\}$ holds, then the class $\mathrm{SBCI}(\mathcal{N})$ is closed with respect to subspace intersection.

Proof. Given two subspaces $\mathcal{S}_1$ and $\mathcal{S}_2 \in \mathrm{SBCI}(\mathcal{N})$, let $\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2$. Let us show first that in our hypothesis $E\mathcal{S}_1 \cap E\mathcal{S}_2 = E\mathcal{S}$. To this aim let x belong to $E\mathcal{S}_1 \cap E\mathcal{S}_2$, i.e. $x = Es_1 = Es_2$ for some $s_1 \in \mathcal{S}_1, s_2 \in \mathcal{S}_2$. Hence $E(s_1 - s_2) = 0$, i.e. $(s_1 - s_2) \in \mathcal{V}^*(\mathcal{N}) \cap \mathrm{Ker} E = \{0\}$. This proves that $E\mathcal{S}_1 \cap E\mathcal{S}_2 = E\mathcal{S}$, and that $E\mathcal{S} \supset E\mathcal{V}^*(\mathcal{N}) + \mathrm{Im}(B)$.

Now, let us remark that $A\mathcal{S} = A(\mathcal{S}_1 \cap \mathcal{S}_2) \subseteq A\mathcal{S}_1 \cap A\mathcal{S}_2 \subseteq (E\mathcal{S}_1 + \mathrm{Im}(B)) \cap (E\mathcal{S}_2 + \mathrm{Im}(B))$. Then, if $x \in (E\mathcal{S}_1 + \mathrm{Im}(B)) \cap (E\mathcal{S}_2 + \mathrm{Im}(B))$, it follows that there exist some $s_1 \in \mathcal{S}_1$, $s_2 \in \mathcal{S}_2$, and b_1, $b_2 \in \mathrm{Im}(B)$ such that $x = Es_1 + b_1 = Es_2 + b_2$. Hence $E(s_1 - s_2) = b_1 - b_2 \in \mathrm{Im}(B)$. Since $s_1 - s_2 \in \mathcal{V}^*(\mathcal{N})$, this implies in particular that $E(s_1 - s_2) \in E\mathcal{S}_2$, that is $E(s_1 - s_2) = E\bar{s}_2$ for some $\bar{s}_2 \in \mathcal{S}_2$. Then $Es_1 \in E\mathcal{S}_1 \cap E\mathcal{S}_2$, $x \in E\mathcal{S}_1 \cap E\mathcal{S}_2 + \mathrm{Im}(B) = E(\mathcal{S}_1 \cap \mathcal{S}_2) + \mathrm{Im}(B)$, and $A\mathcal{S} \subseteq E\mathcal{S} + \mathrm{Im}(B)$. This proves that $\mathcal{S}$ is controlled invariant. □

Given a subspace $\mathcal{L} \subseteq \mathcal{N}$ let $\mathrm{SBCI}(\mathcal{N}, \mathcal{L})$ denote the class of all the Self Bounded Controlled Invariant Subspaces w.r.t. $\mathcal{N}$ that contain $\mathcal{L}$. By the above Proposition, it turns out that, if $\mathcal{V}^*(\mathcal{N}) \cap \mathrm{Ker} E = \{0\}$, then $\mathrm{SBCI}(\mathcal{N}, \mathcal{L})$ contains a minimum element, that will be denoted by $\mathcal{V}_*(\mathcal{N}, \mathcal{L})$. In order to give a procedure for constructing such subspace, after recalling that a subspace $\mathcal{V}$ of $\mathbf{X}$ is said (A,E)-invariant if $A\mathcal{V} \subseteq E\mathcal{V}$, let us state the following results.

Lemma 3.1. Let $\mathcal{V} \subseteq \mathbf{X}$ be an (A,E)-invariant such that $\mathcal{V} \cap \mathrm{Ker} E = \{0\}$, and let $\mathcal{L}$ be a subspace of $\mathcal{V}$. The minimum (A,E)-invariant subspace of $\mathcal{V}$ containing $\mathcal{L}$ is given by the limit of the following sequence:

$$\mathcal{V}_0 = \mathcal{L} \quad (3.1)$$
$$\mathcal{V}_i = \mathcal{V}_{i-1} + (E^{-1}A\mathcal{V}_{i-1} \cap \mathcal{V}). \quad (3.2)$$

Proof. The sequence (3.2) is increasing, and since $\mathcal{V}_i \subseteq \mathcal{V}$, the limit is reached in at most $k+1$ steps, with $k = \dim(\mathcal{V})$. Then, we have $\mathcal{V}_{k+1} = \mathcal{V}_k = \mathcal{V}_k + (E^{-1}A\mathcal{V}_k \cap \mathcal{V})$, and therefore $E^{-1}A\mathcal{V}_k \cap \mathcal{V} \subseteq \mathcal{V}_k$. Now, since $A\mathcal{V}_k \subseteq A\mathcal{V} \subseteq E\mathcal{V}$, $E(E^{-1}A\mathcal{V}_k \cap \mathcal{V})$ is equal to $A\mathcal{V}_k$, so applying E to both sides of the relation obtained previously we have $A\mathcal{V}_k \subseteq E\mathcal{V}_k$.
If $\mathcal{V}' \subset \mathcal{V}$ is an (A,E)-invariant such that $\mathcal{V}' \supset \mathcal{L}$, we have $E\mathcal{V}' \supseteq A\mathcal{V}' \supset A\mathcal{L}$, and hence $E^{-1}E\mathcal{V}' \cap \mathcal{V} \supset E^{-1}A\mathcal{L} \cap \mathcal{V}$. By $\mathcal{V} \cap \mathrm{Ker}E = \{0\}$, we have that $E^{-1}E\mathcal{V}' \cap \mathcal{V} = \mathcal{V}'$, then $\mathcal{V}' \supset \mathcal{V}_1$. Repeating the same argument we obtain that $\mathcal{V}' \supset \mathcal{V}_k$. □

In case $E\mathcal{V}^*(\mathcal{N}) \cap \mathrm{Im}(B) = \{0\}$, all the Controlled Invariant subspaces of $\mathcal{N}$ are Self Bounded and, given $\mathcal{L} \subseteq \mathcal{V}^*(\mathcal{N})$, there exists a minimum among the elements of $\mathrm{CI}(\mathcal{N})$ containing $\mathcal{L}$. A characterization of this element is provided by the following Theorem.

Theorem 3.1. Assume that $\mathcal{V}^*(\mathcal{N}) \cap \mathrm{Ker}E = \{0\}$ and $E\mathcal{V}^*(\mathcal{N}) \cap \mathrm{Im}(B) = \{0\}$. Then, given $\mathcal{L} \subseteq \mathcal{V}^*(\mathcal{N})$, the minimum Controlled Invariant subspace containing $\mathcal{L}$ is the minimum (A+BF,E)-invariant subspace of $\mathcal{V}^*(\mathcal{N})$, where F is any friend of $\mathcal{V}^*(\mathcal{N})$.

Proof. By the Lemma 3.1, the minimum (A+BF,E) invariant contained in $\mathcal{V}^*(\mathcal{N})$ and containing $\mathcal{L}$ is given by the limit $\mathcal{V}_k$ of the sequence $\mathcal{V}_0 = \mathcal{L}$; $\mathcal{V}_i = \mathcal{V}_{i-1} + (E^{-1}A\mathcal{V}_{i-1} \cap \mathcal{V}^*(\mathcal{N}))$. Clearly, $\mathcal{V}_k$ is controlled invariant. In order to show that it is minimum among the elements of $\mathrm{CI}(\mathcal{N})$ containing $\mathcal{L}$, let us consider a subspace $\mathcal{V}', \mathcal{V}' \in \mathrm{CI}(\mathcal{N})$, such that $\mathcal{V}' \supseteq \mathcal{L}$. Since $\mathcal{V}'$ is Self Bounded, by the Proposition 2.1, it follows that $(A + BF)\mathcal{V}' \subset E\mathcal{V}'$, that is $\mathcal{V}'$ is $(A + BF, E)$-invariant and contains $\mathcal{L}$. Since the minimum $(A + BF, E)$-invariant containing $\mathcal{L}$ is $\mathcal{V}_k$, $\mathcal{V}_k$ is contained in $\mathcal{V}'$. □

4. THE DISTURBANCE DECOUPLING PROBLEM

Consider the linear, time invariant, discrete-time singular system S_2 described by the equations:

$$\begin{cases} E\,x(t+1) = A\,x(t) + B\,u(t) + S\,z(t) \\ y(t) = C\,x(t). \end{cases}$$

It is known that the problem of decoupling the input z from the output by means of a feedback consists in making unobservable a controlled invariant subspace, if there exists one, containing $\mathrm{Im}S$ and contained in $\mathrm{Ker}C$. A common strategy is therefore that of checking if $\mathrm{Im}S$ is contained in $\mathcal{V}^*(\mathrm{Ker}C)$ and, in this case, in looking for a feedback F that makes $\mathcal{V}^*(\mathrm{Ker}C)(A + BF)$-invariant. The system is in this case made maximally unobservable, and this may result in the unnecessary loss of other desirable properties. When possible, it is therefore convenient to look for controlled invariant subspaces smaller than $\mathcal{V}^*(\mathrm{Ker}C)$, like, if it exists, $\mathcal{V}_*(\mathrm{Ker}C, \mathrm{Im}S)$.

A situation of this kind is described in the following example.

Let $\Sigma = (E, A, B, C)$ be given by:

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad S = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We have:

$$\mathcal{V}^*(\mathrm{Ker}C) = \mathrm{span} \left(\begin{array}{c|c} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{array} \right)$$

and, since $\mathrm{Im}S \subseteq \mathcal{V}^*(\mathrm{Ker}C)$, any feedback F of the form:

$$F = \begin{pmatrix} -1 & 0 & f_3 & f_4 \end{pmatrix}$$

with f_3, $f_4 \in \mathbb{R}$, that makes the system maximally unobservable, decouples the input z from the output. Since the hypotheses:

$$\mathcal{V}^*(\mathcal{N}) \cap \mathrm{Ker}E = \{0\} \text{ and } E\mathcal{V}^*(\mathcal{N}) \cap \mathrm{Im}B = \{0\}$$

are satisfied, we can consider $\mathcal{V}_*(\mathrm{Ker}C, \mathrm{Im}S)$, and we have:

$$\mathcal{V}_*(\mathrm{Ker}C, \mathrm{Im}S) = \mathrm{span} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

As a consequence, any feedback of the form:

$$F' = \begin{pmatrix} -1 & f_2 & f_3 & f_4 \end{pmatrix}$$

with f_2, f_3, $f_4 \in \mathbb{R}$, actually decouples the input z from the output. Clearly, using F' instead of F we have one more degree of freedom, that can be used for satisfying other requirements.

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