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CONTROLLABILITY OF SEMILINEAR DELAY SYSTEMS

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Sufficient conditions are established for the controllability of semilinear delay systems. The results are obtained by using the Schauder fixed point theorem and generalize the previous results.

1. INTRODUCTION

The problem of controllability of nonlinear systems has been studied by several authors by means of fixed point principle [1]. In [7] Lukes showed that, if the linear system

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$

is controllable, then the perturbed nonlinear system

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) + f(t,\,x(t),\,u(t))$$

is controllable, provided the function f is bounded. The case where the function f is independent of the control parameter u was considered by Vidyasager [8]. He showed that if the function $|f|$ grows slower than $|x|$ as $|x|$ becomes large, then the controllability of linear system implies that of the perturbed system. Dauer [3] obtained several sufficient conditions on the function f for the controllability of perturbed nonlinear systems. Recently Do [5] made another weaker condition on f for the controllability of perturbed system and deduced Dauer's results as a particular case.

Dauer and Gahl [4] considered the controllability on a bounded interval $J = [0, t_1]$ of nonlinear perturbations of the linear delay system

$$\dot{x}(t) = L(x, u)$$

where the operator L is defined by

$$L(x,u) = A(t)\,x(t) + B(t)\,x(t-1) + \int_{-1}^{0} K(t,s)\,x(t+s)\,\mathrm{d}s + C(t)\,u(t) + D(t)\,u(t-h).$$

They showed that, if the linear system is completely controllable, then the perturbed system

$$\dot{x}(t) = L(x,u) + f(t,\, x(t),\, x(t-1),\, u(t),\, u(t-h))$$

is completely controllable provided the function f satisfies certain growth conditions. Several types of controllability for delay systems are considered in the literature [2, 6]. Here the perturbed system is said to be completely controllable on J if, for every continuous function ϕ defined on $[-1,0]$ and every $x_1 \in R^n$ there exists an admissible control function $u(t)$ such that the solution of

$$\begin{aligned} \dot{x}(t) &= L(x,u) + f(t,\, x(t),\, x(t-1),\, u(t),\, u(t-h)), \quad t \in J \\ x(t) &= \phi(t), \qquad\qquad t \in [-1,0] \end{aligned}$$

satisfies $x(t_1) = x_1$. In this paper we shall study the controllability of semilinear delay system, that is the system without delay in control of Dauer and Gahl [4], by suitably adopting the technique of Dauer [3] and Do [5]. Here our control functions are continuous functions.

2. PRELIMINARIES

Consider the semilinear delay system of the form

$$\begin{aligned} \dot{x}(t) &= A(t)\,x(t) + B(t)\,x(t-1) + \textstyle\int_{-1}^{0} K(t,s)\,x(t+s)\,\mathrm{d}s + C(t)\,u(t) \\ &\quad + f(t,\, x(t),\, x(t-1),\, u(t)), \qquad t \in J = [0,t_1] \\ x(t) &= \phi(t) \qquad\qquad \text{on } [-1,0] \end{aligned} \tag{1}$$

where $x \in R^n$, $u \in R^m$ and A, B, K and C are continuous matrix functions with appropriate dimensions and f is continuous. We shall assume that the linear system

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,x(t-1) + \int_{-1}^{0} K(t,s)\,x(t+s)\,\mathrm{d}s + C(t)\,u(t) \tag{2}$$

is controllable. The solution of system (1) on J with $x(t) = \phi(t)$ for $-1 \le t \le 0$ is given by the solution of integral equation:

$$x(t) = x(t,0,\phi) + \int_0^t X(t,s)\,C(s)\,u(s)\,\mathrm{d}s + \int_0^t X(t,s)\,f(s,\, x(s),\, x(s-1),\, u(s))\,\mathrm{d}s$$

where

$$\begin{aligned} x(t,0,\phi) &= X(t)\,\phi(0) + \int_{-1}^{0} X(t,\, s+1)\,B(s+1)\,\phi(s)\,\mathrm{d}s + \\ &\quad + \int_{-1}^{0}\int_{0}^{\tau+1} X(t,s)\,K(s,\, \tau - s)\,\phi(\tau)\,\mathrm{d}s\mathrm{d}\tau \end{aligned}$$

and $X(t,s)$ is an $n \times n$ matrix function satisfying

$$\frac{\partial X(t,s)}{\partial t} = A(t)\,X(t,s) + B(t)\,X(t-1,s) + \int_{-1}^{0} K(t,\tau\,X(t+\tau,s)\,\mathrm{d}\tau$$

for $0 \le s \le t \le t_1$ such that $X(t,t) = I$, the identity matrix and $X(t,s) = 0$ for $t < s$. Further $X(t,s)$ is continuous in the compact region $0 \le s \le t \le t_1$.

Define the controllability matrix by

$$W = W(0,t_1) = \int_0^{t_1} X(t_1,s)\,C(s)\,C^*(s)\,X^*(t_1,s)\,\mathrm{d}s,$$

where the star denotes the matrix transpose. It is clear that x_1 is reachable from the initial function $\phi(t)$ if there exists continuous functions $x(\cdot)$ and $u(\cdot)$ such that

$$u(t) = C^*(t)X^*(t_1,t)W^{-1}\left[x_1 - x(t_1,0,\phi) - \int_0^{t_1} X(t_1,s)f(s,x(s),x(s-1),u(s))\,\mathrm{d}s\right] \quad (3)$$

$$x(t) = x(t,0,\phi) + \int_0^{t} X(t,s)\,[C(s)\,u(s) + f(s,\,x(s),\,x(s-1),\,u(s))]\,\mathrm{d}s \quad (4)$$

and

$$x(t) = \phi(t) \quad \text{on } [-1,0].$$

We must find conditions for the existence of such $x(\cdot)$ and $u(\cdot)$. If $\alpha_i \in L^1(J)$, $i = = 1,2,\ldots,q$ then $\|\alpha_i\|$ is the L^1 norm of $\alpha_i(s)$, that is,

$$\|\alpha_i\| = \int_0^{t_1} |\alpha_i(s)|\,\mathrm{d}s.$$

Next, for our convenience, let us introduce the following notations:

$$\begin{aligned}
K &= \max\left\{\|X(t,s)\| : 0 \le s \le t \le t_1\right\}, \\
k &= \max\left\{\|X(t,s)\,C(s)\|\,t_1,\,1\right\}, \\
a_i &= 6k\left\{\|C^*(s)\,X^*(t_1,s)\|\,\|W^{-1}\|\,\|X(t_1,s)\|\,\|\alpha_i\|\right\}, \\
b_i &= 6K\|\alpha_i\|, \\
c_i &= \max\{a_i, b_i\} \\
d_1 &= 6k\|C^*(s)\,X^*(t_1,s)\|\,\|W^{-1}\|\,[|x_1 - x(t_1,0,\phi)|], \\
d_2 &= 6k|x(t_1,0,\phi)|, \\
d &= \max\{d_1, d_2\}.
\end{aligned} \quad (5)$$

3. MAIN RESULTS

Now let us prove our main result in this section. For this we put $p = (x,y,u)$ and $\|p\| = |x| + |y| + |u|$.

Theorem 3.1. Let measurable functions $\phi_i : R^{2n} \times R^m \to R^+$ $(i = 1, 2, \ldots, q)$ and L^1 functions $\alpha_i : J \to R^+$ $(i = 1, 2, \ldots, q)$ be such that

$$|f(t,p)| \leq \sum_{i=1}^{q} \alpha_i(t)\, \phi_i(p) \quad \text{for every } (t,p) \in J \times R^{2n} \times R^m.$$

Then, the controllability of (2) implies the controllability of (1) if

$$\limsup_{r\to\infty} \left(r - \sum_{i=1}^{q} c_i \sup\{\phi_i(p) : \|p\| \leq r\} \right) = \infty. \tag{6}$$

Proof. Let $Q = C(J;\, R^n \times R^m)$ and define $T : Q \to Q$ as follows

$$T(x,u) = (z,v),$$

where

$$v(t) = C^*(t)X^*(t_1,t)W^{-1}\left[x_1 - x(t_1,0,\phi) - \int_0^{t_1} X(t_1,s) f(s,x(s),x(s-1),u(s))\,\mathrm{d}s\right] \tag{7}$$

$$z(t) = x(t,0,\phi) + \int_0^t X(t,s)\,[C(s)v(s) + f(s,x(s),x(s-1),u(s))]\,\mathrm{d}s \tag{8}$$

and

$$z(t) = \phi(t) \quad \text{on } [-1,0].$$

Under our regularity assumptions of f, T is continuous. Clearly the solutions $u(\cdot)$ and $x(\cdot)$ to (3) and (4) are fixed points of T. We will prove the existence of such fixed points by using the Schauder fixed point theorem. Let

$$\psi_i(r) = \sup\{\phi_i(p) : \|p\| \leq r\}.$$

Since (6) holds, there exists $r_0 > 0$ such that

$$\sum_{i=1}^{q} c_i\, \psi_i(r_0) + d \leq r_0.$$

Now, let

$$Q_{r_0} = \{(x,u) \in Q : \|x\| \leq r_0/3,\ \|u\| \leq r_0/3\}.$$

If $(x,u) \in Q_{r_0}$, from (7) and (8), we have

$$\begin{aligned}
\|v\| \leq\ & \|C^*(t)\,X^*(t_1,s)\|\, \|W^{-1}\| \Bigg[|x_1 - x(t_1,0,\phi)| \\
& + \int_0^{t_1} \|X(t_1,s)\| \sum_{i=1}^{q} \alpha_i(s)\, \phi_i(x(s),\, x(s-1),\, u(s))\,\mathrm{d}s \Bigg] \\
\leq\ & \|C^*(t)\,X^*(t_1,s)\|\, \|W^{-1}\| \Bigg[|x_1 - x(t_1,0,\phi)|
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^{t_1} \|X(t_1, s)\| \left(\sum_{i=1}^{q} \alpha_i(s)\, \psi_i(r_0) \right) \mathrm{d}s \Bigg] \\
&\leq \ (1/6k) \left(d + \sum_{i=1}^{q} c_i\, \psi_i(r_0) \right) \\
&\leq \ (r_0/6k) \leq (r_0/6)
\end{aligned}
$$

and

$$
\begin{aligned}
\|z\| \ &\leq \ |x(t, 0, \phi)| + \int_0^{t} \|X(t, s)\, C(s)\|\, \|v\|\, \mathrm{d}s \\
&\quad + \int_0^{t} \|X(t, s)\| \sum_{i=1}^{q} \alpha_i(s)\ [\phi_i(x(s),\ x(s-1),\ u(s))\, \mathrm{d}s] \\
&\leq \ (d/6) + k\|v\| + K \sum_{i=1}^{q} \|\alpha_i\|\, \psi_i(r_0) \\
&\leq \ (d/6) + k\|v\| + (1/6) \sum_{i=1}^{q} c_i\, \psi_i(r_0) \\
&\leq \ (1/6) + \left(d + \sum_{i=1}^{q} c_i\, \psi_i(r_0) \right) + k\|v\| \\
&\leq \ (r_0/6) + (r_0/6) = r_0/3.
\end{aligned}
$$

Hence, T maps Q_{r_0} into itself. Next, we show that $T(Q_r)$ is equicontinuous for all $r > 0$. To prove this note that for all $(x, u) \in Q_r$ and $s_1,\ s_2 \in J,\ s_1 < s_2$ we have

$$
\begin{aligned}
&\|v(s_1) - v(s_2)\| \leq \|C^*(s_1) X^*(t_1, s_1) - C^*(s_2) X^*(t_1, s_2)\|\, \|W^{-1}\| \Bigg[|x_1 - x(t_1, 0, \phi)| \\
&+ \int_0^{t_1} \|X(t_1, s)\| \sum_{i=1}^{q} \alpha_i(s)\, \phi_i(x(s),\ x(s-1),\ u(s))\, \mathrm{d}s \Bigg] \\
&\leq \|C^*(s_1)\, X^*(t_1, s_1) - C^*(s_2)\, X^*(t_1, s_2)\|\, \|W^{-1}\| \Bigg[|x_1 - x(t_1, 0, \phi)| \\
&+ \|X(t_1, s)\| \sum_{i=1}^{q} \|\alpha_i\|\, \psi_i(r) \Bigg]
\end{aligned}
\tag{9}
$$

and

$$
\begin{aligned}
&\|z(s_1) - z(s_2)\| \leq |x(s_1, 0, \phi) - x(s_2, 0, \phi)| \\
&+ \int_0^{s_1} \|X(s_1, s) - X(s_2, s)\|\, \|C(s)\|\, \|v\|\, \mathrm{d}s + \int_{s_1}^{s_2} \|X(s_2, s)\|\, \|C\|\, \|v\|\, \mathrm{d}s \\
&+ \int_0^{s_1} \|X(s_1, s) - X(s_2, s)\| \sum_{i=1}^{q} \alpha_i(s)\, \psi_i(r)\, \mathrm{d}s + \int_{s_1}^{s_2} \|X(s_2, s)\| \sum_{i=1}^{q} \alpha_i(s)\, \psi_i(r)\, \mathrm{d}s
\end{aligned}
$$

$$\leq |x(s_1,0,\phi) - x(s_2,0,\phi)| + \|X(s_1,s) - X(s_2,s)\|\,\|C\|\,\|v\|\,t_1 + \|X(s_2,s)\|\,\|C\|\,\|v\|\,(s_2 - s_1) + \|X(s_1,s) - X(s_2,s)\| \sum_{i=1}^{q} \|\alpha_i\|\,\psi_i(r) + \|X(s_2,s)\|\,(s_2 - s_1) \sum_{i=1}^{q} \alpha_i(s)\,\psi_i(r). \quad (10)$$

Moreover, for all $(x,u) \in Q_r$,

$$\begin{aligned} \|v\| &\leq \|C^*(t)\,X^*(t_1,s)\|\,\|W^{-1}\| \left[|x_1 - x(t_1,0,\phi)| + \int_0^{t_1} \|X(t_1,s)\| \sum_{i=1}^{q} \alpha_i(s)\,\psi_i(r)\,\mathrm{d}s \right] \\ &\leq \|C^*(t)\,X^*(t_1,s)\|\,\|W^{-1}\| \left[|x_1 - x(t_1,0,\phi)| + \|X(t_1,s)\| \sum_{i=1}^{q} \|\alpha_i\|\,\psi_i(r) \right]. \end{aligned}$$

Thus, the right hand side of (9) and (10) do not depend on particular choices of (x,u). Hence, it is clear that $T(Q_r)$ is equicontinuous for all $r > 0$. By the Ascoli–Arzela theorem, $\overline{T(Q_r)}$ is compact in Q, that is, T is a compact operator. Since Q_{r_0} is nonempty, closed, bounded and convex, by the Schauder fixed point theorem, solutions of (3) and (4) exist. □

4. APPLICATIONS

To apply the above theorem, one has to construct α_i's and ϕ_i's such that (6) is satisfied. These constructions are different for different situations. However, an obvious construction of α_i's and ϕ_i's is easily achieved by letting $q = 1$, $\alpha_1 = \alpha = 1$ and

$$\phi_1(p) = \phi(p) = \sup\{|f(t,p)| : t \in J\}.$$

In this case (6) holds if

$$\liminf_{r\to\infty} (1/r) \sup\{\phi(p) : \|p\| \leq r\} < 1/c_1.$$

The following two corollaries are the direct consequence of the Theorem 3.1.

Corollary 4.1. If f is continuous on $J \times R^{2n+m}$ and

$$\lim_{|p|\to\infty} \frac{|f(t,p)|}{|p|} = 0, \quad \text{uniformly in } t,$$

then (1) is controllable if (2) is controllable.

Corollary 4.2. If $f(t,p)$ is continuous on $J \times R^{2n+m}$, locally bounded in u and

$$\lim_{|u|\to\infty} \frac{|f(t,p)|}{|u|} = 0, \quad \text{uniformly in } t,$$

then (1) is controllable if (2) is controllable.

Corollary 4.3. Suppose that there exist L^1 functions α, β and monotonically nondecreasing functions ϕ, τ, ψ such that

$$|f(t,p)| \le \alpha(t)\left(\phi(|x|) + \tau(|y|) + \psi(|u|)\right) + \beta(t), \quad \text{for all } (t,p) \in J \times R^{2n+m}.$$

Let

$$c = \max\left\{6k\|C^*X^*(t_1,s)\|\,\|W^{-1}\|\,\|X(t_1,s)\|\,\|\alpha\|,\, 6K\|\alpha\|\right\}.$$

Then (1) is controllable if (2) is controllable and

$$\limsup_{r\to\infty}\left(r - c(\phi(r) + \tau(r) + \psi(r)\right) = \infty.$$

In particular this is true if

$$\liminf_{r\to\infty}\left(\phi(r) + \tau(r) + \psi(r)\right)/r < 1/c. \tag{11}$$

Proof. Apply Theorem 3.1 with $q = 2,\ \alpha_1 = \beta,\ \alpha_2 = \alpha$

$$\phi_1(p) = 1 \quad \text{and} \quad \phi_2(p) = \phi(|x|) + \tau(|y|) + \psi(|u|).$$

First, note that $c = c_2$ where c_2 is defined by (5). To prove the corollary, we need to show that the condition (6) holds. However, this is trivial, since

$$\begin{aligned} &\limsup_{r\to\infty}\left(r - \sup_{\|p\|\le r}\left\{c_1 + c_2(\phi(|x|) + \tau(|y|) + \psi(|u|))\right\}\right) \\ \ge\ &\limsup_{r\to\infty}\left(r - c_1 - c_2(\phi(r) + \tau(r) + \psi(r))\right) = \infty. \end{aligned}$$

Hence by Theorem 3.1, the controllability of (2) implies the controllability of (1). □

Corollary 4.4. Consider (1), where

$$|f(t,p)| \le \alpha(t)\left(\|p\|\right) + \beta(t).$$

Here, $\alpha(t),\ \beta(t) \ge 0$, and both belong to $L^1(J)$. Assume (2) is controllable on J. Then there exists an $A_0 > 0$, which depends on only the matrix functions $A(t)$ and $B(t)$, such that (1) is controllable on J provided that $\|\alpha\| \le A_0$.

Proof. Apply the above Corollary 4.3, with

$$\phi(|x|) = |x|, \quad \tau(|y|) = |y|, \quad \psi(|u|) = |u|.$$

From condition (11), we have

$$\lim_{r\to\infty}\left(\phi(r) + \tau(r) + \psi(r)\right)/r = 1/2 \le 1/c = (1/\tilde{c}\|\alpha\|) \quad \text{if} \quad \|\alpha\| < (2/\tilde{c})$$

where $\tilde{c} = \max\left\{6k\|C^*X^*(t_1,s)\|\,\|W^{-1}\|\,\|X(t_1,s)\|,\, 6K\right\}$.

Here Corollary 4.4 hold with $A_0 < (2/\tilde{c})$. □

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