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THE POLE PLACEMENT EQUATION – A SURVEY

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We consider the linear equation $AX + BY = C$ where A, B and C are given polynomials from $K[s]$, the ring of polynomials in the indeterminate s over a field K, and X and Y are unknown polynomials in $K[s]$.

1. MOTIVATION

The equation

$$AX + BY = C \tag{1}$$

has found application in several design problems for linear control systems, including the pole placement design. This problem consists in the following: given a plant with real-rational proper transfer function

$$P(s) = \frac{B(s)}{A(s)},$$

where A and B are coprime polynomials, one seeks to determine a dynamic output feedback controller with a real-rational proper transfer function, say

$$Q(s) = -\frac{Y(s)}{X(s)}$$

such that the closed-loop system has prespecified poles.

Provided A is the characteristic polynomial of the plant and X is that of the controller, then the characteristic polynomial of the closed-loop system, say $C(s)$, which specifies the poles desired, is given by $C = AX + BY$.

Thus the pole placement design is based on equation (1). However not all solution pairs X, Y are of interest: one must take the one in which Y has least degree. This leads to a proper controller whenever one exists.

2. REVIEW OF THEORY

It is well known [1] that $K[s]$ is a principal ideal domain. Thus (1) is solvable if and only if any greatest common divisor of A and B divides C. Writing D for a greatest common divisor of A and B and denoting

$$\bar{A} = \frac{A}{D}, \quad \bar{B} = \frac{B}{D}, \quad \bar{C} = \frac{C}{D}$$

one concludes that (1) has a solution if and only if $\bar{C}$ is a polynomial. Therefore if A and B are coprime then (1) is solvable for *any* C.

Suppose that $\bar{X}, \bar{Y}$ is a particular solution pair of (1). Since the equation is linear, any and all solution pairs of (1) are given by

$$X = \bar{X} - \bar{B}T, \quad Y = \bar{Y} + \bar{A}T,$$

where T varies over $K[s]$. Thus the solution class of (1) is *parametrized* through T in a simple manner.

It is well known [1] that $K[s]$ is a euclidean domain. Therefore if (1) is solvable and $B \neq 0$ there is a unique solution pair $X_{1\,\min}, Y_1$ of (1) such that either $X_{1\,\min} = 0$ or $\deg X_{1\,\min} < \deg \bar{B}$. Further if (1) is solvable and $A \neq 0$ then there is a unique solution pair $X_2, Y_{2\,\min}$ of (1) such that either $Y_{2\,\min} = 0$ or $\deg Y_{2\,\min} < \deg \bar{A}$. These two *least-degree solution* pairs coincide [4] whenever $\deg \bar{A} + \deg \bar{B} > \deg \bar{C}$.

As a result, equation (1) with $A \neq 0$ and $B \neq 0$ can possess solution pairs X, Y of arbitrarily high degree, limited only from below by $\deg X_{1\,\min}$ and $\deg Y_{2\,\min}$.

3. FIXED DEGREE SOLUTIONS

We shall study the class of solutions whose degrees are limited from above. We suppose that A, B and C in (1) are *non-zero* polynomials from $K[s]$ with A and B *coprime*. Hence (1) is solvable. Let

$$p = \deg A, \quad q = \deg B, \quad r = \deg C.$$

If

$$A = a_0 + a_1 s + \ldots + a_p s^p$$

then, for any integer $k \geq p$, we denote

$$\operatorname{vec}_k A = [a_0 \; a_1 \ldots a_p \underbrace{0 \; \ldots \; 0}_{k-p}].$$

The existence result [5] is as follows. Let m, n be non-negative integers and $d = \max(m + p, \; n + q, \; r)$. Then a solution pair X, Y of (1) exists such that

$$X = 0 \;\text{ or }\; \deg X \leq m, \qquad Y = 0 \;\text{ or }\; \deg Y \leq n \tag{2}$$

if and only if $\operatorname{vec}_d C$ is a K-linear combination of $\operatorname{vec}_d A, \operatorname{vec}_d sA, \ldots, \operatorname{vec}_d s^m A, \operatorname{vec}_d B, \ldots, \operatorname{vec}_d s^n B$.

A special case of particular interest concerns the *constant* solutions of (1). Putting $m = n = 0$ we deduce [6] that a solution pair X, Y of (1) exists in K if and only if $\operatorname{vec}_d C$ is a K-linear combination of $\operatorname{vec}_d A$ and $\operatorname{vec}_d B$.

The set of solutions whose degrees are limited from above can be parametrized as follows [5]. Let $m \geq q$ and $n \geq p$. If $n \geq r - q$ then the set of solutions X, Y of (1) that satisfy (2) is given as

$$X = X_{1\,\min} - BT_1, \quad Y = Y_1 + AT_1, \tag{3}$$

where T_1 varies over $K[s]$ and

$$\deg T_1 \leq \min(m - q, n - p);$$

if $m \geq r - p$ then the set of solutions X, Y of (1) that satisfy (2) is given as

$$X = X_2 - BT_2, \quad Y = Y_{2\,\min} + AT_2, \tag{4}$$

where T_2 varies over $K[s]$ and

$$\deg T_2 \leq \min(m - q, \ n - p).$$

Indeed suppose that $n \geq r - q$. Then (3) implies

$$\begin{aligned} \deg X &= q + \deg T_1 \leq m \\ \deg Y &= \max(r - q, \ p + \deg T_1) \leq n \end{aligned}$$

so that $\deg T_1 \leq m - q$ and $\deg T_1 \leq n - p$. In case $m \geq r - p$ then (4) implies

$$\begin{aligned} \deg X &= \max(r - p, \ q + \deg T_2) \leq m \\ \deg Y &= p + \deg T_2 \leq n \end{aligned}$$

and again $\deg T_2 \leq m - q$ and $\deg T_2 \leq n - p$.

We note that at least one of the two conditions, $m \geq r - p$ and $n \geq r - q$, is always satisfied. Of course (3) can be used to parametrize the solution set (2) even if $n < r - q$. Then, however, T_1 has a higher degree than shown and is not completely free in $K[s]$. An analogous statement is true for (4) when $m < r - p$. To illustrate, we parametrize the solution class of

$$X + sY = s^2$$

such that $\deg X \leq 1$ and $\deg Y \leq 1$. Using (3),

$$X = -sT_1, \quad Y = s + T_1, \quad T_1 \ \text{constant}$$

while using (4),

$$X = s^2 - T_2, \ Y = T_2, \ T_2 = s + \tau, \ \tau \ \text{constant}.$$

4. EXAMPLES

Can the double integrator

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u, \quad y = x_1$$

be converted into an harmonic oscillator using a *proportional* output feedback?

The double integrator gives rise to the transfer function

$$P(s) = \frac{1}{s^2}$$

and any harmonic oscillator has the characteristic polynomial

$$C(s) = s^2 + \omega^2$$

for some real constant $\omega > 0$. Thus the answer depends on the polynomial equation

$$s^2 X + Y = s^2 + \omega^2$$

having a constant solution pair X, Y.

Since

$$\begin{aligned} \mathrm{vec}_2 A &= [0 \;\; 0 \;\; 1] \\ \mathrm{vec}_2 B &= [1 \;\; 0 \;\; 0] \\ \mathrm{vec}_2 C &= [\omega^2 \;\; 0 \;\; 1] \end{aligned}$$

the answer is an affirmative: the output feedback $u = -\omega^2 y$ will do the job. The resulting system equations read

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u - \omega^2 x_1, \quad y = x_1.$$

On the other hand, the double integrator cannot be stabilized via proportional output feedback: the polynomial $s^2 X + Y$ is not Hurwitz for any real numbers X and Y.

As the second example, we consider the plant

$$\dot{x}_1 = u - x, \quad y = x$$

and find *all* output feedback controllers that will alter its characteristic polynomial $s + 1$ to $s^2 + 3s + 2$.

These controllers possess the transfer functions

$$Q(s) = -\frac{Y(s)}{X(s)},$$

where X, Y is the solution set of the equation

$$(s + 1)X + Y = s^2 + 3s + 2$$

such that $\deg X = 1$ and $\deg Y \leq 1$.

The condition $m \geq r - p = 1$ is verified. Therefore the solution set is given by

$$X = s + 2 - T_2, \quad Y = (s+1)\,T_2,$$

where T_2 is any real polynomial of degree at most $\min(m-q,\ n-p) = 0$, hence any real constant.

A realization of the parametrized controller set is

$$\begin{aligned} \dot{w} &= (T_2 - 2)\,w + (T_2 - 1)\,y \\ -u &= T_2 w + T_2 y\,. \end{aligned}$$

The case $T_2 = 0$ leads to an unobservable realization while $T_2 = 1$ leads to an uncontrollable realization. A PI controller is obtained when $T_2 = 2$.

If desired, the parameter T_2 can be chosen so that a specific goal is achieved. For example, if the H_∞-norm of the sensitivity function

$$S(s) = \frac{s+2}{s+2-T_2}$$

is not to exceed 1, we should avoid the values $0 < T_2 < 4$.

5. METHODS OF SOLUTION

Equation (1) can be solved in several ways [4]. One can distinguish *parametric* methods (where the polynomials are represented by their coefficients) and *non-parametric* ones (where the polynomials are represented by their functional values.) We shall describe three major parametric methods.

We suppose that A, B and C in (1) are non-zero *real* polynomials with A and B coprime. Hence (1) is solvable. For the sake of simplicity let

$$\deg A = \deg B = N, \quad \deg C = 2N - 1\,.$$

The *Method of Indeterminate Coefficients* [4] converts equation (1) into a system of $2N$ linear equations over the field of real numbers. Suppose we seek the least-degree solution pair X, Y :

$$\deg X \leq N - 1, \quad \deg Y \leq N - 1.$$

The $2N$ coefficients of X, Y satisfy the system of equations

$$[\,\mathrm{vec}_{N-1}X \quad \mathrm{vec}_{N-1}Y\,] \begin{bmatrix} \mathrm{vec}_{2N-1}\,A \\ \cdots \\ \mathrm{vec}_{2N-1}\,s^{N-1}A \\ \mathrm{vec}_{2N-1}\,B \\ \cdots \\ \mathrm{vec}_{2N-1}\,s^{N-1}B \end{bmatrix} = \mathrm{vec}_{2N-1}C\,.$$

The system matrix is a Sylvester matrix and it has full rank since A and B are coprime.

The *Method of Polynomial Reductions* [3] reduces equation (1) to a polynomial equation that is much easier to solve. It consists of the substitutions

$$\begin{aligned} C' &= C - A\,\frac{c_{\deg C}}{a_{\deg A}}\,s^{\deg C - \deg A} \\ C' &= C - B\,\frac{c_{\deg C}}{b_{\deg B}}\,s^{\deg C - \deg B} \\ B' &= B - A\,\frac{b_{\deg B}}{a_{\deg A}}\,s^{\deg B - \deg A} \\ A' &= A - B\,\frac{a_{\deg A}}{b_{\deg B}}\,s^{\deg A - \deg B} \end{aligned}$$

each reducing the degree of one of the polynomials A, B, C. The substitutions are repeated for the new polynomials A', B', C' and will ultimately reduce all A, B, C but one to zero. The resulting equation has a solution $X' = 0$, $Y' = 0$ and the solution pair X, Y of (1) is obtained through the backward substitutions

$$\begin{aligned} X &= X' + \frac{c_{\deg C}}{a_{\deg A}}\,s^{\deg C - \deg A} \\ Y &= Y' + \frac{c_{\deg C}}{b_{\deg B}}\,s^{\deg C - \deg B} \\ X &= X' - Y\frac{b_{\deg B}}{a_{\deg A}}\,s^{\deg B - \deg A} \\ Y &= Y' - X\frac{a_{\deg A}}{b_{\deg B}}\,s^{\deg A - \deg B}. \end{aligned}$$

The process involves the euclidean algorithm for A, B and leads to the least-degree solution pair X, Y.

The *Method of State-space Realization* [2] combines matrix and polynomial operations. We write (1) as

$$X + \frac{B}{A}Y = \frac{C}{A}$$

and determine a reachable state-space realization (F, G, H, J) of the rational function B/A. The N coefficinets of Y satisfy the system of equations

$$\mathrm{vec}_{N-1}Y \begin{bmatrix} H \\ HF \\ \cdots \\ HF^{N-1} \end{bmatrix} = \mathrm{vec}_{N-1}(C \bmod A)$$

and the corresponding X is recovered from (1); it is the least-degree solution pair. The system matrix is an observability matrix and it has full rank since A and B are coprime.

6. NUMERICAL EXPERIENCE

The method of indeterminate coefficients is straightforward and leads directly to a system of linear equations for the coefficients of the unknown polynomials. The method of polynomial reductions solves the polynomial equation by polynomial means and is not suitable for pencil-and-paper calculations, for it requires a large number of logical operations. The method of state-space realization combines the two above: one unknown polynomial is obtained by solving a system of linear equations while the other results from polynomial manipulations.

The comparison of the methods with respect to the arithmetic complexity is quite clear [7]. The fastest is the method of polynomial reductions, where the operations count is proportional to N^2. For the other two methods the arithmetic complexity is proportional to N^3. The slowest method, however, is that of indeterminate coefficients because it leads to a larger system of linear equations than the method of state-space realization.

The comparison of the methods from the precision point of view [7] is not that simple, however. Provided the polynomials A and B have no (especially multiple) roots close to each other, the precison of all three methods is alike. The ill-conditioned data, however, make the method of polynomial reductions fail more often than that of indeterminate coefficients. The method of state-space realization shows no clear-cut tendency, it stays between the two preceding methods.

To conclude, polynomial reductions are fast but sensitive to data, indeterminate coefficients are robust but slow, and the method of state-space realization is universal but second best in each single aspect.

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