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DIFFERENTIAL GEOMETRIC STRUCTURES OF STABLE STATE FEEDBACK SYSTEMS WITH DUAL CONNECTIONS[1]

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This paper gives new approach to investigate differential geometric structures of stable and stable state feedback systems. For this purpose pairs of dual connections are introduced. Some of these connections are found to characterize the geometric structures of stable state feedback systems well.

1. INTRODUCTION

There have been many differential geometric approaches to investigate structures of linear (dynamical) systems (e. g. [1] - [3]). However, it seems that differential geometric structures of feedback systems, which are very important in the engineering sense, have not been deeply studied yet.

As is widely known, the parametrization of stabilizing controllers by Youla et al. [4] and Kučera [5] has given great advantage to the control theory and its applications. Since designers of controllers must optimize various performance indices on the parametrized set of stabilizing controllers, studying its geometric structures gives useful insights.

Recently, it has been shown that the set of stable matrices and the set of stabilizing state feedback gains are diffeomorphic to some kind of vector bundles, and the set of state feedback system matrices is parametrically imbedded as a submanifold of this vector bundle [6].

In this paper, we define metrics and connections on these vector bundles to analyze the differential geometric structures of stable state feedback systems. The main tool used here is *the theory of dual connections*, what is called *Information Geometry*, extensively studied by Amari [8] in statistics. Using this theory, we can reveal the simple structures of the set of stable systems and stabilizing state feedback gains, which can not be elucidated by usual Riemannian geometry.

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In this paper, $PD(n)$, $Skew(n)$ and $\mathcal{S}(n)$ denote, respectively, the set of n positive definite real matrices, skew symmetric real matrices ($M = -M^{\mathrm{T}}$) and stable real matrices (all the eigenvalues are located in the open left half complex plane).

2. PARAMETRIZATION OF STABILIZING STATE FEEDBACK GAINS AND STABLE MATRICES

This section will give parametrizations of stabilizing state feedback gains and stable matrices using Lyapunov equations. Complete derivations of the results in this section can be found in [6].

Consider an n-dimensional linear system with m inputs represented by a state space equation:

$$\dot{x} = Ax + Bu, \tag{2.1}$$

where (A, B) is stabilizable and B is of column full rank.

Let $\mathcal{F}_S(A, B)$ denote the set of *stabilizing state feedback gains* F, i. e., $\mathcal{F}_S(A, B) := \{F | A + BF \in \mathcal{S}(n)\}$.

Definition.

1. Let Q be in $PD(n)$. The set of positive definite matrices P that satisfy the following equation:

$$(I - B\,B^{\dagger})(AP + PA^{\mathrm{T}} + Q)(I - B\,B^{\dagger}) = 0 \tag{2.2}$$

is denoted by $PD(n;\, A, B, Q)$.

2. The set of skew symmetric matrices S that satisfy the following equation:

$$B\,B^{\dagger}\,S\,B\,B^{\dagger} = S, \quad \text{(or equivalently } B\,B^{\dagger}\,S = S) \tag{2.3}$$

is denoted by $Skew(n; B)$.

Here, $\bullet^{\dagger}$ represents a pseudo (Moore–Penrose) inverse of matrix $\bullet$.

Proposition 1. (Parametrization of $\mathcal{F}_S(A, B)$)

i) Let Q be in $PD(n)$. All stabilizing state feedback gains $F \in \mathcal{F}_S(A, B)$ are parametrized by

$$F = -B^{\dagger}(AP + PA^{\mathrm{T}} + Q)\left(I - \frac{1}{2}B\,B^{\dagger}\right)P^{-1} - B^{\dagger}\,S\,P^{-1} \tag{2.4}$$

using $P \in PD(n;\, A, B, Q)$ and $S \in Skew(n; B)$.

ii) The mapping $\psi_Q : PD(n;\, A, B, Q) \times Skew(n;\, B) \to \mathcal{F}_S(A, B)$ defined by (2.4) is diffeomorphic.

iii) The feedback gain F represented as (2.4) satisfies the following Lyapunov equation:

$$(A + B\,F)\,P + P(A + B\,F)^{\mathrm{T}} + Q = 0$$

for $P \in PD(n;\, A, B, Q)$ that is just parametrizing F in (2.4).

Proposition 2.

i) The set $PD(n;\, A, B, Q)$ is an $m(2n-m+1)/2\,(=: N_P)$-dimensional submanifold of $PD(n)$.

ii) The set $Skew(n; B)$ is an $m(m-1)/2\,(=: N_S)$-dimensional vector subspace of $Skew(n)$.

Proposition 3. (Parametrization of $\mathcal{S}(n)$)

i) Let Q be in $PD(n)$. All stable matrices $A_S \in \mathcal{S}(n)$ are parametrized by

$$A_S = -\frac{1}{2}\, Q\, P^{-1} + S\, P^{-1} \tag{2.5}$$

using $P \in PD(n)$ and $S \in Skew(n)$.

ii) The mapping $\phi_Q : PD(n) \times Skew(n) \to \mathcal{S}(n)$ defined by (2.5) is diffeomorphic.

iii) The stable matrix A_S represented as (2.5) satisfies the following Lyapunov equation:

$$A_S P + P\, A_S^{\mathrm{T}} + Q = 0.$$

Remark 1. Proposition 3 ii) shows that the set $\mathcal{S}(n)$ can be treated as a vector bundle that consists of $PD(n)$ as a base manifold and $Skew(n)$ as a fibre of vector space. We use a notation $Skew_P(n)$ to represent each fibre attached to the element P in the base manifold $PD(n)$. Each fibre $Skew_P(n)$ has its own metric depending on P (Theorem 3.2). This is one of the main reason why we will treat $PD(n) \times Skew(n)$ as a vector bundle rather than a mere product set. The interpretation of decomposition (2.5) from point of view of system dynamics is discussed in [6, 9].

Define the set of *state feedback system matrices as*

$$\mathcal{S}_f(A, B) := \{A + B\, F \mid F \in \mathcal{F}_S(A, B)\} \subset \mathcal{S}(n)$$

and a mapping

$$\chi : \mathcal{F}_S(A, B) \ni F \mapsto A + B\, F \in \mathcal{S}_F(A, B).$$

Using ϕ_Q^{-1}, we can characterize structures of $\mathcal{S}_f(A, B)$ in $PD(n) \times Skew(n)$.

From (2.4) we can get

$$A + B\, F = -\frac{1}{2}\, Q\, P^{-1} + (S_0(P) - S)\; P^{-1}, \tag{2.6}$$

where

$$S_0(P) \;:=\; A\, P - B\, B^{\dagger}\, (A\, P + P\, A^{\mathrm{T}} + Q) \left(I - \frac{1}{2}\, B\, B^{\dagger}\right) + \frac{1}{2}\, Q, \tag{2.7}$$

$$P \in PD(n;\, A, B, Q), \quad S \in Skew(n; B).$$

Since $P \in PD(n;\, A, B, Q)$, $S_0(P)$ is proved to be skew symmetric using (2.2) and so is $S_0(P) - S$.

Let $Skew_P(n; B)$ denote the set of all $S \in Skew_P(n)$ that satisfy (2.3). Comparing (2.5) and (2.6), we find how $\mathcal{S}_f(A, B)$ is imbedded in $PD(n) \times Skew(n)$ by ϕ_Q^{-1}:

Proposition 4. The set of stable state feedback system matrices $\mathcal{S}_f(A,B)$ is imbedded in the vector bundle $PD(n) \times Skew(n)$ as follows:

i) in the base manifold $PD(n)$, $\mathcal{S}_f(A,B)$ is restricted to the submanifold $PD(n;\, A, B, Q)$,

ii) in each fibre $Skew_P(n)$ such that $P \in PD(n;\, A, B, Q)$, $\phi_Q^{-1}(\mathcal{S}_f(A,B))$ is restricted to $S_0(P) + Skew_P(n; B)$.

In other words, $\mathcal{S}_f(A,B)$ is diffeomorphic to a submanifold

$$\phi_Q^{-1}(\mathcal{S}_f(A,B)) = \bigcup_{P \in PD(n;A,B,Q)} \{S_0(P) + Skew_P(n;B)\}\,, \tag{2.8}$$

contained in $PD(n) \times Skew(n)$. (See Figure 1).

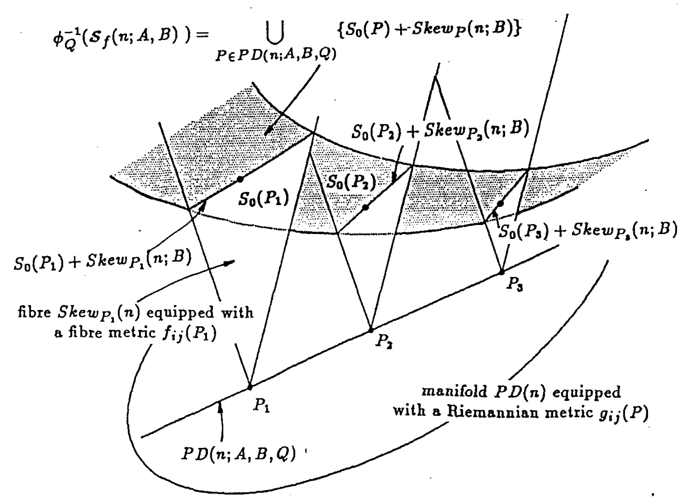

Fig. 1. Geometric structures of $\phi_Q^{-1}(\mathcal{S}_f(A,B))$ in vector bundle $PD(n) \times Skew(n)$.

Example 1. Consider the following linear system:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad \text{where } A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

It is easily verified that (A, B) is controllable and B is of full column rank. Define $Q = I \in \mathbb{R}^{3\times3}$ and represent $P \in PD(3)$ and $S \in Skew(3)$ as

$$P = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_2 & \eta_4 & \eta_5 \\ \eta_3 & \eta_5 & \eta_6 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & -\zeta_1 & -\zeta_2 \\ \zeta_1 & 0 & -\zeta_3 \\ \zeta_2 & \zeta_3 & 0 \end{bmatrix}.$$

Then we can obtain the parameter set $PD(3;\, A, B, I)$ and $Skew(3; B)$ as follows:

Calculate (2.2) and (2.3) using the pseudo-inverse matrix of B:

$$B^\dagger = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

then we get $\eta_3 = -\frac{1}{2}$ and $\zeta_2 = \zeta_3 = 0$. Thus, any $P \in PD(3;\, A, B, I)$ and $S \in Skew(3, B)$ are of the forms:

$$P = \begin{bmatrix} \eta_1 & \eta_2 & -\frac{1}{2} \\ \eta_2 & \eta_4 & \eta_5 \\ -\frac{1}{2} & \eta_5 & \eta_6 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & -\zeta_1 & 0 \\ \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.9}$$

Hence, all stabilizing state feedback gains of (A, B) are expressed using (2.4) and P, S of (2.9) as

$$F = \psi_I(P, S) = \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} + \zeta_1 & -\eta_1 \\ \frac{1}{4} - \zeta_1 & -\frac{1}{2} - \eta_5 & -\eta_2 - \eta_6 \end{bmatrix} \begin{bmatrix} \eta_1 & \eta_2 & -\frac{1}{2} \\ \eta_2 & \eta_4 & \eta_5 \\ -\frac{1}{2} & \eta_5 & \eta_6 \end{bmatrix}^{-1}.$$

From (2.9), we can examine the number of free parameters of $PD(3;\, A, B, I)$ and that of $Skew(3; B)$ are 5 and 1, respectively. These are equal to the dimension of $PD(3;\, A, B, I)$ and $Skew(3; B)$ calculated by Proposition 2.

Finally, we can obtain $S_0(P) \in Skew(3)$ from (2.7) as

$$S_0(P) = \begin{bmatrix} 0 & \frac{1}{4} & -\eta_1 \\ -\frac{1}{4} & 0 & -\eta_2 \\ \eta_1 & \eta_2 & 0 \end{bmatrix}$$

and $S_0(P) + Skew_P(n; B)$ is the set of skew symmetric matrices of the form:

$$\begin{bmatrix} 0 & \frac{1}{4} + \zeta_1 & -\eta_1 \\ -\frac{1}{4} - \zeta_1 & 0 & -\eta_2 \\ \eta_1 & \eta_2 & 0 \end{bmatrix}$$

included in the fibre $Skew_P(n)$. Then, all stable state feedback systems are expressed using (2.6) as

$$A + B\,F = \psi_I\,(P, S_0(P) - S) = \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} + \zeta_1 - \eta_1 & \\ -\frac{1}{4} - \zeta_1 & -\frac{1}{2} & -\eta_2 \\ \eta_1 & \eta_2 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \eta_1 & \eta_2 & -\frac{1}{2} \\ \eta_2 & \eta_4 & \eta_5 \\ -\frac{1}{2} & \eta_5 & \eta_6 \end{bmatrix}^{-1}.$$

The set of $PD(3;\, A, B, I) \times Skew(3; B)$ is imbedded in $PD(3) \times Skew(3)$ in this way.

3. TRANSFORMATION INVARIANT METRICS

To explore metric structures of $PD(n) \times Skew(n)$, we introduce metrics to two vector bundles. One is a tangent bundle $T\,PD(n) = \bigcup_{P \in PD(n)} T_P\,PD(n)$, where $T_P\,PD(n)$ denotes the tangent vector space at a point $P \in PD(n)$. The other is $PD(n) \times Skew(n) = \bigcup_{P \in PD(n)} Skew(n)$ itself. A metric $g(P)$ of $T\,PD(n)$ is called a *Riemannian metric* on $PD(n)$ and we refer to a metric $f(P)$ of $PD(n) \times Skew(n)$ as a *fibre metric*.

For the basis transformations of state space, $\tilde{x}(t) = T\,x(t)$, where $T \in G\,L(n; \mathbb{R})$, the matrices P and S in the parametrization are transformed congruently as

$$(P, S) \to (\tilde{P}, \tilde{S}) = (T\,PT^{\mathrm{T}},\, T\,ST^{\mathrm{T}}). \tag{3.1}$$

For consistency with linear systems theory, we should define metrics on $PD(n) \times Skew(n)$ invariant against the above congruent transformations.

3.1. Transformation invariant Riemannian metric on $PD(n)$

Let E_{pq} be the matrix with one as the (p,q)th element and zero otherwise. Now, we define E_i, the basis matrices of vector space $Sym(n)$, by

$$E_i := E_{\sigma(p,q)} = \begin{cases} E_{pp}, & p = q, \\ E_{pq} + E_{qp}, & p < q. \end{cases}$$

Here, σ is an appropriate rule to assign integers to the pairs (p,q), i.e. $\sigma(p,q) = i$, where $1 \le p \le q \le n$ and $1 \le i \le N := n(n+1)/2$.

Using E_i, we can represent any $P \in PD(n)$ as $P = \sum_{i=1}^{N} \eta^i E_i$ uniquely, where (η^i) belongs to some open subset of $\mathbb{R}^N$ that satisfies the positive definiteness. Hence, we consider (η^i) as a global coordinate system of N-dimensional manifold $PD(n)$. Then, natural basis of tangent vector fields $\partial_i := \partial / \partial\eta^i$ can be identified with E_i, i.e.,

$$\mathcal{X}(PD(n)) \ni \partial_i \sim E_i \in Sym(n), \quad 1 \le i \le N, \tag{3.2}$$

where $\mathcal{X}(PD(n))$ denotes the set of tangent vector fields on $PD(n)$. Using (3.2), we shall hereafter identify $\mathcal{X}(PD(n))$ with the set of $Sym(n)$-valued differentiable function $X(P)$ on $PD(n)$:

$$\mathcal{X}(PD(n)) \ni \sum_{i=1}^{N} a^i(P) \frac{\partial}{\partial\eta^i} \sim X(P) := \sum_{i=1}^{N} a^i(P)\, E_i, \quad a^i(P) \in \mathcal{C}(PD(n)),$$

where $\mathcal{C}(PD(n))$ denotes the set of differentiable functions on $PD(n)$. Similarly we shall identify the tangent vector space at P, which is denoted by $T_P\,PD(n)$, with $Sym(n)$:

$$X_P := \sum_{i=1}^{N} a^i \left(\frac{\partial}{\partial\eta^i}\right)_P \in T_P\,PD(n) \sim X := \sum_{i=1}^{N} a^i\, E_i \in Sym(n), \quad a^i \in \mathbb{R}. \tag{3.3}$$

First we shall make clear what invariance is required for a Riemannian metric on $PD(n)$. A Riemannian metric $g(P) = [g_{ij}(P)]$ defines an inner product $g_P(\bullet, \bullet)$ on each tangent space $T_P\, PD(n)$. To represent $g_P(\bullet, \bullet)$ as an inner product of *Sym*(n), we use the same notation. Then these inner products are defined by

$$g_P(\partial_i, \partial_j) = g_P(E_i, E_j) := g_{ij}(P), \quad 1 \le i \le N,\ 1 \le j \le N. \tag{3.4}$$

Denote the congruent transformation (3.1) by

$$\tau_T : P \longmapsto T\,P\,T^{\mathrm{T}}, \quad T \in GL(n; \mathbb{R}), \tag{3.5}$$

which is induced by the basis transformation of the state space (1.11). Then using the identification (3.3), the differential $\tau_{T*} : T_P\, PD(n) \mapsto T_{\tau_T(P)} PD(n)$ is represented as a transformation in *Sym*(n) by

$$\tau_{T*} : X \longmapsto T\,X\,T^{\mathrm{T}}, \tag{3.6}$$

where X and $T\,X\,T^{\mathrm{T}}$ indicate $X_P \in T_P\, PD(n)$ and $\tau_{T*}(X_P) \in T_{\tau_T(P)} PD(n)$, respectively.

The invariance we require here of a Riemannian metric $g(P)$ is that the following equation:

$$g_P(X_P, Y_P) = g_{\tau_T(P)}\left(\tau_{T*}(X_P),\ \tau_{T*}(Y_P)\right) \tag{3.7}$$

should be satisfied for any $P \in PD(n)$, $X_P, Y_P \in T_P PD(n)$ and $T \in GL(n; \mathbb{R})$. This is equivalent to

$$g_P(X, Y) = g_{TPT^{\mathrm{T}}}\left(T\,X\,T^{\mathrm{T}},\, T\,Y\,T^{\mathrm{T}}\right) \tag{3.8}$$

for any $P \in PD(n)$, $X, Y \in$ *Sym*(n) and $T \in GL(n; \mathbb{R})$.

Now the invariant Riemannian metric on $PD(n)$ is obtained.

Theorem 3.1. Define $g_{ij}(P)$ by

$$g_{ij}(P) := \frac{1}{2}\,\mathrm{tr}\left(P^{-1}E_i P^{-1}E_j\right), \tag{3.9}$$

then $g(P) = [g_{ij}(P)]$ is a Riemannian metric on $PD(n)$ invariant under the basis transformation of the state space.

Proof. Because of (3.4) and (3.9), the inner product on *Sym*(n) is represented by

$$g_P(X, Y) = \frac{1}{2}\,\mathrm{tr}\left(P^{-1}\,X\,P^{-1}\,Y\right). \tag{3.10}$$

Hence the invariance condition (3.8) can be easily confirmed as

$$\begin{aligned} g_{TPT^{\mathrm{T}}}\left(T\,X\,T^{\mathrm{T}},\, T\,Y\,T^{\mathrm{T}}\right) &= \frac{1}{2}\,\mathrm{tr}\left\{T^{-\mathrm{T}}(P^{-1}\,X\,P^{-1}\,Y)\,T^{\mathrm{T}}\right\} \\ = \ & \frac{1}{2}\,\mathrm{tr}\left\{(P^{-1}\,X\,P^{-1}\,Y)\,T^{\mathrm{T}}T^{-\mathrm{T}}\right\} = g_P(X, Y). \end{aligned} \tag{3.11}$$

To show the positive definiteness of $g_{ij}(P)$, consider the basis transformation $T \in GL(n;\mathbb{R})$ satisfying

$$T P T^{\mathrm{T}} = I. \tag{3.12}$$

Let $X := \sum_{i=1}^{N} a^i E_i$, then from the invariance we can get

$$g_P(X,X) = \sum_{i,j=1}^{N} g_{ij}(P)\, a^i\, a^j = \frac{1}{2}\operatorname{tr}\left\{(P^{-1}X)^2\right\} = \frac{1}{2}\operatorname{tr}(X'^2), \quad X' := T X T^{\mathrm{T}}.$$

Since X' is also a symmetric matrix, $\operatorname{tr}(X'^2)$ is identical to the square of the Euclid norm of X', which is always positive except that $X' = 0$. This means $g_{ij}(P)$ is positive definite. Moreover, the differentiability of $g_{ij}(P)$ follows from that of P^{-1}.

Thus, $g(P)$ is proved to be an invariant Riemannian metric on $PD(n)$. □

Example 2. We shall calculate the Riemannian metric on $PD(2)$. Set $\{E_i\}$ as

$$E_1 := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_2 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_3 := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

then P can be represented using coordinate system (η^i) as

$$P := \begin{bmatrix} \eta^1 & \eta^2 \\ \eta^2 & \eta^3 \end{bmatrix}.$$

From the result of Theorem 3.1, we obtain invariant Riemannian metric on $PD(2)$ as

$$[g_{ij}(P)] = \frac{1}{(\eta^1\eta^2 - (\eta^2)^2)^2} \begin{bmatrix} (\eta^3)^2 & -2\eta^2\eta^3 & (\eta^2)^2 \\ -2\eta^2\eta^3 & 2\left\{(\eta^2)^2 + \eta^1\eta^3\right\} & -2\eta^1\eta^2 \\ (\eta^2)^2 & -2\eta^1\eta^2 & (\eta^1)^2 \end{bmatrix}.$$

Let $P(t) = (\eta^i(t)),\ a \le t \le b$ be a differentiable curve on $PD(n)$. The arc length L of $P(t)$ is defined by

$$L := \int_{P(t)} \mathrm{d}s = \int_a^b \left\{\sum g_{ij}(p(t))\, \frac{\mathrm{d}\eta^i}{\mathrm{d}t}\,\frac{\mathrm{d}\eta^j}{\mathrm{d}t}\right\}^{1/2} \mathrm{d}t$$

and *distance* between two points is usually defined by the infimum of the lengths of all differentiable curves connecting them. The distance defined like this is, of course, invariant under the state-space basis transformation. Since the manifold $PD(n)$ equipped with Riemannian metric $g_{ij}(P)$ defined in Theorem 3.1 is proved to be *Riemannian (globally) symmetric space* [11, 12], we can give the distance function explicitly:

Theorem 3.2. Assume P_1 and P_2 are in the Riemannian manifold $(PD(n), g)$ where g is defined in Theorem 3.1. The invariant distance between the points P_1 and P_2 is given by

$$\operatorname{dist}(P_1, P_2) = \left\{\operatorname{tr}[(\log P_T)^2]\right\}^{1/2}, \quad \text{where } P_T := P_1^{-\frac{1}{2}} P_2 P_1^{-\frac{1}{2}}.$$

Proof. See the Appendix.

3.2. Transformation invariant fibre metric of $PD(n) \times Skew(n)$

On the other hand, define the basis matrices $\tilde{E}_\mu$ of $n(n-1)/2$-dimensional vector space $Skew(n)$ by

$$\tilde{E}_\mu := \tilde{E}_{\tilde{\sigma}(p,q)} = E_{pq} - E_{qp}, \quad p < q,$$

where $\tilde{\sigma}$ is an appropriate rule to assign integers to the pair (p,q), i.e. $\tilde{\sigma}(p,q) = \mu$, where $1 \leq p < q \leq n$ and $1 \leq \mu \leq \tilde{N} := n(n-1)/2$. Then any skew symmetric matrix $S = (\tilde{\eta}^\mu)$ in each fibre space $Skew_P(n)$ is represented by

$$S = \sum_{\mu=1}^{\tilde{N}} \tilde{\eta}^\mu \tilde{E}_\mu, \quad \tilde{\eta}^\mu \in \mathbb{R}.$$

Furthermore, a $Skew(n)$-valued differentiable function $S(P)$ on $PD(n)$ is just a cross section of $PD(n) \times Skew(n)$:

$$S(P) = \sum_{\mu=1}^{\tilde{N}} \tilde{\eta}^\mu(P) \tilde{E}_\mu \in \Gamma\left(PD(n) \times Skew(n)\right), \quad \tilde{\eta}^\mu(P) \in \mathcal{C}(PD(n)),$$

where $\Gamma(PD(n) \times Skew(n))$ denotes the set of cross sections of $PD(n) \times Skew(n)$. We regard $\tilde{E}_\mu$ as a (constant) basis cross section.

A fibre metric $f(P) = [f_{\mu\lambda}(P)]$ defines an inner product $f_P(\bullet, \bullet)$ on each fibre $Skew_P(n)$

$$f_P(\tilde{E}_\mu, \tilde{E}_\lambda) := f_{\mu\lambda}(P), \quad \tilde{E}_\mu, \tilde{E}_\lambda \in Skew_P(n).$$

Note that the basis transformation of the state space causes the congruence transformation which maps $S \in Skew_P(n)$ to $TST^{\mathrm{T}} \in Skew_{TPT^{\mathrm{T}}}(n)$, or equivalently,

$$\tau_T : (P, S) \longmapsto \left(TPT^{\mathrm{T}}, TST^{\mathrm{T}}\right).$$

Then we find a fibre metric $f_{\mu\lambda}(P)$ is required to satisfy the invariance such that

$$f_P(S, R) = f_{TPT^{\mathrm{T}}}\left(TST^{\mathrm{T}}, TRT^{\mathrm{T}}\right)$$

for any $P \in PD(n)$, S, $R \in Skew_P(n)$ and $T \in GL(n; \mathbb{R})$.

An invariant fibre metric $f(P) = [f_{\mu\lambda(P)}]$ can be derived in the similar manner to Theorem 3.1.

Theorem 3.3. Define $f_{\mu\lambda}(P)$ by

$$f_{\mu\lambda}(P) := -\frac{1}{2}\mathrm{tr}\left(P^{-1}\tilde{E}_\mu P^{-1}\tilde{E}_\lambda\right),$$

then $f(P) = [f_{\mu\lambda}(P)]$ is a fibre metric of $PD(n) \times Skew(n)$ invariant under the basis transformation of the state space.

Proof. The invariance and the differentiability of $f_{\mu\lambda}(P)$ are proved in the same way of Theorem 2. In contrast to Theorem 2, skew symmetry guarantees the positive definiteness. Consider the same basis transformation (3.12), then

$$f_P(S,S) = \sum_{\mu,\lambda=1}^{\tilde{N}} f_{\mu\lambda}(P)\,\tilde{\eta}^\mu\tilde{\eta}^\lambda = -\frac{1}{2}\mathrm{tr}\left\{(P^{-1}S)^2\right\} = -\frac{1}{2}\mathrm{tr}(S'^2) = \frac{1}{2}\mathrm{tr}(S'^{\mathrm{T}}S') = \frac{1}{2}\|S'\|^2.$$

Hence, $f(P) = [f_{\mu\lambda}(P)]$ is positive definite. □

4. DUAL CONNECTIONS ON $PD(n) \times Skew(n)$

Let t be in an interval $[0, t_1] \subset \mathbb{R}$ and $c : t \mapsto c(t) = P(t)$ be a smooth curve in $PD(n)$ from P_0 to P_1. For all t, consider a linear isomorphism $\Pi_c(t) : T_{P_0}\,PD(n) \rightarrow T_{P(t)}\,PD(n)$ called a *parallel displacement along the curve c*. Let $X(P)$ be a tangent vector field on $PD(n)$, then *covariant derivative* at P_0 for the direction $\dot{c}(0) = \dot{P}(0)$ is obtained from the parallel displacement $\Pi_c(t)$ as

$$\nabla_{\dot{P}(0)}X = \lim_{t\to\infty}\frac{1}{t}\left\{\Pi_c(t)^{-1}X(P(t)) - X(P_0)\right\}.$$

Using the parallel displacement along all the smooth curve on $PD(n)$, we can define the covariant derivative vector field on $PD(n)$.

In the same way, a parallel displacement along the curve c can be introduced to the vector bundle $PD(n) \times Skew(n)$. It is characterized as a linear isomorphism $\tilde{\Pi}_c(t) : Skew_{P_0}(n) \rightarrow Skew_{P(t)}(n)$, and the corresponding covariant derivative for a cross section $S(P) \in \Gamma(PD(n) \times Skew(n))$ is expressed by

$$\tilde{\nabla}_{\dot{P}(0)}S = \lim_{t\to 0}\frac{1}{t}\left\{\tilde{\Pi}_c(t)^{-1}S(P(t)) - S(P_0)\right\}.$$

The above shows that if parallel displacements Π_c in $T\,PD(n)$ and $\tilde{\Pi}_c$ in $PD(n) \times Skew(n)$ are defined for any piecewise smooth curve c in $PD(n)$, we can derive *affine connections* ∇ of $T\,PD(n)$ and *fibre connections* $\tilde{\nabla}$ of $PD(n) \times Skew(n)$, respectively.

Consider two parallel displacements Π_c and Π_c^* of $T\,PD(n)$ defined by

$$\Pi_c(t)\,X = X, \quad \Pi_c^*(t)\,X = P(t)\,P_0^{-1}\,X\,P_0^{-1}\,P(t), \tag{4.1}$$

for any curve c and $X \in Sym(n)$ using the identification (3.3). Let ∇ and ∇^* denote the corresponding affine connections. It is easily proved from (3.9) that these two parallel displacements satisfy

$$g_{P_0}(X,Y) = g_{P(t)}\left(\Pi_c(t)\,X,\ \Pi_c^*(t)\,Y\right), \quad \forall\, X, Y \in Sym(n).$$

Such a pair of parallel displacements (Π_c, Π_c^*) and a pair of the derived connections (∇, ∇^*) are said to be *mutually dual* [8]. The obtained results are as follows.

Theorem 4.1. The covariant derivatives with respect to the parallel displacements Π_c and Π_c^* satisfy

$$\nabla_{E_i} E_j = 0, \quad \nabla^*_{E_i} E_j = -E_i \, P^{-1} \, E_j - E_j \, P^{-1} \, E_i, \tag{4.2}$$

respectively. Here, we are identifying the vector field $\partial \,/\, \partial \eta^i$ and E_i.

Proof. Since the parallel displacement Π_c does not change basis tangent vectors as $\Pi_c \, E_i = E_i$, the equation $\nabla_{E_i} E_j = 0$ obvious.

To obtain the $\nabla^*_{E_i} E_j$, we consider the curve $\gamma : \, P(t)$ defined by

$$P(t) := \tau_{P1/2} \exp(Xt) = P^{\frac{1}{2}} \exp(Xt) \, P^{\frac{1}{2}}. \tag{4.3}$$

The curve $\gamma : \, P(t)$ is found to satisfy

$$P(0) = 0 \qquad \text{and} \qquad \dot{P}(0) = \tau_{P1/2*} \, X = P^{\frac{1}{2}} \, X \, P^{\frac{1}{2}},$$

where $\tau_{P1/2*}$ is the differential of $\tau_{P1/2}$. Hence, to calculate the covariant derivative $\nabla^*_{E_i} E_j$ using the curve $\gamma : \, P(T)$, we shall set

$$X := P^{-\frac{1}{2}} \, E_i \, P^{-\frac{1}{2}}.$$

Then from the definition of covariant derivative,

$$\nabla^*_{E_i} E_j = \lim_{t \to 0} \frac{1}{t} \left\{ \Pi^*_\gamma(t)^{-1} \, E_j - E_j \right\}. \tag{4.4}$$

Since $\Pi^*_\gamma(t)^{-1} \, E_j$ is represented via (4.1) and (4.3) by

$$\Pi^{*-1}_{\gamma(t)} E_j = P \, P(t)^{-1} \, E_j \, P(t)^{-1} \, P = P^{\frac{1}{2}} \exp(-Xt) \, P^{-\frac{1}{2}} E_j \, P^{-\frac{1}{2}} \exp(-Xt) \, P^{\frac{1}{2}},$$

we substitute this expression in (4.4) to get

$$\begin{aligned} \nabla^*_{E_i} E_j &= \frac{\mathrm{d}}{\mathrm{d}t} \left\{ P^{\frac{1}{2}} \exp(-Xt) \, P^{-\frac{1}{2}} \, E_j \, P^{-\frac{1}{2}} \exp(-Xt) \, P^{\frac{1}{2}} \right\} \Big|_{t=0} = \\ &= -P^{\frac{1}{2}} \, X \, P^{-\frac{1}{2}} \, E_j - E_j \, P^{-\frac{1}{2}} \, X \, P^{\frac{1}{2}} \\ &= -E_i \, P^{-1} \, E_j - E_j \, P^{-1} \, E_i. \end{aligned}$$

□

Theorem 4.2.

i) The manifold $PD(n)$ is torsion free and flat with respect to the affine connection ∇. (We shall call the latter properties ∇-*flat*.

ii) The manifold $PD(n)$ is torsion free and flat with respect to the affine connection ∇^*. (We shall call the latter properties ∇^*-*flat*.)

Proof.

i) From (4.2), the coefficients of affine connection ∇, which are defined by $\Gamma_{ijk} := g_P(\nabla_{E_i} E_j, \, E_k)$, vanish. This means the statement i).

ii) First, let $T^*(\bullet,\bullet)$ be the torsion tensor of the affine connection ∇^*. We have

$$T^*_{ijk} := g_P\left(T^*(E_i, E_j),\, E_k\right) = g_P\left(\nabla^*_{E_i} E_j - \nabla^*_{E_j} E_i,\, E_k\right).$$

Here, the coefficients of affine connection ∇^* denoted by Γ^*_{ijk} is obtained by

$$\begin{aligned}\Gamma^*_{ijk}(P) := g_P\left(\nabla^*_{E_i} E_j,\, E_k\right) &= \frac{1}{2}\mathrm{tr}\left\{P^{-1}(-E_i\, P^{-1}\, E_j - E_j\, P^{-1}\, E_i)\, P^{-1}\, E_k\right\} \\ &= -\mathrm{tr}\left(P^{-1}\, E_i\, P^{-1}\, E_j\, P^{-1}\, E_k\right)\end{aligned}$$

using the symmetry of P, E_i, E_j and E_k. Hence, we get

$$T^*_{ijk}(P) = \Gamma^*_{ijk}(P) - \Gamma^*_{jik}(P) = 0.$$

Secondly, recall the definition of Π^*_c, then we find it depends not on the curve it passes along but on the points where it starts and finishes. Furthermore, when the curve is closed, Π^*_c is proved to map a tangent vector $X \in T_{P_0}PD(n)$ to itself. This means ii) is true because the curvature tensors are geometrically interpreted as changes of tangent vectors by parallel displacements along infinitesimally small closed curves. Thus the statement ii) follows. □

Similarly a pair of parallel displacements $(\tilde{\Pi}_c, \tilde{\Pi}^*_c)$ for any curve can be defined on $PD(n) \times \mathit{Skew}(n)$:

$$\tilde{\Pi}_c(t)\, S = S, \qquad \tilde{\Pi}^*_c(t)\, S = P(t)\, P_0^{-1}\, S\, P_0^{-1}\, P(t).$$

And we shall also call $(\tilde{\Pi}_c, \tilde{\Pi}^*_c)$, or a pair of the corresponding fibre connections $(\tilde{\nabla}, \tilde{\nabla}^*)$, *mutually dual* because they satisfy

$$f_{P_0}(S, R) = f_{P(t)}\left(\tilde{\Pi}_c(t),\, S,\, \tilde{\Pi}^*_c(t)\, R\right), \quad \forall\, S,\, R \in \mathit{Skew}_{P_0}(n).$$

We can show the followings similarly to Theorem 4.1 and 4.2.

Theorem 4.3. The covariant derivatives with respect to the parallel displacements $\tilde{\Pi}_c$ and $\tilde{\Pi}^*_c$ satisfy

$$\tilde{\nabla}_{E_i}\, \tilde{E}_\mu = 0, \qquad \tilde{\nabla}^*_{E_i}\, \tilde{E}_\mu = -E_i\, P^{-1}\, \tilde{E}_\mu - \tilde{E}_\mu\, P^{-1}\, E_i,$$

respectively. Here, we are identifying $\tilde{E}_\mu$ as the basis cross section.

Theorem 4.4. The vector bundle $PD(n) \times \mathit{Skew}(n)$ is flat with respect to both fibre connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$. We shall call these properties $\tilde{\nabla}$-*flat* and $\tilde{\nabla}^*$-*flat*, respectively.

Remark 2. Both ∇ and ∇^* are non-metric affine connections. However, we see they qualify $PD(n)$ as a torsion free, flat manifold in Theorem 4.2. Similarly, non-metric fibre connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$ endow the vector bundle $PD(n) \times Skew(n)$ with flatness.

The coordinate system (η^i) is called *affine* with respect to ∇ because its basis vector fields satisfy $\nabla_{E_i} E_j = 0$. The coordinate system $(\tilde{\eta}^\mu)$ is also natural with respect to $\tilde{\nabla}$ in the sense that its basis cross sections satisfy $\tilde{\nabla}_{E_i} \tilde{E}_\mu = 0$. On the other hand, we can also introduce affine and "natural" coordinate systems with respect to ∇^* and $\tilde{\nabla}^*$. These coordinate systems are called *dual coordinate systems* [8]. Such a pair of primal and dual coordinate systems plays important roles in the theory of dual connections, e. g., defining a pseudo-distance called *divergences* [8].

Another remark is that we can define the family of connections using (∇, ∇^*) and $(\tilde{\nabla}, \tilde{\nabla}^*)$ in the same way to [8]. Define the connections depending one parameter $\alpha \in \mathbb{R}$ by

$$\overset{\alpha}{\nabla} := \frac{1-\alpha}{2}\nabla + \frac{1+\alpha}{2}\nabla^*, \qquad \overset{\alpha}{\tilde{\nabla}} := \frac{1-\alpha}{2}\tilde{\nabla} + \frac{1+\alpha}{2}\tilde{\nabla}^*.$$

We call $\overset{\alpha}{\nabla}$ and $\overset{\alpha}{\tilde{\nabla}}$ *α-affine and α-fibre connections*, respectively. The pairs of connections $(\overset{\alpha}{\nabla}, \overset{-\alpha}{\nabla})$ and $(\overset{\alpha}{\tilde{\nabla}}, \overset{-\alpha}{\tilde{\nabla}})$ are mutually dual, respectively. Particularly, affine connection $\overset{0}{\nabla}$ and fibre connection $\overset{0}{\tilde{\nabla}}$ are metric. It is well known that $\overset{0}{\nabla}$ is called Riemannian (Levi–Civita) connection of $PD(n)$.

5. DIFFERENTIAL GEOMETRY OF PARAMETER SPACE FOR $\mathcal{F}_S(A, B)$ AND $\mathcal{S}_f(A, B)$

In the previous sections, we have defined metrics and connections, the fundamental quantities for the differential geometric structures of $PD(n) \times Skew(n)$. Now using these quantities, we shall exploit the geometric structures of vector bundle $PD(n;\, A, B, Q) \times Skew(n; B) = \psi_Q^{-1}(\mathcal{F}_S(A, B))$, which parametrizes $\mathcal{F}_S(A, B)$ and imbedded submanifold $\phi_Q^{-1}(\mathcal{S}_f(A, B))$ in $PD(n) \times Skew(n)$, which parametrizes $\mathcal{S}_f(A, B)$.

In this section, indices $\{i, j, \ldots\}$, $\{a, b, \ldots\}$, $\{\mu, \lambda, \ldots\}$ and $\{\alpha, \beta, \ldots\}$ attached to quantities means that the quantities are components with respect to $PD(n)$, $PD(n; A, B, Q)$, $Skew(n)$ and $Skew(n; B)$, respectively.

5.1. Geometry of $PD(n; A, B, Q) \times Skew(n; B)$ induced from $PD(n) \times Skew(n)$

Using the equation (2.2) which specifies the submanifold $PD(n; A, B, Q)$ in $PD(n)$, we first construct the coordinate system (γ^a) for $PD(n; A, B, Q)$ and then define induced Riemannian metric and connections.

Since (2.2) is linear equations with respect to the components of P, i. e., $\eta = (\eta^i) \in \mathbb{R}^N$, then it can be rewritten as

$$K\,\eta = w, \tag{5.1}$$

where $K \in \mathbb{R}^{(N-N_P)\times N}$, $w \in \mathbb{R}^{N-N_P}$ are some constant matrix and vector determined from (2.2) [6].

Let T be in $GL(N;\mathbb{R})$ which satisfies $KT = [\,I\ 0\,]$ and define $\eta' := T^{-1}\eta$. Then (5.1) is transformed and solved as

$$[\,I\ 0\,]\,\eta' = w, \qquad \eta' = \begin{bmatrix} w \\ \gamma \end{bmatrix}.$$

The free parameter $\gamma = (\gamma^a) \in \mathbb{R}^{N_P}$ is just the coordinate system of the submanifold $PD(n;\ A, B, Q)$. The relation between (η^i) and (γ^a) is

$$\eta(\gamma) = T\,\eta' = [\,T_1\ T_2\,]\begin{bmatrix} w \\ \gamma \end{bmatrix}.$$

Hence, the Jacobian matrix $J := (\partial\eta^i\,/\,\partial\gamma^a) = T_2$ is a constant matrix. Then, the relation between basis tangent vector fields E_i of $PD(n)$ and $E_a \sim \partial\,/\,\partial\gamma^a$ of $PD(n;\ A, B, Q)$ is found to be $E_a = J_a^i\, E_i$.

Using this relation, geometrical quantities on submanifold $PD(n;\ A, B, Q)$ are naturally induced from those of $PD(n)$. The induced metric $g_{ab}(\gamma)$ on $PD(n;\ A, B, Q)$ is given by

$$g_{ab}(\gamma) = J_a^i\, J_b^j\, g_{ij}(\eta). \tag{5.2}$$

Generally, the coefficients $\Gamma_{abc}(\gamma)$ of induced affine connections are obtained by

$$\Gamma_{abc}(\gamma) = g_{P(\gamma)}\left(\nabla_{E_a} E_b,\, E_c\right) = J_a^i\, J_b^j\, J_c^k\, \Gamma_{ijk}(\eta) + \left(\partial_a\, J_b^i\right)\, J_c^j\, g_{ij}(\gamma), \tag{5.3}$$

where $\Gamma_{ijk}(\eta)$ is the coefficients of any affine connection on $PD(n)$.

By means of (5.2), (5.3), we can obtain the following results:

Theorem 5.1. *$PD(n;\ A, B, Q)$ is ∇-flat and ∇^*-flat manifold in itself, namely its Riemann–Christoffel curvature tensor R_{abcd} and R^*_{abcd} vanish.*

Proof. Since Jacobian matrix J in (5.3) is constant, its partial derivatives by γ^a vanish, $\partial_a\, J_b^i = 0$. From this and $\Gamma_{ijk} = 0$ (Theorem 4.1), (5.3) means $\Gamma_{abc} = 0$, i.e., the coefficients of affine connection for submanifold $PD(n;\ A, B, Q)$ vanish. Since Riemann–Christoffel curvature tensors of ∇-connection R_{abcd} is defined by

$$R_{abcd} = g_P\left(R(E_A, E_B)\, E_c,\, E_d\right) = \left(\partial_a\, \Gamma^e_{bc} - \partial_b\, \Gamma^e_{ac}\right)\, g_{ed} + \left(\Gamma_{aed}\, \Gamma^e_{bc} - \Gamma_{bed}\, \Gamma^e_{ac}\right),$$

it also vanishes. This shows $PD(n;\ A, B, Q)$ is ∇-flat. The ∇^*-flatness follows automatically from ∇-flatness [8]. □

In the same way, using the equation (2.3) specifying *Skew*$(n; B)$, we can construct the coordinate system $\tilde\gamma = (\tilde\gamma^\alpha) \in \mathbb{R}^{N_S}$ of *Skew*$(n; B)$. In this fibre case, constant Jacobian matrix denoted by $\tilde J$ is also obtained: $\tilde\eta = \tilde J\,\tilde\gamma$.

We can induce fibre metrics and connections on $PD(n;\ A, B, Q) \times$ *Skew*$(n; B)$. Since $\tilde J$ is constant,

$$f_{\alpha\beta}(\gamma) = \tilde J_\alpha^\mu\, \tilde J_\beta^\lambda\, f_{\mu\lambda}(\eta)$$

$$\tilde\Gamma_{a\alpha\beta}(\gamma) = f_P\left(\tilde\nabla_{E_a}\, \tilde E_\alpha,\, \tilde E_\beta\right) = J_a^i\, \tilde J_\alpha^\mu\, \tilde J_\beta^\lambda\, \tilde\Gamma_{i\mu\lambda} + \left(\partial_a\, \tilde J_\alpha^\mu\right)\, \tilde J_\beta^\lambda\, f_{\alpha\beta} = J_a^i\, \tilde J_\alpha^\mu\, \tilde J_\beta^\lambda\, \tilde\Gamma_{i\mu\lambda}(\eta).$$

Then we can show the following similarly to Theorem 5.1.

Theorem 5.2. The vector bundle $PD(n;\, A, B, Q) \times Skew(n; B)$ is $\tilde{\nabla}$- and $\tilde{\nabla}^*$-flat vector bundle in itself, i. e., its curvatures vanish.

5.2. Geometry of $\phi_Q^{-1}(\mathcal{S}_f(A, B))$ imbedded in $PD(n) \times Skew(n)$

In section 2, we have seen $\mathcal{S}_f(A, B)$ is imbedded in $PD(n) \times Skew(n)$ by ϕ_Q as (2.8). We can get some results using this imbedding.

To see the structures of $PD(n;\, A, B, Q)$ as a canonically imbedded submanifold in $PD(n)$, it is enough to calculate *Euler–Schouten (imbedding) curvature tensor*. Let $\{E_{\overline{k}}\}$ ($\overline{k} = N_P + 1, \ldots, N$) be the basis of orthogonally complement subspace of $T_P\, PD(n;\, A, B, Q)$ in $T_P\, PD(n)$, then Euler–Schouten curvature tensor $H_{ab\overline{k}}$ is defined by

$$H_{ab\overline{k}} = g_P\left(\nabla_{E_a} E_b,\, E_{\overline{k}}\right), \qquad H^*_{ab\overline{k}} = g_P\left(\nabla^*_{E_a} E_b,\, E_{\overline{k}}\right). \tag{5.4}$$

This quantity shows how curved $PD(m;\, A, B, Q)$ is in $PD(n)$.

Theorem 5.3. The submanifold $PD(n;\, A, B, Q)$ is an *autoparallel* submanifold in $PD(n)$ with respect to ∇, namely, its Euler–Schouten (imbedding) curvature is identically vanishes. Hence, $PD(n;\, A, B, Q)$ is *totally geodesic* submanifold with respect to ∇, which means that $PD(n;\, A, B, Q)$ consists of all the ∇-geodesics whose tangent vectors belong to the tangent space $T_P\, PD(n;\, A, B, Q)$.

Proof. Calculating (5.4), Euler–Schouten curvature is

$$H_{ab\overline{k}} = J^i_a\, J^j_b\, J^k_{\overline{k}}\, \Gamma_{jik} + \left(\partial_a\, J^j_b\right)\, J^k_{\overline{k}}\, g_{jk}. \tag{5.5}$$

From the facts $\Gamma_{ijk} = 0$ and (J^k_b) is constant, $H_{ab\overline{k}}$ is found to be zero as Theorem 5.1. Since autoparallel submanifold is always totally geodesic [10], the statement follows. □

Let $\{\tilde{E}_{\overline{\kappa}}\}$, $\overline{\kappa} = N_S+1, \ldots, \tilde{N}$ be the basis of orthogonally complement subspace of $Skew(n; B)$ in $Skew(n)$. On a canonically imbedded submanifold $PD(n; A, B, Q) \times Skew(n; B) \hookrightarrow PD(n) \times Skew(n)$, which has a subvector bundle structure, we can define curvature tensor $\tilde{H}_{a\mu\overline{\kappa}}$ and $\tilde{H}^*_{a\mu\overline{\kappa}}$ in the same manner to Euler–Schouten curvature tensor,

$$\tilde{H}_{a\mu\overline{\kappa}} = f_P\left(\nabla_{E_a} \tilde{E}_\mu,\, \tilde{E}_{\overline{\kappa}}\right), \qquad \tilde{H}^*_{a\mu\overline{\kappa}} = f_P\left(\nabla^*_{E_a} \tilde{E}_\mu,\, \tilde{E}_{\overline{\kappa}}\right).$$

It is also easy to observe only $\tilde{H}_{a\mu\overline{\kappa}}$ vanishes and the subvector bundle $PD(n;\, A, B, Q) \times Skew(n; B)$ consists of $\tilde{\nabla}$-geodesics. However, when it is imbedded into $PD(n) \times Skew(n)$ as the submanifold $\bigcup_{P \in PD(n; A, B, Q)} \{S_0(P) + Skew_P(n; B)\}$ by $\phi_Q^{-1} \circ \chi \circ \psi_Q$, which just embodies the structures of $\mathcal{S}_f(A, B)$ in $\mathcal{S}(n)$, this manifold (not subvector bundle) is generally curved even in $\tilde{\nabla}$-flat sense because of the origin-shifted term $S_0(P)$.

6. CONCLUSIONS

In this paper, we have dually introduced affine and fibre connections on $PD(n) \times$ *Skew*(n), which is diffeomorphic to $\mathcal{S}(n)$. Then, using these connections, geometric structures of stable state feedback systems have been discussed. Connections ∇ and $\tilde{\nabla}$ are proved to characterize them well. Analysis by divergences and applications of obtained results will be found in another place.

APPENDIX: THE PROOF OF THEOREM 3.2

To prove the theorem we need some notions. We illustrate them applying the abstract theory of Riemannian symmetric space [11,12] to our manifold $(PD(n),\, g)$. Consider the following mappings ι and τ on $PD(n)$:

$$\iota : P \mapsto P^{-1}, \quad \tau_T : P \mapsto T\, P\, T^{\mathrm{T}}, \quad T \in GL(n;\mathbb{R}).$$

By differentiating $P\, P^{-1} = I$, we get ι_*, the differential of ι by

$$\iota_* : T_P\, PD(n) \ni X \mapsto \iota_*(X) = -P^{-1}\, X\, P^{-1} \in T_{\iota(P)} PD(n). \tag{A.1}$$

The differential of τ_T denoted by τ_{T^*} is

$$\tau_{T^*} : T_P\, PD(n) \ni X \mapsto \tau_{T^*}(X) = T\, X\, T^{\mathrm{T}} \in T_{\tau_T(P)}\, PD(n). \tag{A.2}$$

Using the Riemannian metric g in Theorem 1, the inner product $\langle \bullet, \bullet \rangle_P$ of $T_P\, PD(n)$ is

$$\langle X, Y \rangle_P = \mathrm{tr}\left(P^{-1}\, X\, P^{-1}\, Y\right), \quad X,\, Y \in T_P\, PD(n). \tag{A.3}$$

Since (A.1) and (A.2) shows that the differential ι_* and τ_{T^*} satisfy

$$\langle X, Y \rangle_P = \langle \iota_*(X),\, \iota_*(Y) \rangle_{\iota(P)}, \quad \langle X, Y \rangle_P = \langle \tau_{T^*}(X),\, \tau_{T^*}(Y) \rangle_{\tau_T(P)},$$

ι and τ_T are called *isometries* of Riemannian manifold $(PD(n),\, g)$.

Define the mappings s_P for each $P \in PD(n)$ by

$$s_P := \tau_{P^{1/2}} \circ \iota \circ \tau_{P^{-1/2}}.$$

It is easily examined that i) s_P is also an isometry, ii) s_P^2 is the identity, however so is not s_P itself, iii) P is an isolated fixed point of s_P. Such Riemannian manifold $(PD(n),\, g)$ equipped with the mapping s_P for each $P \in PD(n)$ is called *Riemannian (globally) symmetric.*

The manifold $PD(n)$ can be identified with a quotient manifold $GL(n;\mathbb{R})\,/\,O(n)$. Denote the Lie algebra of $GL(m;\mathbb{R})$ by $gl(n)$. Lie algebra of $O(n)$ is known to be *Skew*(n). Since $gl(n)$ is decomposed as $gl(n) =$ *Skew*$(n) \oplus$ *Sym*(n), *Sym*(n) is isomorphic to $T_I\, PD(n)$. We denote this isomorphism by $d\pi$:

$$d\pi : \tilde{X} \in Sym(n) \mapsto X = 2\tilde{X} \in T_I\, PD(n).$$

Lemma A.1. [11, p. 173], [12, p. 71] Let $\gamma_X(t)$ be the geodesic satisfying

$$\gamma_X(0) = I \in PD(n), \quad \dot{\gamma}_X(0) = X \in T_I\, PD(n).$$

Then, $\gamma_X(t)$ is given by

$$\gamma_X(t) = \tau_{\exp \tilde{X}\, t}(I) = \exp X\, t, \tag{A.4}$$

where $\tilde{X} = (d\pi)^{-1} X = \frac{1}{2} X$ and exp is exponential mapping of matrices. □

Since Riemannian symmetric space is *complete* [2, 3], the *exponential mapping* Exp_I of $(PD(n),\, g)$:

$$\mathrm{Exp}_I : X \in T_P\, PD(n) \mapsto \gamma_X(1) = \exp X \in PD(n), \tag{A.5}$$

can be defined on the entire $T_I\, PD(n)$. Furthermore, Lemma A.1 means that Exp_I is diffeomorphism between $T_I\, PD(n)$ and $PD(n)$ because exp is.

Proof of Theorem 3.2. We shall consider the distance between P_1 and $P_2 \in PD(n)$. Using the isometry $\tau_{P_1^{-1/2}}$, P_1 and P_2 are transformed as

$$\tau_{P_1^{-1/2}}(P_1) = I, \quad \tau_{P_1^{-1/2}}(P_2) = P_1^{-\frac{1}{2}}\, P_2\, P_1^{-\frac{1}{2}} = P_T.$$

Consequently

$$\mathrm{dist}(P_1, P_2) = \mathrm{dist}(I, P_T)$$

because of the invariant Riemannian metric g. On general Riemannian manifolds (M, g) the distance between p and $\mathrm{Exp}_p X$ is given by $\sqrt{g(p)\,(X, X)}$ [12]. Since, in our case, Exp_I is given from (A.4) and (A.5) by

$$\mathrm{Exp}_I\, X = \exp X,$$

the tangent vector X satisfying $\mathrm{Exp}_P\, X = P_T$ is $\log P_T$. Therefore

$$\mathrm{dist}(I, P_T) = (\langle \log P_T,\, \log P_T \rangle_I)^{1/2} = \left\{ \mathrm{tr}[(\log P_T)^2] \right\}^{1/2}.$$

This proves the theorem. □

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