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LOCALLY HARMONIZABLE COVARIANCE: SPECTRAL ANALYSIS

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In this paper we introduce the notion of locally harmonizable covariance. The motivation is to define a large class of nonstationary processes containing locally stationary and harmonizable cases, for which the covariance admits spectral components.

1. INTRODUCTION

The notion of a locally stationary covariance function is due to Silverman [11]. A covariance K defined on $\mathbb{R}$ is called locally stationary if there are a positive function K_1 and a continuous stationary covariance K_2 such that,

$$K(t,s) = K_1\left(\frac{t+s}{2}\right) K_2(t-s). \tag{1}$$

In this work, the stationarity of the covariance K_2, is replaced by a less restricting notion: the harmonizability in the sense of Loève [4] (strong harmonizability according to Rao [9]). Then we present the spectral analysis of such covariances which are not necessarily bounded.

Remind that a covariance K defined on $\mathbb{R}^2$ is harmonizable when there exists a complex valued measure M on $\mathbb{R}^2$ such that for all t, s in $\mathbb{R}$,

$$K(t,s) = \iint_{\mathbb{R}^2} e^{i(tx-sy)} M(\mathrm{d}x, \mathrm{d}y).$$

A zero mean second order process $X = \{X(t),\ t \in \mathbb{R}\}$ is harmonizable if and only if its covariance $K(t,s) = \operatorname{cov}(X(t), X(s))$ is harmonizable. Then there exists a stochastic measure μ in $\mathbb{R}$ such that

$$X(t) = \int_{\mathbb{R}} e^{itx} \mu(\mathrm{d}x).$$

A continuous stationary covariance is harmonizable.

2. NOTION OF LOCALLY HARMONIZABLE COVARIANCE

Definition 1. A nonzero covariance K is called locally harmonizable when there exist a positive function K_1 and a harmonizable covariance K_2 such that for all t, s in $\mathbb{R}$,

$$K(t,s) \quad = \quad K_1\left(\frac{t+s}{2}\right) K_2(t,s). \tag{2}$$

A second order process is called locally harmonizable when its covariance is locally harmonizable.

For a locally stationary process, decomposition (1) is known to be unique up to a constant factor, the problem of the uniqueness of decomposition (2) of the covariance of a locally harmonizable process is still open to debate.

Every locally stationary covariance as well as every harmonizable covariance is locally harmonizable. We know that the product of two covariances is a covariance and here we can state the following result.

Theorem 1. The product of two locally harmonizable covariances is locally harmonizable.

P r o o f. Indeed, this result is a direct consequence of the fact that the product of two harmonizable covariances is a harmonizable covariance: if

$$\begin{aligned} K(t,s) &= \iint_{\mathbb{R}^2} e^{i(tx-sy)} M(\mathrm{d}x, \mathrm{d}y) \quad \text{and} \\ K'(t,s) &= \iint_{\mathbb{R}^2} e^{i(tx-sy)} M'(\mathrm{d}x, \mathrm{d}y), \end{aligned}$$

according to Fubini theorem, we deduce that

$$K(t,s)\, K'(t,s) \quad = \quad \iint_{\mathbb{R}^2} e^{i(tx-sy)} G(\mathrm{d}x, \mathrm{d}y),$$

where G is the measure defined on $\mathbb{R}^2$ by

$$G(A,B) \quad = \quad \iint_{\mathbb{R}^2} M(A-x, B-y)\, M'(\mathrm{d}x, \mathrm{d}y). \qquad \square$$

Consequently the product of two locally harmonizable processes which are stochastically independent, is locally harmonizable.

In the following, we produce examples of non locally stationary but locally harmonizable processes obtained by linear transformations of locally stationary processes.

Examples. 1. Let Z be a nonzero, locally stationary process such that

$$\int_{\mathbb{R}} (E([Z(x)]^2))^{1/2} \mathrm{d}x \quad < \quad \infty.$$

Then the harmonizable process X defined by

$$X(t) \quad = \quad \int_{\mathbb{R}} e^{itx} \, Z(x) \, \mathrm{d}x,$$

is locally stationary. Furthermore, the process Y_1 defined by $Y_1(t) = h * X(t)$, with $h(x) = \mathbf{1}_{[-1,1]}(x)$, is harmonizable:

$$Y_1(t) \quad = \quad \int_{-1}^{1} X(t-s) \, \mathrm{d}s = \int_{\mathbb{R}} e^{itx} Q(x) \, Z(x) \, \mathrm{d}x,$$

where $Q(x) = 2\sin x/x$. As, there are no functions f, g such that,

$$Q\left(t + \frac{s}{2}\right) Q\left(t - \frac{s}{2}\right) \quad = \quad f(t) \, g(s),$$

Theorem 1 in [8] implies that Y_1 is not locally stationary.

2. If $Y_2(t) = e^{at} X(t)$ with $a \in \mathbb{R} - \{0\}$, then the process Y_2 is locally harmonizable but not harmonizable, as it is not bounded in quadratic mean. If in addition, we assume that

$$\int_{\mathbb{R}} |x| \, (E([Z(x)]^2))^{1/2} \mathrm{d}x \quad < \quad \infty,$$

then the process Y_2', derivative in the mean-square sense of Y_2,

$$Y_2'(t) \quad = \quad e^{at} \int_{\mathbb{R}} e^{itx} Q(x) \, Z(x) \, \mathrm{d}x,$$

with $Q(x) = a + ix$, is not locally stationary but locally harmonizable.

From the previous discussions we deduce the following inclusions of classes, the inclusions being strict.

$$\{\text{Stationary}\} \quad \subset \quad \begin{matrix} \{\text{ Harmonizable}\} \\ \neq \\ \{\text{ Locally stationary }\} \end{matrix} \quad \subset \quad \{\text{ Locally harmonizable }\}$$

3. ASYMPTOTIC SPECTRAL STUDY

Every harmonizable process admits an associated spectrum (see [10]). Michálek [7] proved that a locally stationary function K admits an associated spectrum if and only if $\lim_{t\to\infty} \frac{1}{t} \int_0^t K_1(s) \, \mathrm{d}s$ exists and is finite. More generally in the case of a locally harmonizable process, we have the following asymptotic spectral result.

Theorem 1. Let $K(t,s) = K_1\left(\frac{t+s}{2}\right) K_2(t,s)$ be a locally harmonizable covariance such that K_1 is locally integrable and that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t K_1(s)e^{-isu}\mathrm{d}s \quad = \quad g(u) \tag{3}$$

exists and is finite for every u in $\mathbb{R}$. Then

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t K(s+h,s)e^{-isu}\mathrm{d}s \quad = \quad b(u,h) \tag{4}$$

exists and is finite for every u and h in $\mathbb{R}$. Morever

$$b(u,h) \quad = \quad \iint_{\mathbb{R}^2} e^{\frac{ih(x+y+u)}{2}} g(x-y-u)\, M(\mathrm{d}x,\mathrm{d}y), \tag{5}$$

where M is the measure in $\mathbb{R}^2$ associated with the harmonizable covariance K_2.

Proof. First we note that the function g defined by (3) is $\mathcal{B}(\mathbb{R})$-measurable and bounded. Indeed g, limit of a sequence of $\mathcal{B}(\mathbb{R})$-measurable functions, is $\mathcal{B}(\mathbb{R})$-measurable, and since for every $t > 0$,

$$\left|\frac{1}{t}\int_0^t K_1(s)\, e^{-isu}\mathrm{d}s\right| \quad \le \quad \frac{1}{t}\int_0^t K_1(s)\,\mathrm{d}s,$$

it satisfies $|g(u)| \le g(0)$, for $u \in \mathbb{R}$.

Let h be fixed in $\mathbb{R}$. From Fubini theorem we obtain that, for $t > 0$,

$$\begin{aligned} &\frac{1}{t}\int_0^t K(s+h,s)\, e^{-isu}\mathrm{d}s \\ &\quad = \quad \iint_{\mathbb{R}^2} e^{ihx}\left(\frac{1}{t}\int_0^t K_1\left(s+\frac{h}{2}\right) e^{is(x-y-u)}\mathrm{d}s\right)\; M(\mathrm{d}x,\mathrm{d}y). \end{aligned}$$

After the change of the variable $S = s + h/2$, by hypothesis on K_1, we can deduce that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t K_1\left(s+\frac{h}{2}\right) e^{is(x-y-u)}\mathrm{d}s \quad = \quad e^{\frac{-ih(x-y-u)}{2}} g(x-y-u).$$

Then Lebesgue dominated convergence theorem can be applied and the theorem follows. □

Consequently a locally harmonizable process whose covariance has a decomposition which satisfies the hypothesis of Theorem 1, admits an associated spectrum. The limit $b(u,h)$ in (5) can be interpreted as the cyclic component at the frequency u of the function $s \to K(s+h,s)$ with the fixed time delay h, and we have the Fourier decomposition

$$K(s+h,s) \sim \sum_u b(u,h)\, e^{ius}.$$

Whenever the process is harmonizable, the set $\Lambda = \{u,\ b(u,h) \neq 0 \text{ for some } h \in \mathbb{R}\}$ is countable, and the series converges absolutely and uniformly with respect to h and u in $\mathbb{R}$, towards a covariance K_* (see [1]). This covariance K_* coincides with K if and only if for every h, the function $s \to K(s+h,s)$ is almost periodic in the sense of Bohr. See [1] and references therein for more details on the related theory of the almost periodically correlated processes.

Under the hypotheses of Theorem 1, thanks to the Bochner theorem and [1], there exists a unique family $\{m_u, u \in \mathbb{R}\}$ of complex measures on $\mathbb{R}$ which are absolutely continuous with respect to m_0, and such that $b(u,.)$ is the Fourier transform of m_u. For every u, the measure m_u can be expressed with M and g. Indeed, consider the transformation defined on $\mathbb{R}^2$ by $T_u(x,y) = ((x+y+u)/2, x-y-u)$ and the complex valued measure on $\mathbb{R}^2$, $G_u(A \times B) = M(T_u^{-1}(A \times B))$. Then m_u satisfies the equalities

$$\begin{aligned} m_u(A) &= \iint_{A\times\mathbb{R}} g(y)\, G_u(\mathrm{d}x, \mathrm{d}y) \\ &= \int_{\mathbb{R}} g(y)\, G_u(A, \mathrm{d}y). \end{aligned}$$

When the covariance K is locally stationary, $b(u,\cdot)$ can be expressed in a simpler manner.

Corollary 1. Let $K(t,s) = K_1\left(\frac{s+t}{2}\right) K_2(t-s)$, be a nonzero locally stationary covariance where K_1 is locally integrable. The following two conditions are equivalent:

1. limit (4) exists and is finite for all u and h,
2. limit (4) exists and is finite for every u, and $h = 0$.

Moreover, when these conditions are satisfied, we have

$$b(u,h) = \frac{K_2(h)}{K_2(0)}\, b(u,0)\, e^{\frac{iuh}{2}} = K_2(h)\, e^{\frac{iuh}{2}} g(-u). \tag{6}$$

Proof. For all h, u in $\mathbb{R}$, and every $t > 0$, we have

$$\int_0^t K(s+h,s)\, e^{-isu} \mathrm{d}s = K_2(h) \int_0^t K_1\left(s + \frac{h}{2}\right) e^{-isu}\, \mathrm{d}s.$$

The change of the variable $S = s + \frac{h}{2}$ implies that

$$\int_0^t K_1\left(s+\frac{h}{2}\right) e^{-isu} \mathrm{d}s = e^{\frac{iuh}{2}} \int_{h/2}^{t+h/2} K_1(s)\, e^{-isu} \mathrm{d}s.$$

As $K(s,s) = K_1(s)\, K_2(0)$ is not identically null , we can conclude the equivalence between the two conditions. When these conditions are satisfied, we can readily prove (6). □

Under the hypotheses of Corollary 1, the function K_2 is the Fourier transformation of a nonnegative bounded measure m defined in $\mathbb{R}$ and

$$\begin{aligned} b(u,h) &= g(-u)\int_{\mathbb{R}} e^{i(x+\frac{u}{2})h} m(\mathrm{d}x) \\ &= g(-u)\int_{\mathbb{R}} e^{ixh} m\left(\mathrm{d}x-\frac{u}{2}\right), \end{aligned}$$

thus $m_u(A) = g(-u)\, m(A-u/2)$. When the covariance K is harmonizable, $K = K_2$, (3) is satisfied with $g(u) = \mathbf{1}_{\{0\}}(u)$. Thus for all u, h in $\mathbb{R}$, we get

$$\begin{aligned} b(u,h) &= \iint_{D_u} e^{ixh} M(\mathrm{d}x,\mathrm{d}y) \\ m_u(A) &= M((A\times\mathbb{R})\cap D_u) \end{aligned}$$

where $D_u = \{(x,y), x-y=u\}$. More generally, we can state

Corollary 2. Suppose that $K(t,s) = K_1\left(\frac{s+t}{2}\right) K_2(t,s)$ is a locally harmonizable covariance such that

$$K_1(s) = \int_{\mathbb{C}} e^{sz} F(\mathrm{d}z)$$

where F is a scalar measure on the set of complex numbers $\mathbb{C}$, with support included in the left half-plane. Thus for all u, h in $\mathbb{R}$,

$$\begin{aligned} b(u,h) &= \lim_{t\to\infty}\frac{1}{t}\int_0^t K(s+h,s)\, e^{-isu}\mathrm{d}s \\ &= \iint_{\mathbb{R}^2} e^{\frac{i(x+y+u)}{2}} F(i(x-y-u))\, M(\mathrm{d}x,\mathrm{d}y), \\ m_u(A) &= \iint_{A\times\mathbb{R}} F(iy)\, G_u(\mathrm{d}x,\mathrm{d}y), \end{aligned}$$

where G_u has been defined above.

Proof. Let u in $\mathbb{R}$, Fubini theorem implies that

$$\frac{1}{t}\int_0^t K_1(s)e^{isu}\mathrm{d}s = \int_{\mathbb{C}}\left(\frac{1}{t}\int_0^t e^{s(x+i(u+y))}\mathrm{d}s\right) F(\mathrm{d}z),$$

where $z = x+iy$. Since for every $x \geq 0$ and every y in $\mathbb{R}$, we have

$$\sup_{t>0}\left|\frac{1}{t}\int_0^t e^{s(x+i(u+y))}\mathrm{d}s\right| \leq 1,$$

and

$$\begin{aligned} \lim_{t\to\infty}\frac{1}{t}\int_0^t e^{s(x+i(u+y))}\mathrm{d}s &= 0 \quad \text{if } x+i(u+y)\neq 0, \\ &= 1 \quad \text{if } x+i(u+y)=0. \end{aligned}$$

Then Lebesgue dominated convergence theorem can be applied, and we obtain that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t K_1(s) e^{-isu} \mathrm{d}s \quad = \quad F(iu).$$

From Theorem 1, we can deduce the expressions of $b(u,h)$ and $m_u(A)$. □

Let $X = X_1 X_2$ be the product of two stochastically independent processes, such that X_1 is a symmetric process, that is, its covariance function $K_1(t+s)$ is a continually exponentially convex (see Michálek [5], Getoor [3] and Loève [4]), and such that X_2 is a harmonizable process. Thus there exists a probability measure F on $\mathbb{R}$ such that for all t, s in $\mathbb{R}$,

$$E(X_1(t)\overline{X_1(s)}) \quad = \quad \int_{\mathbb{R}} e^{(t+s)u} F(\mathrm{d}u).$$

If the support of F is contained in $(-\infty, 0]$, then the covariance K of X verifies the conditions of Corollary 1, it admits an associated spectrum, and for all u, h in

$$b(u,h) \quad = \quad F(0) \iint_{D_u} e^{ixh} M(\mathrm{d}x, \mathrm{d}y).$$

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