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THE DEGENERATION OF REACHABLE SINGULAR SYSTEMS UNDER FEEDBACK–TRANSFORMATIONS[1]

HEIDE GLÜSING-LÜERSSEN AND DIEDERICH HINRICHSEN

In this paper we consider the feedback-action for reachable singular systems and its limit phenomena. Firstly we introduce the invariants which parametrize the feedback-orbits. These results were published in part a few years ago independently by different authors (see [1, 6, 7]). Then we characterize the orbit closures in the space of all reachable systems in terms of these invariants. This gives a contribution to the answer of the question: how does a system might look like, which occurs as the limit of a sequence of feedback-transformations, applied to a given system?

We consider the singular system

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0, \tag{1}$$

described by the triple $(E, A, B) \in \mathbb{K}^{n\times(2n+m)}$, where $\mathbb{K}$ is the field of real or complex numbers. Let

$$\mathcal{S}_{n,m} = \{(E, A, B) \in \mathbb{K}^{n\times(2n+m)} \mid \mathrm{rk}[sE - tA, B] = n, \ \forall\ (s,t) \neq (0,0)\} \tag{2}$$

be the set of all reachable systems. The feedback-action on $\mathcal{S}_{n,m}$ is given by the group

$$\mathcal{F} = \left\{ \left(M, \begin{bmatrix} N & 0 \\ F & R \end{bmatrix} \right) | M,\ N \in Gl_n(\mathbb{K}), R \in Gl_m(\mathbb{K}),\ F \in \mathbb{K}^{m\times n} \right\}$$

and the algebraic action

$$\begin{array}{rccc} \Psi : & \mathcal{F} \times \mathcal{S}_{n,m} & \longrightarrow & \mathcal{S}_{n,m} \\ & \left(\left(M, \begin{bmatrix} N & 0 \\ F & R \end{bmatrix} \right), (E, A, B) \right) & \longmapsto & (E', A', B') \end{array}$$

with

$$(E', A', B') = (MEN^{-1}, M(A + BR^{-1}F)N^{-1}, MBR^{-1}). \tag{3}$$

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We write: $(E,A,B)\overset{\text{fb}}{\sim}(E',A',B')$ iff there exists $\left(M,\begin{bmatrix} N & 0 \\ F & R\end{bmatrix}\right)\in\mathcal{F}$ such that (3) holds.

Since the regularity condition

$$\det(sE-A)\not\equiv 0, \tag{4}$$

which is necessary for (1) being an *admissible* system, can be violated by the feedback-action Ψ, we omit this restriction. It will be possible to interpret our results also for admissible systems.

For our aim to investigate the limit systems under feedback-transformations we need a precise description of the invariants of the action Ψ. Thus we repeat the results given in [1]: let $\lambda_1\geq\ldots\geq\lambda_m$ be the minimal column indices of $[sE-A,B]$ and $\gamma_1\geq\ldots\geq\gamma_l$ the minimal indices of the module $\{x\in\mathbb{K}[s]^n \mid \exists u\in\mathbb{K}[s]^m : (sE-A)x(s)=Bu(s)\}$, where $l=\text{rk}B$. Then it can be shown: $\gamma_i\leq\lambda_i\leq\gamma_i+1$ for $i\in\underline{l}:=\{1,\ldots,l\}$ and $\lambda_j=0$ for $j=l+1,\ldots,m$. The indices $\lambda_i, i\in\underline{l}$, which satisfy $\lambda_i=\gamma_i$, are called *singular controllability indices*, all the others are the *regular controllability indices*. It holds:

a) $\sum_{i=1}^m\lambda_i=\text{rk}E$,

b) $\sum_{i=1}^l\gamma_i=n-l$,

c) $n-\text{rk}E=\#\{i\in\underline{l}\mid\lambda_i \text{ is a singular index}\}$.

With a system $(E,A,B)\in\mathcal{S}_{n,m}$ we associate the list of controllability indices

$$\text{ci}(E,A,B)=(\varrho;\sigma)=(\varrho_1,\ldots,\varrho_{m-n+r};\sigma_1,\ldots,\sigma_{n-r})$$

with $\varrho_1\geq\ldots\geq\varrho_{m-n+r}$ the regular and $\sigma_1\geq\ldots\geq\sigma_{n-r}$ the singular indices and $\text{rk}E=r$. The controllability indices together with their additional property of regularity or singularity constitute a complete invariant for the feedback-action on $\mathcal{S}_{n,m}$. It holds:

$$(E,A,B)\overset{\text{fb}}{\sim}(E',A',B')\Longleftrightarrow \text{ci}(E,A,B)=\text{ci}(E',A',B').$$

Observe that by property c) a system $(E,A,B)\in\mathcal{S}_{n,m}$ with $E\in Gl_n$ only has regular indices, these are just the familiar controllability indices of the state space system $(I,E^{-1}A,E^{-1}B)$.

Let $\mathcal{P}(r;m,m-n+r)$ be the set

$$\Big\{(\varrho;\sigma)=(\varrho_1,\ldots,\varrho_{m-n+r};\sigma_1,\ldots,\sigma_{n-r})\in\mathbb{N}_0^m \mid \varrho_j\geq\varrho_{j+1}, \\ \sigma_j\geq\sigma_{j+1}, \sum_{i=1}^{m-n+r}\varrho_i+\sum_{i=1}^{n-r}\sigma_i=r\Big\}$$

and

$$\mathcal{P}_{n,m}=\bigcup_{r=n-m}^{n}\mathcal{P}(r;m,m-n+r).$$

Then for each $(\varrho;\sigma) = (\varrho_1,\dots,\varrho_{m-n+r};\sigma_1,\dots,\sigma_{n-r}) \in \mathcal{P}(r;m,m-n+r)$ there exists a system $(E,A,B) \in \mathcal{S}_{n,m}$ with $\mathrm{ci}(E,A,B) = (\varrho;\sigma)$. This can be seen via a canonical form analogously to the Brunovsky-form for controllable state space systems:

A canonical form of an arbitrary system $(E,A,B) \in \mathcal{S}_{n,m}$ with $\mathrm{rk}E = r$ can be achieved by transforming (E,A,B) via the feedback-group to

$$\left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}, \begin{bmatrix} B_1 & B_2 \\ 0 & I_{n-r} \end{bmatrix}\right). \tag{5}$$

Then $(I_r, A_1, [B_1, B_2])$ is a controllable state space system of order r with the same controllability indices as the given system (E,A,B). The singular indices are associated with the columns of B_2, the regular indices are associated with the columns of B_1.

Observe that there exists an admissible system in each feedback-orbit

$$\mathcal{O}(\varrho;\sigma) = \{(E,A,B) \in \mathcal{S}_{n,m} \mid \mathrm{ci}(E,A,B) = (\varrho;\sigma)\}.$$

With this parametrization of the orbits we can study the limit phenomena of the feedback-action: we will describe those orbits $\mathcal{O}(\varrho;\sigma)$, which are in the closure $\overline{\mathcal{O}(\bar\varrho;\bar\sigma)}$ in $\mathcal{S}_{n,m}$, endowed with the standard Euclidian topology. Our goal is to formulate the *adherence order* as a partial order on the set $\mathcal{P}_{n,m}$. For this we introduce the set

$$\mathcal{K}(r;m) = \left\{(c_1,\dots,c_m) \mid c_i \in \mathbb{N}_0, \sum_{i=1}^{m} c_i = r\right\}$$

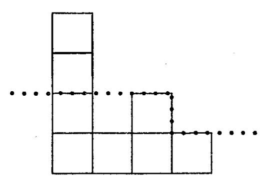
of all *combinations* of r into m numbers. For $c = (c_1,\dots,c_m)$, $c' = (c'_1,\dots,c'_m) \in \mathcal{K}(r;m)$ define

$$r_{ij}(c) = \sum_{\substack{k=1 \\ c_k \le i-1}}^{m} c_k + i\,\#\{k \in \underline{j} \mid c_k \ge i\} + (i-1)\,\#\{k \in \underline{m}\setminus\underline{j} \mid c_k \ge i\} \tag{6}$$

and let

$$c \preceq c' \iff r_{ij}(c) \le r_{ij}(c') \text{ for all } (i,j) \in \underline{r} \times \underline{m}. \tag{7}$$

Observe that $r_{ij}(c)$ can easily be computed from the *Young-diagram* of c: e.g. for $c = (4,1,2,1) \in \mathcal{K}(8;4)$ it is



the Young-diagram of c and $r_{2,3} = 6 = r_{2,4}$ is just the number of blocks below the dotted line.

It is easy to see that for *partitions* c and c' (i. e. those for which $c_j \geq c_{j+1}$, $c'_j \geq c'_{j+1}$) the partial order $\preceq$ is just the *dominance order*, since in this case it holds:

$$c \preceq c' \iff \sum_{j=1}^{k} c_j \geq \sum_{j=1}^{k} c'_j \text{ for all } k \in \underline{m}.$$

Remember that for controllable state space systems the dominance order of the controllability indices is equivalent to the adherence order of the corresponding feedback-orbits (see [3]).

Now we are in the position to formulate the main result of this note. It generalizes the above mentioned equivalence between the dominance order and the adherence order in the state space situation to the space of reachable singular systems.

Theorem 1. Let $\mathcal{O}(\varrho;\sigma)$, $\mathcal{O}(\bar{\varrho};\bar{\sigma}) \subset \mathcal{S}_{n,m}$ be two feedback-orbits with $(\varrho;\sigma) \in \mathcal{P}(r;m,m-n+r)$, $(\bar{\varrho};\bar{\sigma}) \in \mathcal{P}(\bar{r};m,m-n+\bar{r})$. Then it follows:

a) If $r = \bar{r}$, then $\mathcal{O}(\varrho;\sigma) \subseteq \overline{\mathcal{O}(\bar{\varrho};\bar{\sigma})} \iff (\varrho;\sigma) \preceq (\bar{\varrho};\bar{\sigma})$.

b) If $r < \bar{r} = r + s$ put $l = m - n + r$. Let $(\bar{\varrho};\bar{\sigma}) = (\bar{\varrho}_1, \ldots, \bar{\varrho}_{l+s}; \bar{\sigma}_1, \ldots, \bar{\sigma}_{n-r-s})$, then it is $\mathcal{O}(\varrho;\sigma) \subset \overline{\mathcal{O}(\bar{\varrho};\bar{\sigma})}$ iff there exists a decomposition

$$\{1, \ldots, l+s\} = \{i_1, \ldots, i_s\} \cup \{j_1, \ldots, j_l\}$$

such that $(\varrho;\sigma) \preceq P(\bar{\varrho}_{j_1}, \ldots, \bar{\varrho}_{j_l}; \bar{\varrho}_{i_1} - 1, \ldots, \bar{\varrho}_{i_s} - 1, \bar{\sigma}_1, \ldots, \bar{\sigma}_{n-r-s})$, where the list $P(\bar{\varrho}_{j_1}, \ldots, \bar{\sigma}_{n-r-s})$ consists of the given indices as regular and singular ones with both parts in decreasing order, i. e. the list is an element of $\mathcal{P}(r;m,m-n+r)$.

Proof. Part a) of the theorem can be proven by using a standard-form as in (5) and applying results of [4]. In [4] it is considered the feedback-action for state space systems with the restriction of only unipotent matrices as input transformations. This action preserves the given ordering of the controllability indices and is thus suitable for the study of the orbits $\mathcal{O}(\varrho;\sigma)$.

Part b) is quite technical but straightforward to prove, it also uses the standard-form (5). A proof of Theorem 1 can be found in [2, section 2.4].

Observe that part b) of the theorem says, that in the case of order reduction in the limit process the only *additional* effect, which might occur, is the degeneration of some arbitrary regular indices to singular ones.

The results of the theorem remain valid if we restrict our consideration to admissible systems. □

A conjecture about the characterization of orbit closures is left for future investigations. It would simplify the results of Theorem 1. We hope that the following is true: for $(\varrho;\sigma)$, $(\bar{\varrho};\bar{\sigma}) \in \mathcal{P}_{n,m}$ as in Theorem 1 it is

$$\begin{aligned}\mathcal{O}(\varrho;\sigma) \subseteq \overline{\mathcal{O}(\bar{\varrho};\bar{\sigma})} \quad \iff \quad & (\varrho;\sigma) \preceq (\bar{\varrho};\bar{\sigma}) \text{ and} \\ & P'(\varrho_1, \ldots, \varrho_{m-n+r}, \sigma_1 + 1, \ldots, \sigma_{n-r} + 1) \\ & \quad \preceq \bar{P}(\bar{\varrho}_1, \ldots, \bar{\varrho}_{m-n+\bar{r}}, \bar{\sigma}_1 + 1, \ldots, \bar{\sigma}_{n-\bar{r}} + 1),\end{aligned}$$

where $\preceq$ is defined as in (6), (7) and P', $\bar{P}$ order the given m indices decreasingly.

We are planning to investigate also the feedback-action together with the operations of strong equivalence, applied to singular systems. For this study the recent work of [5] about degeneration under pencil transformations might be helpful.

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Dr. Heide Glüsing-Lüerßen, Universität Oldenburg, Fachbereich 6 – Mathematik, Postfach 2503, 26111 Oldenburg. Germany.

Prof. Dr. Diederich Hinrichsen, Institut für Dynamische Systeme, Universität Bremen, Postfach 330440, D-2800 Bremen. Germany.