NONLINEAR STABILIZATION BY ADDING INTEGRATORS[1]

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In this paper, we study the global stabilization, by means of smooth state feedback, of systems (S) obtained by adding an integrator to a general nonlinear system $(\sum)$. We show how to compute the stabilizing feedback for (S) when a stricte Lyapunov function for $(\sum)$ is difficult to find.

1. INTRODUCTION

In this paper we deal with the global stabilization of nonlinear control systems of the form:

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = u \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^p$ and f is a smooth vector field such that $f(0,0) = 0$.

It is well known [3, 6, 7] that if the subsystem:

$$\dot{x} = f(x,v) \tag{2}$$

where v is the input, is globally asymptotically stabilizable (G.A.S) by means of feedback law $v = k(x)$, where k is of class C^r, $r \geq 1$ then system(1) is G.A.S. Moreover, if it is possible to construct a Lyapunov function V such that

$$\langle \nabla V, f(x,k(x)) \rangle < 0 \quad \forall x \in \mathbb{R}^n, \; x \neq 0 \tag{3}$$

(V is said a strict Lyapunov function for system (2) and exists by the Lyapunov inverse theorem) then a stabilizing feedback $u(x,y)$, which depends on k and V, is explicitly given. However it is not easy to find a function V satisfying (3) even if one knows that the origin is a globally asymptotically stable equilibrium point for the closed loop system

$$\dot{x} = f(x,k(x)). \tag{4}$$

The goal of this paper is to weaken these hypotheses. Theorem 1 shows that, to find a feedback stabilizer for system (1), we do not need to have a strict Lyapunov

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function for (2). Theorem 2 shows how to asymptotically stabilize system(1) without stabilizing system (2).

We recall that the relationship between the stabilizability of (2) and (1) is an open problem when system (2) is stabilizable by means of continuous feedback (not C^1). This problem was addressed in [1] and [2] from the local stabilization point of view. The authors proved that the local stabilizability of (2) is equivalent to the local stabilizability of (1) if $n = p = 1$ and f is a real analytic function.

2. MAIN RESULTS

Before stating the first theorem we introduce the following notations and definitions:

Definition 1. We shall say that system (2) is of LA SALLE-Type (L-T) if there exist:

1. a function $k : \mathbb{R}^n \to \mathbb{R}^p$ of class $C^r (r \geq 1)$ with $k(0) = 0$
2. a function $V : \mathbb{R}^n \to \mathbb{R}$ of class C^1, definite positive and proper such that:
   i) $X \cdot V(x) \leq 0 \quad \forall x \in \mathbb{R}^n$ where $X(x) = f(x, k(x))$ and $X \cdot V$ is the Lie-derivative of V along the trajectories of the vector field X
   (here: $X \cdot V(x) = \langle \nabla V, X(x) \rangle$ where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^n$)

   ii) The largest invariant set contained in $E = \{x \in \mathbb{R}^n | X \cdot V(x) = 0\}$ is the origin of $\mathbb{R}^n$.

Definition 2. A continuously differentiable scalar function $V : \mathbb{R}^n \to \mathbb{R}$ is a weak Lyapunov function for

$$\dot{x} = X(x)$$

if V is positive definite proper and

$$X \cdot V(x) \leq 0 \quad \forall x \in \mathbb{R}^n.$$

By a *proper function* we mean a function

$$V : \mathbb{R}^n \to \mathbb{R}$$

such that $\{x \in \mathbb{R}^n | V(x) \leq \xi\}$ is compact for each $\xi > 0$.

Through this paper $\|.\|$ will denote the usual Euclidian norm in $\mathbb{R}^p$, $X_t(\cdot)$ is the flow of the vector field X defined on $\mathbb{R}^n$.

Remarks.

1. System (2) is of (L-T) $\Leftrightarrow \dot{x} = f(x, k(x))$ is globally asymptotically stable (see [4]).
2. It is often easier to find V satisfying (i) and (ii) then a function V satisfying (3) (Mechanical systems are well known examples of this situation).

Theorem 1. If system (2) is of (L-T) then system (1) is also of (L-T) and the stabilizing feedback is

$$u = -y + k(x) + \mathrm{d}k(x).f(x,y) - G(x,y)^{\mathrm{T}} \cdot \nabla V(x). \tag{5}$$

($^{\mathrm{T}}$ = transpose)
where

$$f(x,y) = f(x,k(x)) + G(x,y) \cdot (y - k(x)). \tag{6}$$

Remark. One can choose for G the matrix:

$$G(x,y) = \int_0^1 \frac{\partial f}{\partial y}(x, ty + (1-t)k(x))\mathrm{d}t. \tag{7}$$

Proof. (2) is of L-T so there exist k and V satisfying i) and ii) of the above definition. Let us denote $X(x) = f(x,k(x))$ and Ω the largest invariant set by X contained in $E = \{x \in \mathbb{R}^n | X \cdot V(x) = 0\}$. By hypotheses $\Omega = \{0\}$.
Let

$$Z(x,y) = \begin{pmatrix} f(x,y) \\ u(x,y) \end{pmatrix}$$

where

$$u(x,y) = -y + k(x) + \mathrm{d}k(x) \cdot f(x,y) - G(x,y)^{\mathrm{T}} \cdot \nabla V(x)$$

and define (see [7])

$$W(x,y) = V(x) + \frac{1}{2}\|y - k(x)\|^2$$

W is of class C^1, definite positive and proper

$$\dot{W}(x,y) = Z \cdot W(x,y) = \langle Z(x,y), \nabla W(x,y)\rangle = X \cdot V(x) - \|y - k(x)\|^2 \leq 0$$

Note that all trajectories of the closed-loop system are bounded because W is proper and its derivative is nonpositive.
Let

$$\begin{aligned} \tilde{E} &= \{(x,y) \in \mathbb{R}^{n+p} | Z \cdot W(x,y) = 0\} \\ &= \{(x,y) \in \mathbb{R}^{n+p} | y = k(x) \text{ and } X \cdot V(x) = 0\}. \end{aligned}$$

According to LaSalle's theorem (see [4] pp. 66–67) all solutions tend to $\tilde{\Omega}$ the largest invariant set by Z contained in $\tilde{E}$. To prove Theorem 1 it remains to that $\tilde{\Omega}$ is the origin of $\mathbb{R}^{n+p}$.
On $\tilde{\Omega}$ the vector field Z is given by:

$$Z(x,k(x)) = \begin{pmatrix} X(x) \\ Y(x) \end{pmatrix}$$

where $Y(x) = \mathrm{d}k(x) \cdot f(x,k(x)) - G(x,k(x))^{\mathrm{T}} \cdot \nabla V(x)$ and $X(x) = f(x,k(x))$

$$\begin{cases} \dot{x} = f(x,k(x)) = X(x) \\ \dot{y} = \mathrm{d}k(x) \cdot f(x,k(x)) - G(x,k(x))^{\mathrm{T}} \cdot \nabla V(x). \end{cases}$$

Let $(x(t), y(t))$ be a solution of the above system starting at $(x(0), y(0))$, the initial condition $\in \tilde{\Omega}$. Since $\tilde{\Omega}$ is Z-invariant we have $(x(t), y(t)) \in \tilde{\Omega}$ for all $t \geq 0$ but we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(x(t)) = X(x(t))$$

hence $x(t) = X_t(x)$ where $X_t(\cdot)$ is the flow of the vector field X defined on $\mathbb{R}^n$.

Consider, now, the following set:

$$M = \{x \in \mathbb{R}^n | (x, k(x)) \in \tilde{\Omega}\}.$$

If $x \in M$ then $(x, k(x)) \in \tilde{\Omega}$ and $(x(t), y(t)) \in \tilde{\Omega}$ since $\tilde{\Omega}$ is invariant, this implies $(X_t(x), y(t)) \in \tilde{\Omega}$ but $y(t) = k(X_t(x))$.

So we have shown: $x \in M \Rightarrow (X_t(x), k(X_t(x))) \in \tilde{\Omega} \Rightarrow X_t(x) \in M$.

This proves that M is X-invariant and since M is contained in E we have $M = \{0\}$ and then $\tilde{\Omega} = \{(0,0)\}$ which completes the proof of Theorem 1. □

Example 1. Consider the following system which evolves in $\mathbb{R}^4$:

$$\begin{cases} \dot{x}_1 = x_2 x_3 + b_1 y \\ \dot{x}_2 = -x_1 x_3 + b_2 y \\ \dot{x}_3 = b_3 y \\ \dot{y} = u \end{cases} \tag{8}$$

where $b_3 \neq 0$ and $(b_1, b_2) \neq (0,0)$.
The subsystem

$$\begin{cases} \dot{x}_1 = x_2 x_3 + b_1 y \\ \dot{x}_2 = -x_1 x_3 + b_2 y \\ \dot{x}_3 = b_3 y \end{cases} \tag{9}$$

is the "famous" system of the angular velocity of a symmetric rigid body. In [5, 7] it is shown that (9) is smoothly globally asymptotically stabilizable. Furthermore, it is shown in [5] that the system (9) is of L-T with the two polynomial functions

$$\begin{aligned} k = \; & -b_3\, x_3 - \frac{\left({x_1}^2 + {x_2}^2 + x_3\right)\,\left(2 b_3 + 4\,(b_1 x_1 + b_2 x_2)\right) P(x_3)}{2} \\ & - \frac{b_3 ({x_1}^2 + {x_2}^2 + x_3)^2 P'(x_3)}{2} \end{aligned}$$

$$V = \frac{{x_3}^2 + \left({x_1}^2 + {x_2}^2 + x_3\right)^2 P(x_3)}{2}$$

where

$$P(x) = \frac{{b_3}^4}{4\,\left({b_1}^2 + {b_2}^2\right)} + {b_3}^2\, x + 2\,\left({b_1}^2 + {b_2}^2\right)\, x^2$$

so all the assumptions of Theorem 1 are satisfied and then our result can be the stabilizing feedback for system (8) using the formula (5). Note that a strict Lyapunov function for system (9) has never been found and may be difficult to construct so the results of [3, 6, 7] cannot be applied to stabilize system (8).
Generally, to compute the stabilizing feedback for systems of the form:

$$\begin{cases} \dot{x} = f(x) + g(x)y \\ \dot{y} = u \\ x \in \mathbb{R}^n, \; y \in \mathbb{R}^p, \; g(x) \in M_{n,p}(C^\infty(\mathbb{R}^n, \mathbb{R})) \end{cases}$$

where the subsystem $\dot{x} = f(x) + g(x)y$ is stabilizable via the Jurdjevic–Quinn's method, one can apply Theorem 1 but not the results based on strict Lyapunov function.

For the following result we suppose that f is a smooth (i.e. C^∞) vector field.

Theorem 2. If there exist a smooth feedback k (not necessarily a stabilizing one) for system (2) and a smooth function V, which is definite positive and proper, such that:

i) $X \cdot V(x) \le 0 \quad \forall\, x \in \mathbb{R}^n$ where $X(x) = f(x, k(x))$

ii) The set

$$S = \{x \in \mathbb{R}^n | X^{s+1} \cdot V(x) = X^s \cdot Y_i \cdot V(x) = 0, \; s \in \mathbb{N}, \; i = 1, \dots, p\}$$

where $Y_i = \dfrac{\partial f}{\partial y_i}(x, k(x))$, is reduced to the set $\{0\}$.

then system (1) is G.A.S and the stabilizing feedback:

$$u = -y + k(x) + \mathrm{d}k(x) \cdot f(x,y) - G(x,y)^{\mathrm{T}} \cdot \nabla V(x) \tag{10}$$

where G is defined by the formula (7).

Proof. We take $u = \mathrm{d}k(x) \cdot f(x,y) - G(x,y)^{\mathrm{T}} \cdot \nabla V(x) + v$ where v is a new input so system(1) can be written:

$$\begin{pmatrix} \dot{x} \\ y \end{pmatrix} = Z(x,y) + B(x,y) \cdot v = Z(x,y) + \sum_{i=1}^{p} v_i B_i$$

where

$$Z(x,y) = \begin{pmatrix} f(x, k(x)) + G(x,y) \cdot (y - k(x)) \\ \mathrm{d}k(x) \cdot f(x,y) - G(x,y)^{\mathrm{T}} \cdot \nabla V(x) \end{pmatrix}$$

and $B_i(x,y) = e_{n+i}$.

Introduce

$$W(x,y) = V(x) + \frac{1}{2} \|y - k(x)\|^2.$$

W is smooth, definite positive and proper

$$Z \cdot W(x,y) = X \cdot V(x) \leq 0.$$

According to [5], the above system is globally asymptotically stabilizable by the feedback

$$v = -B \cdot W(x,y)$$

if the set

$$A = \left\{(x,y) \in \mathbb{R}^{n+p} | Z^{s+1} \cdot W(x,y) = Z^s \cdot B_i \cdot W(x,y) = 0, s \in \mathbb{N}, i = 1, \ldots, p\right\}$$

is reduced to the origin of $\mathbb{R}^{n+p}$.
Since $Z.W(x,y) = X \cdot V(x)$ and $B \cdot W = y - k(x)$ we can write:

$$A = \{(x,y) \in \mathbb{R}^{n+p} | y = k(x) \text{ and } x \in C\}$$

where:

$$\begin{aligned} C = \quad & \{x \in \mathbb{R}^n | X \cdot V(x) = Z^{s+1} \cdot W(x,k(x)) = Z^s \cdot B_i \cdot W(x,k(x)) = 0, \\ & s \geq 1, \ i = 1, \ldots, p\}. \end{aligned}$$

We shall show that $C = S$. We have

$$\begin{aligned} & B \cdot W = y - k(x) \\ & Z \cdot B_i \cdot W = Z \cdot (y_i - k_i(x)) = \langle Z, \nabla(y_i - k_i(x))\rangle . \end{aligned}$$

For $(x,y) \in A$ the vector field Z is (since $y = k(x)$):

$$Z = \begin{pmatrix} X(x) \\ \mathrm{d}k(x) \cdot f(x,k(x)) - G(x,k(x))^{\mathrm{T}} \cdot \nabla V(x) \end{pmatrix}$$

so

$$\begin{aligned} \langle Z, \nabla(y_i - k_i(x))\rangle = \quad & -\langle X, \nabla k_i(x)\rangle \\ & + \left\langle \mathrm{d}k(x) \cdot f(x,k(x)) - G(x,k(x))^{\mathrm{T}} \cdot \nabla V(x), e_{n+i}\right\rangle . \end{aligned}$$

But

$$\begin{aligned} & \left\langle \mathrm{d}k(x) \cdot f(x,k(x)) - G(x,k(x))^{\mathrm{T}} \cdot \nabla V(x), e_{n+i}\right\rangle = \\ & \sum_{j=1}^{n} \frac{\partial k_i}{\partial x_j}(x) f_j(x,k(x)) - \sum_{j=1}^{n} \frac{\partial f_j}{\partial y_i}(x,k(x)) \frac{\partial V}{\partial x_j}(x) \\ & = \langle \nabla k_i(x), X(x)\rangle - \left\langle \nabla V(x), \frac{\partial f}{\partial y_i}(x,k(x))\right\rangle . \end{aligned}$$

Thus

$$Z \cdot B_i \cdot W(x,y) = -Y_i \cdot V(x)$$

where

$$Y_i = \frac{\partial f}{\partial y_i}(x,k(x)).$$

Now $$Z^2 \cdot B_i \cdot W(x,y) = Z \cdot (Z \cdot B_i \cdot W)(x,y) = -Z \cdot (Y_i \cdot V(x))$$

and since $Y_i \cdot V(x)$ is independent of y we have:

$$Z \cdot (Y_i \cdot V(x)) = X \cdot Y_i \cdot V(x)$$

so

$$Z^2 \cdot B_i \cdot W(x,y) = -X \cdot Y_i \cdot V(x)$$

and by induction we prove that for any integer $s \geq 1$ and any $(x,y) \in A$:

$$Z^s \cdot B_i \cdot W(x,y) = -X^{s-1} \cdot Y_i \cdot V(x), \quad i = 1, \ldots, p. \tag{11}$$

A similar computation shows that we have also for any integer s and any $(x,y) \in A$:

$$Z^{s+1} \cdot W(x,y) = X^{s+1} \cdot V(x) \tag{12}$$

The equalities (11) and (12) show that $C = S$ so Theorem 2 is proved. □

Example 2. Consider the following system:

$$\begin{cases} \dot{x} = \sin(xy) = f(x,y) \\ \dot{y} = u \\ (x,y) \in \mathbb{R}^2, \ u \in \mathbb{R}. \end{cases} \tag{13}$$

To stabilize this system we can try to stabilize first the system

$$\dot{x} = \sin(xy)$$

where y is considered as the control. This system is obviously asymptotically stabilizable by a smooth function $y = k(x)$ and after we can use Theorem 1 to stabilize system (13) but the feedback resulting is complicated.
Alternatively we can stabilize system (13) by a simpler smooth feedback if we apply Theorem 2 as follow:

Introduce

$$V(x) = \frac{1}{2}x^2 \quad \text{and} \quad k(x) = 0$$

we have

$$X \cdot V(x) = 0, \quad Y = \frac{\partial f}{\partial y}(x, k(x)) = x, \quad Y \cdot V(x) = 0 \Leftrightarrow x = 0.$$

This shows that V and k satisfy the hypotheses of Theorem 2 so system (13) is (G.A.S) by means of the feedback law:

$$u(x,y) = \begin{cases} -y - x\dfrac{\sin(xy)}{y} & \text{if } y \neq 0 \\ -x^2 & \text{otherwise} \end{cases}$$

Example 3.

$$\begin{cases} \dot{x}_1 = y(x_1 - x_2) \\ \dot{x}_2 = y(x_2 + y) \\ \dot{y} = v \\ x = (x_1, x_2) \in \mathbb{R}^2, \; y \in \mathbb{R}, \; v \in \mathbb{R}. \end{cases} \tag{14}$$

First, consider the reduced system

$$\begin{cases} \dot{x}_1 = u(x_1 - x_2) \\ \dot{x}_2 = u(x_2 + u) \end{cases} \tag{15}$$

where u is regarded as the control. Here

$$f(x,u) = \begin{pmatrix} u(x_1 - x_2) \\ u(x_2 + u) \end{pmatrix}, \quad \text{and} \quad \frac{\partial f}{\partial u}(x,u) = \begin{pmatrix} x_1 - x_2 \\ x_2 + 2u \end{pmatrix}.$$

If we choose $u = k(x_1, x_2) = 0$ and $V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$ then all the hypotheses of Theorem 2 are satisfied. In fact, with the notations used in the proof, we have

$$X.V(x_1, x_2) = 0, \quad Y = \frac{\partial f}{\partial u}(x, k(x)) = \begin{pmatrix} x_1 - x_2 \\ x_2 \end{pmatrix}$$

$$\left\langle \frac{\partial f}{\partial u}(x, k(x)), \nabla V(x) \right\rangle = x_1^2 - x_1 x_2 + x_2^2.$$

So $Y \cdot V(x) = 0 \Leftrightarrow x_1 = x_2 = 0$.
A stabilizer for (14), computed using (10), is

$$u = -y - x_1(x_1 - x_2) - x_2(x_2 + y).$$

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