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ON SUFFICIENT CONDITIONS FOR THE STABILITY OF DYNAMIC INTERVAL SYSTEMS

KAREL SLADKÝ[1]

In this note sufficient conditions for the stability of continuous- and discrete-time dynamic interval systems are investigated. In particular, we focus our attention on stability conditions based on the extensions of Gershgorin's theorem, i.e. Gershgorin's theorem is applied after some similarity transformation, cf. [1], [2] and [6]. We show that the tests on stability and stability margins of dynamic interval systems suggested in [2] and in [6] can be considerably improved.

1. INTRODUCTION

An interval matrix is a real matrix in which all the elements are known only to the extent that each belongs to specified closed interval. In particular, an $r \times r$ interval matrix A_I is in fact a set of real matrices

$$A_I = \{A = [a_{ij}] : a_{ij} \in [b_{ij}, c_{ij}], \quad i,j = 1,\dots,r\},$$

where $b_{ij} \le c_{ij}$ are given real numbers. Let $B = [b_{ij}]$, $C = [c_{ij}]$, and hence $A_I = [B \quad C]$.

The dynamic interval system is defined as

$$\dot{x}(t) = A\,x(t), \quad x(t_0) = x_0 \qquad \text{where} \quad A \in A_I \tag{1a}$$

for the continuous-time case, and as

$$x(k+1) = A\,x(k), \quad x(0) = x_0 \quad \text{where} \quad A \in A_I \tag{1b}$$

for the discrete-time case.

In the present paper we deal with the analysis of stability and marginal stability of dynamic interval systems. The system (1a) is (asymptotically) stable (i.e. $\lim_{t\to\infty} x(t) = 0$) if for every $A \in A_I$ all the eigenvalues of A have negative real parts. The system (1a) is said to be stable with stability margin h, where $h \ge 0$ (or to have the degree of stability h), if for every $A \in A_I$ the real part of any eigenvalue of A is less than $-h$.

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Similarly the discrete-time system (1b) is (asymptotically) stable if for every matrix $A \in A_I$ the modulus of any eigenvalue of A is less than one. The system (1b) is said to be stable with stability margin h, where $0 \leq h < 1$ (or to have the degree of stability h), if the modulus of every eigenvalue of A is less than $1 - h$.

The stability analysis of dynamic interval systems is very important in the robust controller design. In recent years, stability of dynamic interval systems has been studied by many authors and some sufficients conditions for the stability have been obtained, cf. e. g. [1], [2], [4], [5], [6], [7], [8], [9] and [10].

In this note we shall closely follow the approaches used in Argoun [1], Juang and Shao [6], and Chen [2] to construct the tightest stability conditions. These approaches are based on an extension of the well-known Gershgorin's theorem, i. e. Gershgorin's theorem is applied after some similarity transformation of the interval matrix. Comparing with the approaches based on a direct application of Gershgorin's theorem, methods based on the extensions Gershgorin's theorem do not suffer from a shortcoming that the "end points" c_{ii} (diagonal entries of C) must be negative (resp. less than one) if the continuous-time model (1a) (resp. discrete-time model (1b)) is considered. The paper by Chen [2] in an elegant way reviews and improves many previous results on stability of continuous-time interval systems. In particular, Chen [2] improves the stability conditions proposed by Juang and Shao [6] and shows that tighter stability conditions may be obtained by suitably manipulating of some scaling parameters. Moreover, it is shown in [2] that the continuous-time system (1a) is stable with a given margin h, if the spectral radius of certain matrix is less than one.

In this note we show that the test on stability and stability margins suggested in [2] can be improved and that an analogous procedure can be also used for discrete-time systems. In particular, the improved test procedure immediately yields the tightest stability margin and the "optimal" scaling parameters can be calculated. Furthermore, we present a novel algorithm to compute a sequence of stability margins (for both continuous- and discrete-time systems) converging monotoneously to the tightest stability margin, along with a sequence of scaling parameters converging to the "optimally" selected scaling parameters.

The paper is organized as follows. Preliminaries are given in Section 2. Our main results will be presented in Section 3. Examples and comparison of the presented results with the work of Juang and Shao [6], and Chen [2] are given in Section 4. Conclusions are made in Section 5.

2. PRELIMINARIES

In this section we shall briefly review the results given by Juang and Shao [6] and their improvements given by Chen [2]. The results of [6] are the correct version of erroneous results of Argoun [1] based on an extension of Gershgorin's theorem. Recall that according to Gershgorin's theorem every eigenvalue λ of an $r \times r$ matrix $A = [a_{ij}]$ must be contained in at least one of the circles given by the inequalities $|\lambda - a_{ii}| \leq \sum\limits_{j=1, j\neq i}^{r} |a_{ij}|$, (for $i = 1, \ldots, r$) and hence also $\mathrm{Re}(\lambda) \leq$

$\mathrm{Re}\,(a_{ii}) + \sum_{j=1,j\neq i}^{r} |a_{ij}|, \quad |\lambda| \leq |a_{ii}| + \sum_{j=1,j\neq i}^{r} |a_{ij}|$ for at least one $i = 1, \ldots, r$.

For a matrix $A \in A_I = [B \quad C]$ we shall write $A = A_0 + \delta A$ where

$$A_0 = \frac{1}{2}(B + C), \quad \Delta A = \frac{1}{2}(C - B) \quad \text{and} \quad |\delta A| \leq \Delta A$$

($|.|$ represents the matrix with modulus elements; symbols $\geq$, $>$ in a matrix relation are considered componentwise).
After the similarity transformation T^{-1} and T we get

$$T^{-1} A\, T = T^{-1} A_0\, T + T^{-1} \delta A\, T.$$

Selecting T such that $T^{-1} A_0 T$ is is a Jordan form we get for $A \in A_I$ such that $A = A_0 + \Delta A = C$

$$T^{-1} A\, T = J + T^{-1} \Delta A\, T,$$

where $J = \Lambda + E = T^{-1} A_0 T$ is the **Jordan** form of A_0, $\Lambda = \mathrm{diag}[\lambda_{11}, \lambda_{22}, \ldots, \lambda_{rr}]$ with λ_{ii} being an eigenvalue of A_0.
Denoting

$$F = E + |T^{-1}|\ \Delta A\ |T|$$

with $F = [f_{ij}]$, we get for $A \in A_I$

$$T^{-1} A\, T \leq T^{-1} A_0\, T + |T^{-1}|\, |\delta A|\, |T| \leq T^{-1} A_0\, T + |T^{-1}|\, \Delta A\, |T| = \Lambda + F.$$

In what follows we shall assume that the matrix F is positive, i.e. $F > 0$. Since the eigenvalues of the matrices $T^{-1} A\, T$ and Λ are the same, for every eigenvalue λ of $A \in A_I$ we have $\mathrm{Re}\,(\lambda) \leq \mathrm{Re}\,(\lambda_{ii}) + \sum_{j=1}^{r} f_{ij}$ (for $i = 1, \ldots, r$). Hence we can readily formulate sufficient conditions for the stability of continuous-time interval systems. In particular (cf. Theorem 2 of [6]), the system (1a) is stable with stability margin h, if

$$\mathrm{Re}\,(\lambda_{ii}) + \sum_{j=1}^{r} f_{ij} < -h, \qquad \text{for all} \quad i = 1, \ldots, r. \tag{2}$$

Similar result[2] can be also obtained for the discrete-time systems. Since for every eigenvalue λ of $A \in A_I$ we have $|\lambda| \leq |\lambda_{ii}| + \sum_{j=1}^{r} f_{ij}$ (for $i = 1, \ldots, r$), the system (1b) is stable with stability margin h (cf. Theorem 3 of [6]), if

$$|\lambda_{ii}| + \sum_{j=1}^{r} f_{ij} < 1 - h, \qquad \text{for all} \quad i = 1, \ldots, r. \tag{3}$$

Moreover, since eigenvalues are invariant under similarity transformations, sufficient stability conditions (2.1), (2.2) can be further improved by employing a suitable similarity transformation. To this order observe that after the similarity transformation U^{-1} and U, where $U = \mathrm{diag}\,[u_1, \ldots, u_r]$ (with $u_i > 0$, $\forall i = 1, \ldots, r$),

[2]**For the sake of brevity, unless otherwise stated, these and subsequent results will be stated only in terms of the rows of the matrix F, analogous results can be stated in terms of the columns of the matrix F.**

applied on $T^{-1}A\,T \leq \Lambda + F$, for every eigenvalue λ of $A \in A_I$ we have

$$\mathrm{Re}\,(\lambda) \leq \mathrm{Re}\,(\lambda_{ii}) + u_i^{-1} \sum_{j=1}^{r} f_{ij}\, u_j, \quad \text{resp.} \quad |\lambda| \leq |\lambda_{ii}| + u_i^{-1} \sum_{j=1}^{r} f_{ij}\, u_j \quad \text{for } i = 1, \ldots, r.$$

Hence we can conclude (cf. Theorem 3.8 of [2]) that the continuous-time system (1a) is stable with stability margin h, $h \geq 0$, if there exist $u_1, \ldots, u_r$ ($u_i > 0$) such that

$$\mathrm{Re}\,(\lambda_{ii}) + \sum_{j=1}^{r} f_{ij} \frac{u_j}{u_i} < -h, \qquad \text{for all} \quad i = 1, \ldots, r. \tag{4}$$

Similarly for the discrete-time models we can conclude that the system (1b) is stable with stability margin h, $0 \geq h < 1$, if there exist $u_1, \ldots, u_r$ ($u_i > 0$) such that

$$|\lambda_{ii}| + \sum_{j=1}^{r} f_{ij} \frac{u_j}{u_i} < 1 - h, \qquad \text{for all} \quad i = 1, \ldots, r. \tag{5}$$

Of course, if $u_i \equiv 1$ for all $i = 1, \ldots, r$ we obtain the conditions of [6], however tighter stability conditions may be obtained by manipulating the values u_i's.

Moreover, Chen [2] gives also some condition under which (2.3) is fulfilled. To this order define a matrix

$$\Gamma_h = [\gamma_{ij}], \quad \text{where} \quad \gamma_{ij} = \frac{f_{ij}}{|\mathrm{Re}\,(\lambda_{ii}) + h|} \quad \text{for} \quad i, j = 1, \ldots, r. \tag{6}$$

According to Theorem 3.10 of [2], the system (1a) is stable with a margin h, if $\mathrm{Re}\,(\lambda_{ii}) + h < 0$ for $i = 1, \ldots, r$ and the spectral radius of Γ_h is less than one.

In the present paper we show that the above stability criterion can be considerably improved. In particular, we show that a sufficient condition for the stability of the system (1a), along with the tightest stability margin in (2.3), can be obtained by calculating the spectral radius of some matrix M (cf. Theorem 1). The optimal scaling parameters in (2.3) are elements of a right eigenvector corresponding to the spectral radius of the matrix M. Similar procedures can be also used for discrete-time systems (cf. Theorem 3). Furthermore, in Theorem 2, resp. Theorem 4, we show how to find (by a simple algorithmic procedure) the values $\hat{u}_i$'s giving the tightest stability margin h in (2.3), resp. (2.4). Throughout the paper we shall assume that the system is stable if $A = A_0$.

In what follows, we shall also need some basic properties of nonnegative matrices and of matrices with nonnegative off-diagonal elements. Recall that, by the well-known Perron–Frobenius theorem (cf. e.g. [3]), the spectral radius of a nonnegative matrix is equal to its largest positive eigenvalue (called the Perron eigenvalue) and the corresponding eigenvector (called the Perron eigenvector) can be selected nonnegative. In case that the matrix is irreducible, the Perron eigenvalue is simple, the Perron eigenvector is unique up to a multiplicative constant and can be selected positive. A nonnegative irreducible matrix is acyclic if the Perron eigenvalue is the unique eigenvalue with the largest modulus.

Similarly, for a matrix with nonnegative off-diagonal elements the eigenvalue with the largest real part is real and the corresponding eigenvector (called the Perron eigenvector) can be selected nonnegative. Moreover, if the matrix is irreducible, this eigenvalue is simple, the Perron eigenvector is unique up to a multiplicative constant and can be selected positive. Observe that all the above mentioned facts of matrices with nonnegative off-diagonal elements trivially follow from the corresponding properties of nonnegative matrices.

3. MAIN RESULTS

In this section we improve the stability conditions provided in Juang and Shao [6] and in Chen [2]. Furthermore, we suggest simple iterative procedures (both for continuous- and discrete-time systems) that generate sequences of stability margins, based on (2.3) or (2.4), converging monotoneously to the tightest stability margin.

First we shall analyze sufficient conditions for the stability of continuous-time dynamic interval systems. Let $\alpha = -\min_{i=1,\dots,r} (\mathrm{Re}\,(\lambda_{ii}) + f_{ii})$ and introduce the (nonnegative) matrix

$$M = [m_{ij}] \quad \text{where} \quad m_{ij} = (\mathrm{Re}\,(\lambda_{ii}) + \alpha)\, \delta_{ij} + f_{ij}, \tag{7}$$

(δ_{ij} denotes the Kronecker symbol).
Observe that since the similarity transformation T is fixed (and selected such that $T^{-1}A_0T$ is a Jordan form), also all λ_{ii}'s and f_{ii}'s are fixed and hence the number α and the matrix M are well defined. Since $F > 0$ the matrix M is irreducible; moreover, we shall aşsume that the matrix M is acyclic (if M were cyclic, it suffices only to choose $\alpha > -\min_{i=1,\dots,r} (\mathrm{Re}\,(\lambda_{ii}) + f_{ii})$, cf. Example 2 of Section 4). In what follows, $\rho(M)$ is reserved for the spectral radius of M. Since we assume that the system (1a) is stable if $A = A_0$, all the eigenvalues of A_0 have negative real parts, i.e. if continuous-time system (1a) is considered, we assume that $\mathrm{Re}\,(\lambda_{ii}) < 0$ for $i = 1, \dots, r$.

The following theorem improves the results provided in Theorem 3.10 of Chen [2].

Theorem 1. The system (1a) is stable if $\rho(M) < \alpha$. Then $h^\star = \alpha - \rho(M)$ is the least upper bound on the tightest stability margin of the system (1a) that can be produced by (2.3), and "optimally" selected scaling parameters in (2.3) are elements of the right eigenvector $u(M)$ corresponding to $\rho(M)$. Furthermore, if $h > h^\star$ (2.3) cannot hold for positive u_j's ($j = 1, \dots, r$).

P r o o f. Introducing the (positive column) vector $u = [u_i]$, diagonal matrix $\tilde{\Lambda} = \mathrm{diag}\,[\mathrm{Re}\,(\lambda_{ii})]$, condition (2.3) can be written in a matrix form

$$(\tilde{\Lambda} + F) \cdot u < -h \cdot u \iff M \cdot u < (\alpha - h) \cdot u. \tag{8}$$

By the Perron–Frobenius theorem $M \cdot u(M) = \rho(M) \cdot u(M) > 0$, $v(M) \cdot M = \rho(M) \cdot v(M) > 0$, ($u(M)$, resp. $v(M)$, is a right, resp. left Perron eigenvector), and (3.2) holds for $u = u(M)$ and any $h > 0$ such that $h < \alpha - \rho(M) = h^\star = -\sigma$.

Observe that σ is the eigenvalue with the largest real part of the matrix $(\tilde{\Lambda} + F)$ with nonnegative off-diagonal elements, hence also $(\tilde{\Lambda} + F) \cdot u(M) = \sigma \cdot u(M)$.
To finish the proof suppose that $M \cdot \tilde{u} < (\alpha - h) \cdot \tilde{u}$ for some $h > h^\star$ and positive $\tilde{u}$. On premultiplying the above matrix inequality by $v(M)$ we immediately get

$$v(M) \cdot M \cdot \tilde{u} = \rho(M) \cdot v(M) \cdot \tilde{u} < (\alpha - h) \cdot v(M) \cdot \tilde{u} \quad \Longrightarrow \quad \rho(M) < \alpha - h$$

that contradicts $h > h^\star$; hence (2.3) can be fulfilled for $u > 0$ only if $h \in (0, h^\star)$. □

Now we present an algorithmic procedure generating an increasing sequence of stability margins converging to the tightest stability margin that can be produced by (2.3).

Theorem 2. Let for $i = 1, \ldots, r$ $\{u_i(n), n = 0, 1, \ldots\}$ be defined recursively by

$$u_i(n+1) = (\mathrm{Re}\,(\lambda_{ii}) + \alpha)\, u_i(n) + \sum_{j=1}^{r} f_{ij}\, u_j(n) \qquad \text{where} \quad u_j(0) > 0 \tag{9}$$

and let

$$-\, h(n) = \max_{i=1,\ldots,r} \frac{\mathrm{Re}\,(\lambda_{ii})\, u_i(n) + \sum_{j=1}^{r} f_{ij}\, u_j(n)}{u_i(n)}. \tag{10}$$

Then

i) $\lim_{n\to\infty} u_i(n)/u_1(n) = \hat{u}_i$ exists for $i = 1, \ldots, r$,

ii) fulfillment of condition $h(n) > 0$ for some $n = 0, 1, \ldots$ is sufficient for stability of the system (1a), and under this condition $h(n)$ is a stability margin of the system (1a),

iii) the sequence $\{h(n), n = 0, 1, \ldots\}$ is nondecreasing (and if $M > 0$ even increasing) and $h(n) \to h^\star$ as $n \to \infty$,
where

$$\mathrm{Re}\,(\lambda_{ii}) + \sum_{j=1}^{r} f_{ij}\, \frac{\hat{u}_j}{\hat{u}_i} = -h^\star \quad \text{for } i = 1, \ldots, r \tag{11}$$

and $h^\star$ is the least upper bound on the tightest stability margin of the system (1a) that can be produced by (2.3).

Proof. Let us introduce the vector $u(n) = [u_i(n)]$, set $\rho = \rho(M)$ and recall that $M = (\tilde{\Lambda} + F + \alpha I)$. Iterating (3.3) we get $u(n) = M^n\, u(0)$. Since the matrix M is irreducible and acyclic, there exists $\lim_{n\to\infty} \rho^{-n}\, M^n = M^\star > 0$ (cf. e.g. [3]).

To establish part i) observe that $\lim_{n\to\infty} \rho^{-n}\, u(n) = M^\star u(0)$, and hence $\lim_{n\to\infty} u_i(n)/u_1(n) = \hat{u}_i > 0$ for $i = 1, \ldots, r$ (and also $\lim_{n\to\infty} u_j(n)/u_k(n)$ $\forall\, j, k = 1, \ldots, r$) exist. Moreover, the r-column vector $\hat{u} = [\hat{u}_i]$ is an eigenvector of M corresponding to ρ.

Part ii) follows immediately from (2.3) or it is a direct consequence of part iii).

To establish part iii) observe that from (3.4)

$$(-h(n) + \alpha)\, u(n) \geq (\tilde{\Lambda} + F + \alpha\, I)\, u(n) \tag{12}$$

(recall that $\geq$ means that equality holds in (3.6) for at least one row).
Premultiplying (3.6) by $M = (\tilde{\Lambda} + F + \alpha\, I)$ and employing (3.3) we conclude that

$$-\, h(n)\, u(n+1) \geq (\tilde{\Lambda} + F)\, u(n+1). \tag{13}$$

Observe that if $M > 0$ a strict inequality holds in (3.7) implying that $-h(n+1)\, u(n+1) \geq (\tilde{\Lambda} + F)\, u(n+1)$ for some $h(n+1) > h(n)$. However, $\{h(n), n = 0, 1, \ldots\}$ is bounded since by part i) $\{u_j(n)/u_i(n), n = 0, 1, \ldots\}$ must be bounded for any $i, j = 1, \ldots, r$. Hence $h(n) \to h^\star$ as $n \to \infty$. To finish the proof observe that on letting $n \to \infty$ from (3.6) we can conclude that $(-h^\star + \alpha)\, \hat{u} \geq M\, \hat{u} = \rho\, \hat{u}$. Hence $-h^\star$ is the eigenvalue of $(\tilde{\Lambda} + F)$ with the largest real part and (3.5) must hold. □

Now we shall focus our attention on sufficient conditions for the stability of discrete-time interval systems. Since we assume that the system (1b) is stable if $A = A_0$, all the eigenvalues of the matrix A_0 lie in the unit disc; hence if the discrete-time system (1b) is considered, we assume (cf. (2.4)) that $|\lambda_{ii}| < 0$ for all $i = 1, \ldots, r$.

The following theorem is a discrete-time version of Theorem 1. Let

$$P = [p_{ij}] \quad \text{where} \quad p_{ij} = |\lambda_{ii}|\delta_{ij} + f_{ij}, \quad \text{for } i, j = 1, \ldots, r \tag{14}$$

and let $\rho(P)$ be the spectral radius of P and $u(P)$ the corresponding right eigenvector.

Theorem 3. The system (1b) is stable if $\rho(P) < 1$. Then $h^\star = 1 - \rho(P)$ is the tightest stability margin on the system (1b) that can be produced by (2.4), and "optimally" selected parameters in (2.4) are elements of the right eigenvector $u(P)$. Furthermore, if $h > 1 - \rho(P)$ (2.4) cannot hold for positive u_j's ($j = 1, \ldots, r$).

Proof. Let $\Lambda = \operatorname{diag}[\lambda_{11}, \ldots, \lambda_{rr}]$. Condition (2.4) can be written in a matrix form as

$$(|\Lambda| + F) \cdot u < (1 - h) \cdot u \quad \Longleftrightarrow \quad P \cdot u < (1 - h) \cdot u \tag{15}$$

By the Perron–Frobenius theorem $P \cdot u(P) = \rho(P) \cdot u(P) > 0$, $v(P) \cdot P = \rho(P) \cdot v(P) > 0$ and (3.9) holds for $u = u(P)$ and any $1 - h > \rho(P)$.
To finish the proof suppose that $P \cdot \tilde{u} < (1 - h) \cdot \tilde{u}$ for some $1 - h > \rho(P)$ and positive $\tilde{u}$. On premultiplying the above matrix inequality by $v(P)$ we immediately get

$$v(P) \cdot P \cdot \tilde{u} = \rho(P) \cdot v(P) \cdot \tilde{u} < (1 - h) \cdot v(P) \cdot \tilde{u} \quad \Longrightarrow \quad \rho(P) < 1 - h$$

that contradicts our assumption. □

The following theorem presents an algorithmic procedure for generating an increasing sequence of stability margins converging to the tightest stability margin that can be produced by (2.4).

Theorem 4. Let for $i = 1, \dots, r$ $\{u_i(n), n = 0, 1, \dots\}$ be defined recursively by

$$u_i(n+1) = |\lambda_{ii}|\, u_i(n) + \sum_{j=1}^{r} f_{ij}\, u_j(n) \qquad \text{where } u_j(0) > 0 \tag{16}$$

and let

$$g(n) = \max_{i=1,\dots,r} \frac{|\lambda_{ii}|\; u_i(n) + \sum_{j=1}^{r} f_{ij}\, u_j(n)}{u_i(n)} \tag{17}$$

Then

i) $\lim_{n\to\infty} u_i(n)/u_1(n) = \hat{u}_i$ exists for $i = 1, \dots, r$,

ii) fulfillment of condition $g(n) < 1$ for some $n = 0, 1, \dots$ is sufficient for stability of the system (1b), and under this condition $h(n) = 1 - g(n)$ is a stability margin of the system (1b),

iii) the sequence $\{g(n), n = 0, 1, \dots\}$ is nonincreasing (if $P > 0$ even decreasing) and $g(n) \to g^\star$ as $n \to \infty$,

where

$$g^\star\, \hat{u}_i = |\lambda_{ii}|\, \hat{u}_i + \sum_{j=1}^{r} f_{ij}\, \hat{u}_j \quad \text{for } i = 1, \dots, r \tag{18}$$

and $h^\star = 1 - g^\star$ is the least upper bound on the tightest stability margin of the system (1b) that can be produced by (2.4).

Proof. The proof is strictly similar to that of Theorem 2. For the sake of simplicity we write only ρ instead of $\rho(P)$. Iterating (3.10) we get $\lim_{n\to\infty} \rho^{-n}\, u(n) = P^\star u(0)$, where $P^\star = \lim_{n\to\infty} \rho^{-n}\, P^n > 0$, and hence $\lim_{n\to\infty} u_i(n)/u_1(n) = \hat{u}_i$ exists for $i = 1, \dots, r$. From (3.11) we immediately get that $g(n)\, u(n) \ge P\, u(n)$ and after premultiplying this inequality by P we conclude that $g(n+1)\, u(n+1) \ge P\, u(n+1)$ for some $g(n+1) \le g(n)$ (and $g(n+1) < g(n)$ if $P > 0$). However, $g(n)$'s are positive and hence $g(n) \to g^\star$ as $n \to \infty$. Letting $n \to \infty$ in (3.11), we get $g^\star \hat{u} \ge P\, \hat{u} = \rho\, \hat{u}$, hence $g^\star = \rho$. □

4. ILLUSTRATIVE EXAMPLES

In this section we compare our methods on concrete examples with the approaches of Juang and Shao [6] and of Chen [2]. The following examples are borrowed from [6].

Example 1. Consider the continuous-time dynamic system (1a) with the following interval matrix

$$A_I = \begin{bmatrix} [-4.1 & -3.5] & [\ \ 1.3 & 1.9] \\ [\ \ 0.3 & 0.9] & [-4.5 & -3.9] \end{bmatrix}$$

As it is stated in [6]

$$\tilde{A} = \begin{bmatrix} -3 & 0 \\ 0 & -5 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & -0.8 \\ 0.5 & 0.6 \end{bmatrix}, \qquad T^{-1} = \begin{bmatrix} 0.6 & 0.8 \\ -0.5 & 1 \end{bmatrix},$$

$$F = |T^{-1}|\,\Delta A\,|T| = \begin{bmatrix} 0.63 & 0.59 \\ 0.68 & 0.63 \end{bmatrix}$$

From (2.1), we get

$$\mathrm{Re}\,(\lambda_{11}) + \sum_{j=1}^{2} f_{1j} = -3 + 1.22 = -1.78$$

$$\mathrm{Re}\,(\lambda_{22}) + \sum_{j=1}^{2} f_{2j} = -5 + 1.31 = -3.69$$

so (cf. [6]), we know that the system is stable with stability margin 1.78.

Now we check the stability and stability margins according to methods suggested in Chen [2]. To this order we need to construct the matrix Γ_h given by (2.5) for the considered stability margin h. Choosing $h = 0,\ 2$ we get

$$\Gamma_0 = \begin{bmatrix} 0.21 & 0.196 \\ 0.1360 & 0.126 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 0.63 & 0.59 \\ 0.230 & 0.21 \end{bmatrix}$$

and for the spectral radii we have $\rho(\Gamma_0) = 0.323$, $\rho(\Gamma_2) = 0.84$. Since the both $\rho(\Gamma_0)$, $\rho(\Gamma_2)$ are less than one, the system is stable and has at least stability margin 2.

Now let us compare this test procedure with the approaches suggested in Theorem 1. We only need to construct the matrix M given by (3.1). We get $\alpha = 4.37$ and hence $M = \begin{bmatrix} 2 & 0.59 \\ 0.680 & 0 \end{bmatrix}$ with $\rho(M) = 2.183$. By Theorem 1 the system is stable and $h^\star = \alpha - \rho(M) = 2.187$ is the tightest stability margin that can be produced by (2.3).

Now we apply the iterative procedure suggested in Theorem 2. Since $\alpha = 4.37$, by (3.3) for $n = 0, 1, \ldots$

$$\begin{aligned} u_1(n+1) &= 2\,u_1(n) + 0.59\,u_2(n) \\ u_2(n+1) &= 0.68\,u_1(n) \end{aligned}$$

Setting $u_1(0) = u_2(0) = 1$, the obtained values are displayed in the following table:

n	0	1	2	3	4
$u_1(n)$	1	2.59	5.58	12.20	26.63
$u_2(n)$	1	0.68	1.76	3.79	8.30
$h(n)$	1.78	1.78	2.18	2.18	2.18

Conclusion: The considered system is stable with stability margin 2.18.

Note. Observe that since every $A \in A_I$ has nonnegative off-diagonal elements, then $A_1 \geq A_2$ (where $A_1, A_2 \in A_I$) implies that $\sigma(A_1) > \sigma(A_2)$. (This property can be easily verified since the eigenvector correspoding to $\sigma(A)$ can be selected positive, cf. e.g. [3].) Hence, within the interval matrix A_I, $\hat{A} \in A_I$, where $\hat{A} = \begin{bmatrix} -3.5 & 1.9 \\ 0.9 & -3.9 \end{bmatrix}$, is the matrix with the largest real eigenvalue, and for every $A \in A_I$ the real part of any eigenvalue of A is nonngreater than $\sigma(\hat{A}) = -2.375$. Hence 2.37 is the largest stability margin (up to two decimal points) of the considered system.

Example 2. Consider the continuous-time dynamic system (1a) with the following interval matrix

$$A_I = \begin{bmatrix} [-3.80 & -3.20] & [\ 0.7 & 1.3] \\ [-0.55 & 0.05] & [-2.8 & -2.2] \end{bmatrix}.$$

As it is stated in [6]

$$\tilde{\Lambda} = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & -0.8 \\ 0.5 & 0.6 \end{bmatrix}, \qquad T^{-1} = \begin{bmatrix} 0.6 & 0.8 \\ -0.5 & 1 \end{bmatrix},$$

$$F = |T^{-1}|\,\Delta A\,|T| = \begin{bmatrix} 0.63 & 1.59 \\ 0.68 & 0.63 \end{bmatrix}.$$

From (2.1), we obtain

$$\begin{aligned} \operatorname{Re}(\lambda_{11}) + \sum_{j=1}^{2} f_{1j} &= -3 + 2.22 = -0.78 \\ \operatorname{Re}(\lambda_{22}) + \sum_{j=1}^{2} f_{2j} &= -3 + 1.31 = -1.69 \end{aligned}$$

so (cf. [6]), we know that the system is stable with stability margin 0.78.

However, using the approach suggested in Theorem 1, we get $\alpha = 2.37$, $M = \begin{bmatrix} 0 & 1.59 \\ 0.680 & 0 \end{bmatrix}$, $\rho(M) = 1.04$ and the system is stable with margin $h^\star = \alpha - \rho(M) = 1.33$.

Now we apply the iterative procedure suggested in Theorem 2. Since $\alpha = 2.37$, however, then the matrix $M = \begin{bmatrix} 0 & 1.59 \\ 0.68 & 0 \end{bmatrix}$ and is obviously cyclic, we choose $\alpha = 3.37$ and apply the iterative procedure suggested in Theorem 2. We have for $n = 0, 1, \ldots$

$$\begin{aligned} u_1(n+1) &= u_1(n) + 1.59\,u_2(n) \\ u_2(n+1) &= 0.68\,u_1(n) + u_2(n) \end{aligned}$$

The obtained values are displayed in the following table (we set $u_1(0) = u_2(0) = 1$):

n	0	1	2
$u_1(n)$	1	2.59	5.26
$u_2(n)$	1	1.68	3.44
$h(n)$	0.78	1.32	1.33

Conclusion: The considered system is stable with stability margin 1.33.

Example 3. Consider the discrete-time dynamic system (1b) with the following interval matrix

$$A_I = \begin{bmatrix} [-0.20 \quad 0.16] & [-0.34 \quad 0.02] \\ [-0.24 \quad 0.12] & [-0.16 \quad 0.20] \end{bmatrix}.$$

As it is stated in [?]

$$A = \begin{bmatrix} -0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & -0.8 \\ 0.5 & 0.6 \end{bmatrix}, \qquad T^{-1} = \begin{bmatrix} 0.6 & 0.8 \\ -0.5 & 1 \end{bmatrix},$$

$$F = |T^{-1}|\,\Delta A\,|T| = \begin{bmatrix} 0.38 & 0.36 \\ 0.41 & 0.38 \end{bmatrix}.$$

From (2.2), we get

$$\begin{aligned} |\lambda_{11}| + \sum_{j=1}^{2} f_{1j} &= |-0.1| + 0.74 = 0.84 \\ |\lambda_{22}| + \sum_{j=1}^{2} f_{2j} &= 0.1 + 0.79 = 0.89 \end{aligned}$$

so (cf. [?]), the system is stable with stability margin 0.11.

Now we check the stability and stability margins according to methods of Theorem 3. We only need to construct the matrix P given by (3.8), hence $P = \begin{bmatrix} 0.39 & 0.36 \\ 0.41 & 0.39 \end{bmatrix}$ and $\rho(P) = 0.794$. Since $\rho(P)$ is less than one, the system is stable at least with the margin 0.116.

Now we apply the iterative procedure suggested in Theorem 4. We have for $n = 0, 1, \ldots$

$$\begin{aligned} u_1(n+1) &= |-0.1|\,u_1(n) + 0.74\,u_2(n) \\ u_2(n+1) &= 0.1\,u_1(n) + 0.79\,u_2(n). \end{aligned}$$

Setting $u_1(0) = u_2(0) = 1$, the obtained values are displayed in the following table:

n	0	1	2
$u_1(n)$	1	0.84	0.742
$u_2(n)$	1	0.89	0.787
$h(n)$	0.11	0.115	0.116

Conclusion: The considered system is stable with stability margin 0.116.

Note. The obtained stability margin is not satisfactory in this particular case. For example, considering $\tilde{A} \in A_I$, where $\tilde{A} = \begin{bmatrix} -0.20 & -0.34 \\ -0.24 & -0.16 \end{bmatrix}$, we get $\rho(\tilde{A}) = 0.467$ and hence 0.53 seems to be close to the tightest margin of the considered system. Obviously, the spectral radius of every $A \in A_I$ is nongreater than $\rho(B) = 0.486$ of the matrix $B = \begin{bmatrix} 0.20 & 0.34 \\ 0.24 & 0.20 \end{bmatrix}$, obtained by replacing elements of A_I by their maximum possible absolute values. Hence we are sure that the stability margin of the considered interval matrix A_I does not exceed 0.51.

5. CONCLUSIONS

We have suggested simple iterative procedures generating sequences of stability margins of both continuous- and discrete-time dynamic interval systems. The obtained results improve the sufficient conditions for stability margin h suggested in Juang and Shao [?] and are also the tightest in the class of stability margins discussed in Chen [?]. We have stated our results only in terms of the matrix rows, analogous procedures can be stated in terms of the matrix columns; however, the same margins will be obtained since the suggested procedures generate the tightest margins within the considered class of stability margins. Examples show that the margins obtained on the base of extended Gershgorin's theorem can be sometimes worse than margins obtained by other methods. The obtained results are useful to robust control design.

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Ing. Karel Sladký, CSc., Ústav teorie informace a automatizace AV ČR (Institute of Information Theory and Automation – Academy of Sciences of the Czech Republic), Pod vodárenskou věží 4, 182 08 Praha 8. Czech Republic.