KYBERNETIKA — VOLUME 30 (1994), NUMBER 4, PAGES 461–470

ON nth ORDER DIFFERENTIAL EQUATIONS OVER HARDY FIELDS

A. RAMAYYAN

Some properties of nth order Differential Equations over Hardy Fields are studied. A characterization of a nonhomogeneous linear differential equation to be **nonoscillatory** is also given.

1. INTRODUCTION

First and second order differential equations over Hardy fields have been studied by M. Boshernitzan [1–3], V. Maric [9] and M. Rosenlicht [10–14]. In this paper we present some theorems concerning nth order differential equations over Hardy fields which include the extension of some of the results given in [2] and a necessary and sufficient condition for a nonhomogeneous nth order linear ordinary differential equation over a Hardy field to be nonoscillatory. Our results are much more general than the corresponding results of M. Boshernitzan [2].

These problems have applications in control theory, specially for linear dynamical systems described by ordinary differential equations of nth order, with time varying coefficients. The behaviour of solution for $t \to \infty$ (oscillatory or nonoscillatory, stable or unstable) can be investigated by the mathematical device of ordered fields, valuation and Hardy fields. In this way, the known results of polynomial approach for time-invariant system can be generalized.

Most basic definitions and facts on Hardy fields are collected in the next section.

2. DEFINITIONS AND PRELIMINARIES

2.1. L-field of Hardy. G. H. Hardy [7, Page 17] considered a class L of logarithmic-exponential functions (L-functions in short). These are real single valued functions, defined for all values of x greater than some definite value, by a finite combination of the ordinary algebraic symbols (viz $+$, $-$, $\times$, $\div$, $\surd$) and the functional symbols $\log(\cdots)$ and $\exp(\cdots)$, operating on the variable x and on real constants. For example, $f(x) = \sqrt{x} + \log x + \sqrt{x^2+1} + 3e^x$ is a L-function. It is to be observed that the result of working out the value of the function, by substituting x in the formula defining it, is to be real at all stages of the work. It is important to exclude such a function as $\frac{1}{2}\left\{\exp\left(\sqrt{-x^2}\right)+\exp\left(-\sqrt{-x^2}\right)\right\}$, which with a suitable interpretation of the roots, is equal to $\cos x$. Any L-function is ultimately continuous, of constant sign and monotonic, and tends, as $x \to \infty$, to infinitely, or to zero, or to some other definite limit [7, Theorem 13]. This class L is closed under differentiation, integration and composition, and forms a field [1]. This field is called L-field of Hardy. Extending the notion of L-field, M. Boshernitzan [2] and M. Rosenlicht [10] developed the theory of Hardy fields. To understand its properties we need to know the concept of germs of functions and ordered fields.

2.2. Germs of functions. Let B denote a class of continuous real valued functions $f(x)$ which are defined for sufficiently large x. We can identify functions which agree for large x and so B is the class of germs of functions at infinity [1, Page 237]. For example, an L field is the field of germs of functions obtained from the field of rational functions of one variable by repeated adjunction of real valued algebraic functions, logarithm of positive functions and exponential of functions.

2.3. Ordered field. An integral domain $(D, +, \cdot)$ is said to be ordered if D contains a subset D_+ such that

i) D_+ is closed with respect to addition and multiplication as defined in D.

ii) For all $a \in D$, one and only one of $a = 0$, $a \in D_+$, $-a \in D_+$ holds. (Principle of Trichotomy.)

It should be noted that every field is an integral domain. So a field $(F, +, \cdot)$ is said to be ordered if it is ordered as an integral domain. For example, the field of rational numbers is ordered and the L-field is ordered, the positive elements being those that are ultimately positive (i. e. positive for sufficiently large values of $x \in R$).

2.4. Hardy fields. A Hardy field K [10] is a set of germs of real valued functions on deleted neighbourhoods of $+\infty$ in R (or, which is the same, on positive half lines in R) that is closed under differentiation and that form a field under the usual addition and multiplication of germs. Examples of Hardy fields are any subfield of R (viz, Q, R) and the field of rational functions of one variable $R(x)$, where each real number is identified with a constant germ and x is the germ determined by the identity function on R. L-field of Hardy and the fields $R(x, e^x)$ generated by R and the functions indicated in the brackets, are also Hardy fields. More generally, if K is a Hardy field and $f(x)$ a germ such that f is algebraic over K or $f'(x) \in K$ or $\frac{f'(x)}{f(x)} \in K$ then $K(f)$ is a Hardy field.

If K is a Hardy field and f a non zero element of K then K contains $\frac{1}{f}$ which implies $f(x) \neq 0$, if $x \in R$ is sufficiently large. Since $f'(x) \in K$, f is differentiable for sufficiently large $x \in R$ therefore continuous, and therefore $f(x)$ is always positive or always negative for x sufficiently large. Thus each $f \in K$ is ultimately either zero, or always positive or always negative. The same being true for $f' \in K$, each $f \in K$ is ultimately monotonic. In particular, for each $f \in K$, $\lim_{x\to\infty} f(x)$ exists as an element of $R \cup \{+\infty, -\infty\}$. A Hardy field is an ordered field, its positive elements being those that are ultimately positive, that is positive for x sufficiently large x. Each non zero element of a Hardy field ultimately has a constant sign. The functions such as e^x, $\log x$ and polynomials in x belong to a Hardy field.

A Hardy field is called maximal if it is not a proper subfield of any other Hardy field. Any maximal Hardy field is unbounded [5, Theorem 1.1].

Boshernitzan [3, Page 1] defines E as the intersection of all maximal Hardy fields. It contains the L-field of Hardy. It is closed under integration and composition [1, Section 6].

For any Hardy field K, $E(K)$ denotes the intersection of all maximal Hardy fields containing K. $K \subset E(K)$, $E \subset E(K)$ and $E = E(R)$ where R denotes the field of real constants [2, Page 132].

If $E(K) = K$, K is said to be perfect. E is thus the minimal perfect Hardy field and $E(K)$ is the minimal perfect Hardy field containing K [3, Page 1]. If K is perfect it is closed under integration; if $f \in K$ then $\exp(f) \in K$ and $E \subset K$ [2, Lemma 11.6].

Germs f, g of continuous real valued functions on positive half-lines in R which are nowhere zero on some half-line and are such that $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ are either 0, or $\pm\infty$, will be called comparable if on some half-line, each of $|f|$, $|g|$ is bounded above and below by suitable integral powers of the other. Comparability is an equivalence relation among such germs. The rank of a Hardy field is a number of its comparability classes. For example, the Hardy field $R(x, e^x)$ has rank 2 with x and e^x representatives of its comparability classes.

Any Hardy field K has a canonical valuation [10 and 11]. It is a homomorphism ν from the multiplicative group $K^* = K - \{0\}$ of K onto an ordered abelian group (the value group) $\nu(K^*)$. The kernel of ν consists of all $f \in K^*$ such that $\lim_{x\to\infty} f(x)$ is finite and nonzero, while $\nu(f) > 0$ if and only if $\lim_{x\to\infty} f(x) = 0$ and $\nu(f) < 0$ if and only if $\lim_{x\to\infty} f(x) = \pm\infty$. Let a, $b \in K^*$, we write $\nu(a) > \nu(b)$ if $\lim_{x\to\infty} \frac{a(x)}{b(x)} = 0$ and $\nu(a) \geq \nu(b)$ if $\lim_{x\to\infty} \frac{a(x)}{b(x)}$ is finite. Also, if a, $b \in K^*$ and $\nu(a)$, $\nu(b) \neq 0$ then $\nu(a) > \nu(b)$ if and only if $\nu(a') > \nu(b')$.

3. MAIN RESULTS

Throughout we shall assume that a germ means a germ of nontrivial linearly independent solution at infinity of (1) or (2) (given below) unless otherwise specified. It should be noted that n solutions at infinity of (1) namely $y_1(x), y_2(x), \ldots, y_n(x)$, are said to be linearly independent solutions at infinity iff their Wronskian is not equal to zero. Let us consider the nth order linear homogeneous equation

$$L(y) = y^{(n)} + a_1\, y^{(n-1)} + \cdots + a_n(x)\, y = 0 \tag{1}$$

and the nonhomogeneous equation

$$L(y) = f(x) \tag{2}$$

where $a_i(x),\ f(x) \in K$ $(i = 1, 2, \ldots, n)$.
A nontrivial solution $y = s(x)$ of (1) or (2) at infinity is said to be an oscillating germ

at infinity if $s(x_k) = 0$ for a sequence $\{x_k\}$ such that $\lim_{k\to\infty} x_k = \infty$. Otherwise it is called a nonoscillating germ at infinity. The equation (1) or (2) is said to be nonoscillatory if all the nontrivial solutions at infinity are nonoscillating germs at infinity. Germs such as $x^2 \sin\left(\frac{1}{x}\right)$ are nonoscillating though it has infinite number of zeros in $[0, 1)$ and no zeros in $[1, \infty)$.

Example 1. If the roots of characteristic polynomial of the nth order linear differential equation (l.d.e.) with constant coefficients are all real then it can be shown that the equation is nonoscillatory and all its solutions belong to some Hardy field K.

Theorem 1. Let K be any Hardy field. Then the nonoscillating germs of the equation (1) over K belong to $E(K)$.

Proof. We know that the nonoscillating germ of any second order linear homogeneous equation over K belongs to $E(K)$ [2, Theorem 16.8]. So it follows that the nonoscillating germ of any linear differential equation over K belong to $E(K)$ irrespective of its order. Hence the theorem follows.

Example 2. When $n = 4$, consider the l.d.e. $y^{(4)} - y = 0$ over any Hardy field K. Then the nonoscillating germs at infinity $\{e^x, e^{-x}\}$ belong to $E(K)$.

Remark 1. The above theorem fails, if the l.d.e. is not taken over K. For instance consider the following example.

Example 3. $e^{\sin x}$ is a nonoscillating solution of the differential equation

$$y''' - \cos x\, y'' + 2\sin x\, y' + \cos x\, y = 0$$

but $\{e^{\sin x}\} \neq E(K)$ for any K.

Corollary 1.1. If $y_1, y_2, \ldots, y_p$ are linearly independent nonoscillating germs of the equation (1) over K then they lie on the Hardy field $K(y_1, y_2, \ldots, y_p) \supset K$, whose rank is at most $r + p$ $(p \leq n)$ where rank $(K) = r$.

The first part follows by [10, Cor. 1 of Theorem 2] and the second part follows from the definition of rank.

Theorem 2. Let K be a perfect Hardy field. Then the solution of (2) lie in K provided all the solutions of equation (1) lie in K.

Proof. Let $y_1, y_2, \ldots, y_n$ be the nonoscillating solutions of (1) and y_p be the particular solution of equation (2). To prove the theorem it is enough if we prove $y_p(x) \in K$. It can be shown by the method of variation of constants that $y_p(x)$ has the form

$$y_p(x) = \sum_{k=1}^{n} y_k(x) \int^x \frac{W_k(t)\, f(t)\, \mathrm{d}t}{W(y_1, \ldots, y_n)(t)}, \tag{3}$$

where $W(y_1, y_2, \dots, y_n)(t)$ is the Wronskian of $y_1, y_2, \dots, y_n$ and W_k is the determinant obtained from $W(y_1, \dots, y_n)$ by replacing the kth column $(y_k, y'_k, \dots, y_k^{(n-1)})$ by $(0, 0, \dots, 0, 1)$. The terms on the right hand side belong to K since K is closed with respect to integration [2, Lemma 11.6]. It follows that $y_p(x) \in K$. □

Corollary 2.1. The equation $y^{(n)}(x) = f(x)$ where $f(x) \in K$ cannot have oscillating solutions at infinity. The proof is obvious.

Example 4. When $n = 3$, consider the equation

$$P(x) = x^3\, y''' - 3x^2\, y'' + 6x\, y' - 6y = x^2$$

over the field $E = E(R)$, the minimal perfect Hardy field. It can be shown that the solution basis for the homogeneous equation $P(x) = 0$ is $\{x, x^2, x^3\}$. It is clear that any solution of $P(x) = 0$ and of the nonhomogeneous equation $P(x) = x^2$ also lie on E. Thus Theorem 2 is verified.

Remark 2. Theorem 1 and 2 are the generalization of Theorems 16.8 and 16.10 of [2] respectively.

Remark 3. Theorem 2 leads us to establish a necessary and sufficient condition for a nth order l.d.e. over K to be nonoscillatory. To prove the theorem we require the following four propositions whose proofs are obvious so we omit them.

Proposition 1. Linear combination of finite number of nonoscillating germs in K is a nonoscillating germ in K.

Proposition 2. Product of two nonoscillating germs in K is a nonoscillating germ in K.

Proposition 3. Quotient $\frac{f(x)}{g(x)}$ $(g(x) \neq 0)$ of two nonoscillating germs in a nonoscillating germ in K.

Proposition 4. Let K be a perfect Hardy field. If $f(x) \in K$ is a nonoscillating germ then $\int^x f(t)\,dt \in K$ is a nonoscillating germ.

Remark 4. As it can be seen easily as given in the example below we note that any nth order nonhomogeneous l.d.e. with nonoscillatory homogeneous part need not be nonoscillatory.

Consider the l.d.e. of third order

$$x^3\, y''' - 3x^2\, y'' + 6x\, y' - 6y = \sin(\log x)$$

in $C'''(a, \infty)$ $(a > 0)$. It can have a particular solution which is oscillating though the homogeneous part is nonoscillatory.

In contrast to this situation in the classical theory of l.d.es, we are able to get the following necessary and sufficient condition for a nonhomogeneous nth order l.d.e. over K.

Theorem 3. The nonhomogeneous equation (2) over a perfect Hardy field K is nonoscillatory if and only if the homogeneous equation (1) over K is nonoscillatory.

Proof. Let equation (2) be nonoscillatory, then by definition all its solutions at infinity are nonoscillating germs and so in particular the homogeneous equation (1) is nonoscillatory. This proves the "if" part of the theorem.

To prove the other half, assume that the homogeneous equation (1) to be nonoscillatory. Now y_p is given by (3) which is clearly nonoscillating in K by Proposition 1, 2, 3, 4 and so the equation (2) is nonoscillatory. □

Definition. Consider the nth order Euler's equation

$$M(y) = x^n\, y^{(n)} + a_1\, x^{n-1}\, y^{(n-1)} + \cdots + a_n\, y = 0 \tag{4}$$

where $a_1, a_2, \ldots, a_n$ are real constants. Then the polynomial q given by

$$q(r) = r(r-1)\cdots(r-n+1) + a_1\, r(r-1)\cdots(r-n+2) + \cdots + a_n \tag{5}$$

is called the indicial polynomial for (4).

Corollary 3.1. The nonhomogeneous Euler's equation $M(y) = f(x)$ over the Hardy field K is nonoscillatory if and only if the indicial equation (5) has real roots.

Example 5. When $n = 3$, consider the equation

$$y''' - 3y'' + 3y' - y = e^{2x} \quad \text{over } E$$

which is nonoscillatory over E with $\{e^x,\, xe^x,\, x^2\, e^x\}$ as a solution basis for the homogeneous part. It can be shown that $y_p \in E$. This verifies the theorem.

Definition. If V is a finite dimensional vector space, an ordered basis for V is a finite sequence of vectors which is linearly independent and spans V. It can be proved that any n linearly independent vectors in an n-dimensional space V are a basis for V.

Theorem 4. If the equation (1) is nonoscillatory over K, there exists an ordered basis $\{y_1, \ldots, y_n\}$ ($y_i > 0$ in K for V the vector space of all solutions at infinity of the equation (1) such that

$$\nu(y_1) \geq \nu(y_2) \geq \nu(y_3) \geq \cdots \geq \nu(y_n).$$

Also the Wronskian $W(y_1, \ldots, y_n) = c\exp\left(-\int^x a_1(t)\,\mathrm{d}t\right)$ where $c \neq 0$ is a constant, belongs to K when K is perfect.

Proof. By hypothesis, there exists n linearly independent nonoscillating germs $y_1, \ldots, y_n$ for the equation (1). Clearly they form an ordered basis. If $y_1 < y_2 < y_3 < \cdots < y_n$,

$$\lim_{x\to\infty}\left(\frac{y_i}{y_{i+1}}\right) \quad \text{is finite.}$$

Therefore $\nu(y_i) \geq \nu(y_{i+1})$ $(i = 1, 2, \ldots, n-1)$.
An easy calculation gives the Wronskian and it belongs to K since K is perfect. □

Remark 5. In the example (4) the ordered basis is $\{x, x^2, x^3\}$ with $\nu(x) > \nu(x^2) > \nu(x^3)$. An ordered basis do not exist in the case of oscillatory l.d.es in $c^k(a, \infty)$.

Theorem 5. Let K be a perfect Hardy field. If the equation (1) is nonoscillatory over K with an ordered basis $\{y_1, \ldots, y_n\}$ $(y_i > 0,\ i = 1, 2, \ldots, n)$ for its solution space such that

$$\nu(y_1) > \nu(y_2) > \cdots > \nu(y_n)$$

then for all (k, j) $(j > k,\ j, k = 1, 2, \ldots, n)$,

$$\text{i)}\quad \sum_{j>k} \frac{W(y_j, y_k)}{y_j^2} < 0 \qquad \text{ii)}\quad W(y_k, y_j) > 0$$

$$\text{iii)}\quad \int_c^\infty W(y_k, y_j)\,\phi(y_k, y_j)\,\mathrm{d}t \quad \text{is convergent}$$

when $\phi(y_k, y_j)$ is any of the following functions or their finite linear combinations

$$\frac{1}{y_j^2}, \qquad \frac{1}{y_k^2 + y_j^2}, \qquad \frac{\exp\left(\frac{y_k}{y_j}\right)}{y_j^2}, \qquad \frac{\sin\left(\frac{y_k}{y_j}\right)}{y_j^2}, \qquad \frac{\cos\left(\frac{y_k}{y_j}\right)}{y_j^2}.$$

$$\text{iv)}\quad \int_c^\infty \frac{W(y_k, y_j)}{y_k\, y_j}\,\mathrm{d}t \quad \text{is convergent if } \tfrac{y_j}{y_k} \text{ belongs to the kernel of } \nu$$

$$\text{v)}\quad \int_c^\infty \frac{W(y_k, y_j)}{y_k^2}\,\mathrm{d}t \quad \text{is divergent.}$$

It should be noted that $\sin f$ and $\cos f \in K$ when $\nu(f) \geq 0$. [2, Lemma 11.6] and $\exp(f) \in K$ for all $f \in K$.

Proof. Since $\nu(y_k) > \nu(y_j)$, the positive function $\frac{y_k}{y_j}$ approaches zero and hence decreases. So. $\left(\frac{y_k}{y_j}\right)' < 0$. Therefore

$$\frac{W(y_j, y_k)}{y_j^2} < 0.$$

This gives (i) and (ii).

To prove (iii) consider $\left(\frac{y_k}{y_j}\right)' = \frac{W(y_j,y_k)}{y_j^2}$.

Integrating with respect to t from c to β and taking $\beta \to \infty$ we have

$$\int^{\infty} \frac{W(y_k, y_j)\,\mathrm{d}t}{y_j^2}$$

is finite since $\nu(y_k) > \nu(y_j)$. Thus the first part of (iii) is proved when

$$\phi(y_k, y_j) = \frac{1}{y_j^2}.$$

Similarly the other results can be proved. □

Theorem 6. Under the conditions of Theorem 5, for any (y_k, y_j) if $\nu(y_k) > \nu(y_j) < 0)$ $(j,\, k = 1, 2, \ldots, n,\ j > k)$ and $W(y_k, y_j)$ is a positive decreasing function or a constant then if $y_k'(x)$ is non-decreasing, $y_k(x)$ is nonincreasing.

Proof. By hypothesis, $\left(\frac{y_k}{y_j}\right)' = \frac{W(y_j,y_k)}{y_j^2}$.

Integrate with respect to "t" from x to β and making $\beta \to \infty$, and using $\nu(y_k) > \nu(y_j)$

$$y_k(x) = y_j(x) \int_x^{\infty} \frac{W(y_k, y_j)\,\mathrm{d}t}{y_j^2}.$$

Therefore

$$\begin{aligned} y_k'(x) &= y_j'(x) \int_x^{\infty} \frac{W(y_k, y_j)\,\mathrm{d}t}{y_j^2} - \frac{W(y_k, y_j)\,(x)}{y_j(x)} \\ &\leq \int_x^{\infty} \frac{y_j'(t)\, W(y_k, y_j)\,\mathrm{d}t}{y_j^2} - \frac{W(y_k, y_j)\,(x)}{y_j(x)} \end{aligned}$$

since $y_j'(x)$ is nondecreasing. If $W(y_k, y_j) = \text{constant}$, $y_k'(x) \leq 0$. If $W(y_k, y_j)$ is a positive decreasing function, using Bonnet's form of second mean value theorem,

$$y_k'(x) \leq 0.$$

This prove the theorem. □

Remark 6. Theorem 5 (iii) (1) and (v) are generalization of Lemma 16.6 of [2]. The other parts and Theorem 6 are new to the literature.

Remark 7. Theorem 5 and 6 are valid in $c'(a, \infty)$ with appropriate changes.

Theorem 7. Let K be a perfect Hardy field and y_1 in K, be any nontrivial solution of equation (1) over K. If the solutions at infinity of the first reduced equation (R_1) of order $(n-1)$,

$$y_1\, V^{(n-1)} + \cdots + \left[n\, y_1^{(n-1)} + (n-1)\, y_1^{(n-2)} + \cdots + a_{n-1}\, y_1\right] V = 0 \qquad (R_1)$$

lie in K, then all the solutions of (1) lie in K and K contains $R(y_1,\, u_2 y_1, \ldots, u_n y_1)$ which is of finite rank where

$$u_k = \int^x V_k(t)\,\mathrm{d}t \quad (k = 2, 3, \ldots, n)$$

and $V_2, V_3, \ldots, V_n$ are linearly independent solutions of (R_1).

If $\nu(u_k) > 0$ then $\nu(u_k\, y_1) > \nu(y_1)$ where ν is the canonical valuation on K.

Proof. If $y = u\, y_1$ is a solution of equation (1) then $L(y_1\, u) = 0$. The coefficient of u in this equation is just $L(y_1) = 0$. Therefore if $V = u'$, this is a linear equation of order $(n-1)$ in V. Call this (R_1). That is

$$y_1\, V^{(n-1)} + \cdots + \left[n\, y_1^{(n-1)} + (n-1)\, y_1^{(n-2)} + \cdots + a_{n-1}\, y_1\right] V = 0. \qquad (R_1)$$

The coefficient of $V^{(n-1)}$ is y_1. Since $y_1(x) \neq 0$ on the half interval I for sufficiently large x, (R_1) has $(n-1)$ linearly independent solutions $V_2, V_3, \ldots, V_n$ on I. If $u_k(x) = \int^x V_k(t)\,\mathrm{d}t$ $(k = 2, 3, \ldots, n)$, the functions $y_1,\, u_2\, y_1, \ldots,\, u_n\, y_1$ are solutions of (1). Since $V_2, V_3, \ldots, V_n$ belong to the Hardy field K which is perfect, $u_2, u_3, \ldots, u_n$ belong to K. So it follows that all the solutions of (1) belong to K.

Further K contains $R(y_1,\, u_2\, y_1, \ldots,\, u_n\, y_1)$ which has at most n comparability classes and so has a finite rank.

Since

$$\lim_{x\to\infty} \frac{u_k\, y_1}{y_1} = \lim_{x\to\infty} u_k = 0,$$

$$\text{it follows} \quad \nu(u_k\, y_1) > \nu(y_1).$$

□

Corollary 7.1. Under the conditions of the Theorem 7 if V_1 is a solution of the first derived equation (R_1) and $Z_2, Z_3, \ldots, Z_{n-1}$ are the solutions of the subsequent derived equations $R_2, R_3, \ldots, R_{n-1}$ then

$$W(y_1, y_2, \ldots, y_n) = (-1)^{n(n-1)/2}\, y_1^n\, V_1^{n-1}\, Z_2^{n-2} \cdots Z_{n-1},$$

where $y_1, y_2, \ldots, y_n$ are linearly independent solutions of the equation (1) and $W(y_1, y_2, \ldots, y_n)$ denotes the Wronskian of $y_1, \ldots, y_n$. If $y_1, V_1, Z_1, \ldots, Z_n \in K$, then all the solutions of (1) belong to K.

The first part of the Corollary follows by [8, page 127] and the second part follows trivially.

ACKNOWLEDGEMENT

I am thankful to Prof. D. Somsundaram, Department of Mathematics, Madras University P. G. Centre, Salem for constant encouragement and useful suggestions. The author would like to thank the referees for their valuable suggestions in particular for pointing out that the results could be used in Control Theory.

(Received September 11, 1991.)

REFERENCES

[1] M. Boshernitzan: An extension of Hardy's class of orders of infinity. J. Analyse Math. *39* (1981), 235–255.
[2] M. Boshernitzan: New orders of finity. J. Analyse Math. *41* (1982), 130–167.
[3] M. Boshernitzan: Second order differential equations over Hardy fields. J. London Math. Soc. *35* (1987), 2, 109–120.
[4] M. Boshernitzan: Orders of infinity generated by difference equations. Amer. J. Math. *106* (1984), 1067–1089.
[5] M. Boshernitzan: Hardy fields and existence of trans exponential functions. Acquationes Math. *30* (1986), 258–280.
[6] E. A. Coddington: An Introduction to Ordinary Differential Equations. Prentice Hall of India Private Ltd., New Delhi 1974.
[7] G. H. Hardy: Orders of Infinity. Cambridge University Press, London 1954.
[8] A. R. Forsyth: A Treatise on Differential Equations. Sixth edition. Mac Millan and Co. Ltd., New York 1954.
[9] V. Maric: Asymptotic behaviour of solutions of a nonlinear differential equation of the first order. J. Math. Anal. Appl. *38* (1972), 187–192.
[10] M. Rosenlicht: Hardy fields. J. Math. Anal. Appl. *93* (1983), 297–311.
[11] M. Rosenlicht: The rank of a Hardy field. Trans. Amer. Math. Soc. *280* (1983), 659–671.
[12] M. Rosenlicht: Rank change on adjoint real powers to Hardy fields. Trans. Amer. Math. Soc. *284* (1984), 829–836.
[13] M. Rosenlicht: Growth properties of functions in Hardy fields. Trans. Amer. Math. Soc. *299* (1987), 261–272.
[14] M. Rosenlicht: Asymptotic solutions of $y'' = F(x)\,y$. (Unpublished.)

Prof. Dr. A. Ramayyan, Department of Mathematics, Madras University P. G. Centre, Salem-636011, Tamil Nadu. India.