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TESTING LINEARITY AND MODELLING NONLINEAR TIME SERIES

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This paper discusses some of the recent developments in testing linearity of a time series against the alternative that the series has been generated by a nonlinear process. The focus is on testing linearity against parametric alternatives. Special attention is given to the situations in which the parametric nonlinear model only is identified under the alternative but not under the null hypothesis of linearity. The use of some of the linearity tests in the modelling of nonlinear series is considered and illustrated with an example.

1. INTRODUCTION

Modelling possibly nonlinear time series is not an easy task. Once a model builder leaves the linear world behind the choice in principle is immense. In some disciplines such as physics the model choice may not be a major problem because much of the theory is nonlinear, and as a rule this theory is sufficient for specifying the structure of the model. In some other fields such as economics the theory often leaves the functional form of the model open. In such situations linear approximations to the unknown functional form have been very popular when economic relationships have been estimated from data. Quite a few such approximations have turned out to be satisfactory representations of the underlying economic phenomenon. Before trying nonlinear approximations it is therefore advisable to find out whether a linear model offers an adequate representation of the data or not. Only if the latter is true should nonlinear models be considered. This argument makes linearity testing an integral part of nonlinear modelling of time series. In this paper the emphasis lies on tests against parametric nonlinear alternatives which will also be called parametric tests for short. This area has developed rather rapidly since the contribution of Pagan [20]. His paper was perhaps the first one to stress the importance of the score or Lagrange multiplier approach in linearity testing.

Tests against a nonspecified alternative have actually been more popular in the applications than those against a completely specified one. In economics this has been partly because testing linearity has been an aim as such, and rejection of this hypothesis has often not been followed up by any model building exercise. However, rejecting the null hypothesis against an unspecified alternative is generally not very helpful if the ultimate aim is to build a workable model for the phenomenon in question. In this paper it is shown that some of the parametric tests can be very useful in nonlinear time series model building. Other accounts describing these ideas include Granger and Teräsvirta [12], Teräsvirta [26], and Teräsvirta, Tjøstheim, and Granger [27]. The last-mentioned paper also considers nonparametric testing and modelling techniques which for reasons of space will be omitted here.

The contents of the paper are as follows. Section 2 is devoted to linearity testing. Section 3 shows how some of the ideas of the previous section can be applied to testing the parameter constancy in linear models. Section 4 briefly mentions some other approaches to linearity testing. The use of parametric linearity tests in nonlinear model building are discussed in Section 5, and Section 6 contains an application to Wolf's sunspot number series. Section 7 concludes.

2. LINEARITY TESTING

Consider the following nonlinear model

$$y_t = f(w_t, \nu_t, \theta_1) + g(w_t, \nu_t, \theta_2)\, u_t \tag{2.1}$$

where $g(w_t, \nu_t, \theta_2) \geq 0$, $w_t = (1, y_{t-1}, \ldots, y_{t-p})'$, $\nu_t = (u_{t-1}, \ldots, u_{t-q})'$, and $u_t \sim \mathrm{nid}(0, \sigma^2)$. Following Granger and Teräsvirta [12, p. 8], if f can be expressed as a linear function of its variables

$$f(w_t, \nu_t, \theta_1) = \theta_{11}' w_t + \theta_{12}' \nu_t \tag{2.2}$$

then (2.1) is called linear in mean. If, in addition $g \equiv$ constant, then (2.1) has a *complete linear representation*. Otherwise (2.1) is nonlinear. Linearity testing thus involves testing both the hypothesis that the conditional mean of y_t is linear and the constancy of g within (2.1). However, the discussion will concern a more restricted case where the model can be written as

$$y_t = \beta' w_t + f_1(w_t, \nu_t, \theta) + u_t \tag{2.3}$$

where f_1 is at least twice continuously differentiable in a neighbourhood of $\theta = 0$ and $f_1(w_t, \nu_t, 0) = 0$. Furthermore, the errors are assumed homoskedastic, i.e., g is a constant. The latter assumption is made for the case of exposition, but it is also fairly common practice to assume homoskedastic errors when testing linearity in mean. Making the tests robust for the situation where this assumption does not hold is discussed in Granger and Teräsvirta [12, Chapter 6]. The structure of (2.3) implies that linearity in mean is a nested alternative within the nonlinear model, which simplifies the testing problem. The linearity hypothesis is $H_0 : \theta = 0$.

Because (2.3) is linear under the null hypothesis it is natural to apply the score or Lagrange Multiplier principle to test the null hypothesis. If this is done the estimation of (2.3) under the alternative is avoided, which is the whole point. See Engle [9] and Godfrey [11] for more discussion. In this standard situation, the test statistic has the form

$$LM = (1/\hat{\sigma}^2)\sum_{t=1}^{T}\hat{u}_t\,\hat{z}_t'\left(\sum_{t=1}^{T}\hat{z}_t\,\hat{z}_t' - \sum_{t=1}^{T}\hat{z}_t\,w_t'\left(\sum_{t=1}^{T}w_t\,w_t'\right)^{-1}\sum_{t=1}^{T}w_t\,\hat{z}_t'\right)^{-1}\sum_{t=1}^{T}\hat{z}_t\,\hat{u}_t \tag{2.4}$$

where $\hat{u}_t = y_t - \hat{\beta}' w_t$ ($\hat{\beta}$ is an OLS estimator of β under H_0), $\hat{\sigma}^2 = (1/T)\sum_{t=1}^{T}\hat{u}_t^2$, and $\hat{z}_t = \frac{\partial}{\partial\theta} f_1(w_t, \nu_t, 0)$.

Under the null hypothesis (2.4) has an asymptotic χ^2 distribution with k degrees of freedom where k is the dimension of θ. The same result is obtained in the absence of normality if the errors are iid and the moments implied by (2.4) exist; see for instance White [33, Chapter 4] or Luukkonen et al. [18].

The test may also be carried out as follows.

(i) Estimate (2.3) under the null hypothesis using ordinary least squares, estimate the residuals $\hat{u}_t$, and compute the sum of squared residuals $SSR_0 = \sum_{t=1}^{T}\hat{u}_t^2$.

(ii) Regress $\hat{u}_t$ on w_t and z_t and compute the sum of squared residuals SSR_1.

(iii) Compute the test statistic

$$F(k, T-k-p-1) = \frac{(SSR_0 - SSR_1)/k}{SSR_1/(T-k-p-1)}.$$

The F-version of the test is recommended in the small samples whenever k is not small, because its size tends to be closer to the nominal size than that of the χ^2 test while the power remains good. See e.g. Harvey [14, pp. 174–175]. As an example consider a bilinear model in which

$$f_1(w_t, \nu_t, \theta) = w_t'\Theta\nu_t, \quad \theta = \mathrm{vec}\,(\Theta).$$

A total of $pq - k$ elements of θ are assumed zero a priori. Stage (ii) of the test simply consists of regressing u_t on w_t and the $y_{t-i}\,u_{t-j}$ terms corresponding to the k nonzero elements of θ. This is the linearity test against bilinearity discussed in Weiss [32] and Saikkonen and Luukkonen [24].

The above set-up is simple, but unfortunately it does not cover all the interesting nonlinear time series models appearing in the literature. Consider the following model

$$y_t = \beta' w_t + f_1(w_t;\, \theta_1, \theta_2) + u_t \tag{2.5}$$

where $f_1(w_t;\, 0, \theta_2) = 0$. The null hypothesis of linearity is $H_0 : \theta_1 = 0$ whereas the alternative is $H_1 : \theta_1 \neq 0$. The problem is that (2.5) is only identified or estimable under H_1 but not under H_0. An interesting special case of (2.5) is

$$y_t = \beta' w_t + (\theta_{21}' w_t)\, F(w_t;\, \theta_1, \theta_{22}) + u_t \tag{2.6}$$

where $\theta_2 = (\theta_{21}', \theta_{22}')'$. If $\theta_{21} = (\theta_{210}, \ldots, \theta_{21p})'$ with $\theta_{210} = 0$ and

$$F(w_t;\, \theta_1, \theta_{22}) = 1 - \exp\left\{-\theta_1\, y_{t-1}^2\right\}, \quad \theta_1 > 0 \tag{2.7}$$

with $\theta_{22} = 0$, (2.6) becomes the exponential autoregressive (EAR) model discussed for instance in Haggan and Ozaki [13]. Another example is obtained by setting

$$F(w_t;\, \theta_1, \theta_{22}) = (1 + \exp\{-\theta_1(y_{t-d} - \theta_{22}\})^{-1} - 1/2, \quad \theta_1 > 0 \tag{2.8}$$

where $F + 1/2$ is the so-called transition function of the logistic smooth transition autoregressive (LSTAR) model; for discussion see Chan and Tong [5], Luukkonen et al. [19], Granger and Teräsvirta [12], and Teräsvirta [26]. Writing

$$y_t = \beta' w_t + \theta_1' w_t\, F(y_{t-d};\, \theta_2) + u_t \tag{2.9}$$

where $F(y_{t-d};\, \theta_2) = 0$ for $y_{t-d} \le \theta_2$ and $F(y_{t-d};\, \theta_2) = 1$ for $y_{t-d} > \theta_2$, it is seen that the threshold autoregressive (TAR) model of Tong [29,30] with two regimes and threshold variable y_{t-d} fits into this framework as well.

Davies [6,7] considered the above testing problem in a general setting. His solution was the following. Fix the nuisance parameter θ_2 and derive the test, call the test statistic $LM(\theta_2)$. Use $LM^* = \sup_{\theta_2} LM(\theta_2)$ as the (conservative) test statistic. Because the analytic null distribution of LM^* is usually not available obtain the critical value of LM^* at a given significance level through a suitable approximation.

In this paper I discuss another solution which is feasible for models of type (2.6) if the transition function $F(w_t; \theta_1, \theta_{22})$ is sufficiently regular. It is based on approximating F by a suitable Taylor expansion. As an example, consider the EAR model (2.6) with (2.7) and assume that under the linearity hypothesis (2.6) is stationary. Write the transition function (2.7) using the first-order Taylor expansion as follows

$$F(w_t;\, \theta_1, 0) = F'(w_t;\, 0, 0)\, \theta_1 + R(w_t;\, \theta_1, 0) \tag{2.10}$$

where $F'(w_t;\, 0, 0) = y_{t-1}^2$. Substituting the right-hand side of (2.10) for F in (2.6) and reparameterizing yields

$$y_t = \tilde{\beta}' w_t + \delta' \tilde{w}_t y_{t-1}^2 + u_t^* \tag{2.11}$$

where $u_t^* = u_t$ under the linearity hypothesis $H_0 : \theta_1 = 0$ and $\tilde{w}_t = (y_{t-1}, \ldots, y_{t-p})'$. Furthermore, $\delta = \theta_1 \delta_1$ so that linearity may be tested within (2.11), the null hypothesis being $H_0' : \delta = 0$. This leads to a standard test, and again the F-version of the test is recommended instead of the $\chi^2(p)$ test if the time series is not long. If the errors are iid but not normal, the moment assumption $E\, u_t^6 < \infty$ is required for the asymptotic theory to apply.

If the nonlinear alternative (2.6) is a smooth transition autoregressive model with transition function (2.8), the corresponding approximation of (2.6) is

$$y_t = \tilde{\beta}' w_t + \delta' \tilde{w}_t y_{t-d} + u_t^* \tag{2.12}$$

where $\delta = \theta_1 \delta_1$. The linearity hypothesis is $H_0' : \delta = 0$ in (2.12).

This theory does not work if the nonlinear model is a threshold autoregressive model, because in that case F is a step function of y_{t-d}, and this introduces a discontinuity in the likelihood function. Note, however, that the two-regime TAR model is a special case of the LSTAR model (2.6) with (2.8) and is obtained by letting $\theta_1 \to \infty$ in (2.8). The test based on (2.12) is applicable in this limiting case as well. The small-sample power simulations is Petruccelli [21] showed that it has reasonable power when the true model is a TAR.

3. TESTING PARAMETER CONSTANCY

The aforementioned theory is also applicable in multivariate cases; for discussion see Granger and Teräsvirta [12]. An important special case is the one in which the transition function of (2.6) has time as transition variable. For instance, the logistic transition function has the form

$$F(t;\, \theta_1, \theta_{22}) = \left(1 + \exp\{-\theta_1(t - \theta_{22})\}\right)^{-1}, \quad \theta_1 > 0. \tag{3.1}$$

The corresponding auxiliary regression for testing $H_0 : \theta_1 = 0$ against $\theta_1 > 0$ becomes

$$y_t = \beta' w_t + \delta_1' w_t t + u_t^*. \tag{3.2}$$

Using the asymptotic theory in Lai and Wei [16] it can be shown that the asymptotic null distribution of the usual test statistic for testing $H_0' : \delta_1 = 0$ in (3.2) is chi-squared with $p + 1$ degrees of freedom. This requires iid errors, the stationarity of (3.2) under H_0' and the existence of the second moments. For more discussion see Lin and Teräsvirta [17].

Combining (2.6) with (3.1) and testing linearity is useful because it amounts to constructing a parametric test against structural change in a linear model; see Antoch and Hušková [1] for a review of this change-point problem. The approach allows the parameter change to be continuous, which often is a feasible assumption in areas such as econometrics. If the null is rejected an additional advantage is that the alternative may be estimated. This gives the investigator an idea of where in the sample the parameter constancy breaks down and whether the structural change is continuous or rather resembles a single break. By a suitable choice of the transition function different types of structural change may be postulated and detected. The logistic function (3.1) is just one example and more general shapes are possible. A detailed discussion and examples can be found in Lin and Teräsvirta [17].

4. TESTS WITHOUT A SPECIFIC NONLINEAR ALTERNATIVE

As the focus here is on parametric linearity tests many important developments in testing linearity are neglected. Tests based on procedures for detecting structural change such as the CUSUM test of Brown, Durbin, and Evans [4] may be applied to testing linearity against threshold autoregression if the observations are rearranged according to the values of the transition variable. There also exists tests without a specific nonlinear alternative such as tests based on the bispectrum (see Priestley

[22], for a review) or the Brock–Dechert–Scheinkman (BDS) test of independence based on the correlation dimension. The latter is usually applied to the residuals of a linear model, and a rejection of the iid hypothesis is interpreted as evidence in favour of undetected nonlinear structure in the data. For discussion and examples see for instance Brock, Hsieh, and LeBaron [3]. Another important class of tests without a specific alternative are nonparametric linearity tests. The idea is to compare the best linear and nonparametric predictor (both based on the same information set) of a variable and reject linearity if the distance between them is sufficiently large. In the simulation experiments of Hjellvik and Tjøstheim [15] who recently developed tests based on this idea such tests behaved very well. For recent surveys discussing nonparametric tests of independence and linearity see Tjøstheim [28] and Teräsvirta, Tjøstheim, and Granger [27].

It should be stressed, however, that some of the linearity tests presented as tests without a specific alternative can also be interpreted as Lagrange multiplier tests against parametric alternatives. The Regression Specification Error Test or RESET (Ramsey, [23]) and the linearity test of Tsay [31] are examples of such tests. The LM interpretation helps one to find out when the test is powerful and, conversely, against which alternatives it cannot be expected to perform well. For more discussion, see Granger and Teräsvirta [12, Section 6.3].

5. USE OF LINEARITY TESTS IN MODEL BUILDING

In this section I shall describe the use of parametric linearity tests in nonlinear time series model building. For successful use of data-based modelling techniques it is necessary to restrict the class of nonlinear models under consideration. Here it is assumed that if the model generating the data is nonlinear it can only be a STAR model:

$$y_t = \varphi' w_t + (\theta' w_t)\, F(y_{t-d};\, \gamma, c) + u_t \tag{5.1}$$

where $w_t = (1,\, y_{t-1}, \ldots, y_{t-p})'$, $\varphi = (\varphi_0, \varphi_1, \ldots, \varphi_p)'$, and $\theta = (\theta_0, \theta_1, \ldots, \theta_p)'$. F is a bounded continuous function of y_{t-d}. More specifically, it is assumed that (5.1) is either a logistic STAR (LSTAR) model (see Section 2) with

$$F(y_{t-d};\, \gamma, c) = \left(1 + \exp\{-\gamma(y_{t-d} - c)\}\right)^{-1}, \quad \gamma > 0 \tag{5.2}$$

or an exponential STAR (ESTAR) model

$$F(y_{t-d};\, \gamma, c) = 1 - \exp\left\{-\gamma(y_{t-d} - c)^2\right\}, \quad \gamma > 0. \tag{5.3}$$

The ESTAR model is a generalization of the EAR model discussed in Section 2. Write (5.1) as

$$y_t = (\varphi + \theta F)'\, w_t + u_t. \tag{5.4}$$

The model can be interpreted as an autoregressive model whose local dynamics depend on y_{t-d}. The transition function (5.2) is a monotonically increasing function of y_{t-d}, so that the "parameter vector" of (5.4) changes from φ to $\varphi + \theta$ with this transition variable. For the ESTAR model the change is symmetric about c. The parameter vector equals φ at $y_{t-d} - c$ and approaches $\varphi + \theta$ for both low and high values of y_{t-d}. The LSTAR and ESTAR models taken together are thus capable of characterizing rather different types of nonlinear behaviour. Note that in most applications the delay parameter d is unknown and has to be determined from the data.

In the following I shall present a data-based modelling strategy for building STAR models. A more detailed exposition can be found in Teräsvirta [26]. It consists of specification, estimation, and evaluation of a STAR model and thus is similar in character to the linear ARMA model building approach in Box and Jenkins [2]. I shall begin with the specification. It consists of three stages:

(i) Specify a linear autoregressive model for $\{y_t\}$.

(ii) Test linearity for different values of d and if rejected determine d using the test results.

(iii) Choose between LSTAR and ESTAR.

Stage (i) forms the starting-point for testing linearity and is necessary because the lag length p is generally unknown. It can be carried out by applying a suitable order selection criterion such as AIC. The second stage can be performed using another auxiliary regression

$$y_t = \beta' w_t + \delta_1' \tilde{w}_t y_{t-d} + \delta_2' \tilde{w}_t y_{t-d}^2 + \delta_3' \tilde{w}_t y_{t-d}^3 + u_t \tag{5.5}$$

where the linearity hypothesis equals $H_0' : \delta_1 = \delta_2 = \delta_3 = 0$. Testing this hypothesis guarantees power against both LSTAR and ESTAR simultaneously. The motivation for including the fourth-order terms $\delta_3' \tilde{w}_t y_{t-d}^3$ is given in Teräsvirta [26]. The test is carried out for different values of $d \in D = \{1, \ldots, d_0\}$. If the null is rejected, the d corresponding to the smallest p-value among the tests is selected. The reason for this is that if there is a correct d among the alternatives considered, the power of the test is maximized against it. The test may have power against other alternatives as well but is on the average less powerful against them. Proving the consistency of this selection procedure is difficult because the true alternative is nonlinear, but simulations in Teräsvirta [26] support the notion. They also show that the specification procedure as a whole works reasonably well.

The third stage is based on (5.5) and the knowledge of δ_j, $j = 1, 2, 3$, as functions of the parameters of the original STAR model. These vectors are different functions of φ, θ, γ and c in the LSTAR case compared to the ESTAR model, and this fact can be used in selecting between the two families. The choice is based on the outcome of the following test sequence of nested hypotheses:

$$\begin{aligned} H_{03} &: \delta_3 = 0 \\ H_{02} &: \delta_2 = 0 | \delta_3 = 0 \\ H_{01} &: \delta_1 = 0 | \delta_2 = \delta_3 = 0. \end{aligned}$$

The decision rule is as follows. If the p-value for rejecting H_{02} is less than that of the two other tests, choose an ESTAR model. Otherwise choose an LSTAR model. The foundation of this decision rule is discussed in Teräsvirta [26], where it is also shown that the procedure works well already in small samples. Thus the auxiliary regression (5.5) which is a consequence of the Taylor expansion approach to linearity testing when the model is only identified under the alternative is also a useful tool in nonlinear STAR model specification.

The same approach is also applicable at the evaluation stage after a STAR (either LSTAR or ESTAR) model has been estimated. An appropriate question to ask is whether the model has successfully captured all the nonlinear features in the data or not. The latter case calls for respecification of the model. A parametric test of no remaining nonlinearity is obtained as follows. Define an additive STAR model

$$y_t = \varphi' w_t + (\theta_1' w_t)\, F_1(y_{t-d}, \gamma_1, c_1) + (\theta_2' w_t)\, F_2(y_{t-d_2}, \gamma_2, c_2) + u_t \tag{5.6}$$

where γ_1, $\gamma_2 > 0$ and F_1 and F_2 are either of LSTAR or ESTAR type. Assume that a single STAR model has been consistently estimated. To test the adequacy of the STAR model approximate F_2 with a third-order Taylor expansion as above to cover both the LSTAR and ESTAR alternatives. The auxiliary regression for the test has the form

$$\hat{u}_t = \beta' \hat{z}_t + \delta_1' \tilde{w}_t y_{t-d_2} + \delta_2' \tilde{w}_t y_{t-d_2}^2 + \delta_3' \tilde{w}_t y_{t-d_2}^3 + u_t^* \tag{5.7}$$

where $\tilde{z}_t = (\partial \hat{u}_t / \partial \varphi_0,\, \partial \hat{u}_t / \partial \varphi_1, \ldots, \partial \hat{u}_t / \partial c_1)'$, and $\hat{u}_t$ is the residual consistently estimated under the assumption that (5.6) is an ordinary STAR model, i.e., that $\gamma_2 = 0$. This null hypothesis is equivalent to $H_0' : \delta_1 = \delta_2 = \delta_3 = 0$ in (5.7). The standard asymptotic theory applied when the null model is stationary and ergodic and its parameters consistently estimated: the details are found in Eitrheim and Teräsvirta [8]. The result is that the LM type statistic has an asymptotic $\chi^2(3p)$ distribution under H_0'.

6. APPLICATION

In this section the above theory is applied to the well-known Wolf's sunspot numbers, 1700–1979. At present there is no theory available for specifying the dynamics of the process generating the sunspots. As a result, various nonlinear models have been fitted to this series. Tong [30, Section 7.3] reviews these developments. I shall consider the square-root transformed data as in Ghaddar and Tong [10]. Let y_t be the tth observation of the original series. Then the transformed observation is $x_t = 2\left((y_t + 1)^{1/2} - 1\right)$, i.e., the Box-Cox transformation of the original observation plus one with the transformation parameter $\lambda = 1/2$. The transformed series is graphed in Figure 1.

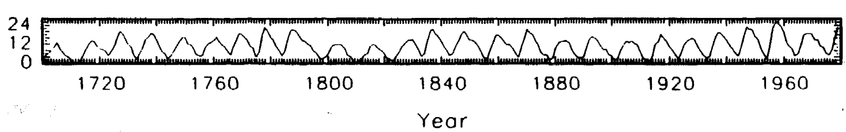

Fig. 1. Transformed Wolf's sunspot numbers, 1700–1979.

I shall characterize the dynamics of the series by a STAR model. The exposition here follows Teräsvirta (1993). To specify a STAR model the first step is to find a linear autoregressive model, and applying AIC yields an AR(9) model. Linearity is tested next using (5.5) with $p = 9$ in w_t. The results are in Table 1. It is seen that the p-value is minimized for $d = 2$, and this value is selected as the delay. The test sequence of the third stage is carried out for $\hat{d} = 2$, and by far the strongest rejection is that of H_{01}. This leads to the selection of an LSTAR model. The model is usually reduced at the estimation stage by imposing exclusion restrictions and re-estimating the parameters, and that has also been done in this example. The final estimated model is

Table 1. Obtained p-valued of tests of linearity for the transformed sunspot series and of no remaining nonlinearity in LSTAR model (6.1) for the transformed sunspot series.

Delay	1	2	3	4	5	6	7	8	9
(a) Linearity tests									
Null hypothesis									
$H_0 : \delta_1 = \delta_2 = \delta_3 = 0$ in (5.5)	0.059	2×10^{-5}	6×10^{-4}	0.015	0.059	0.016	0.0026	0.0042	0.030
$H_{03} : \delta_3 = 0$		0.41							
$H_{02} : \delta_2 = 0 \mid \delta_3 = 0$		0.033							
$H_{01} : \delta_1 = 0 \mid \delta_2 = \delta_3 = 0$		3×10^{-6}							
(b) Tests of no remaining nonlinearity									
Null hypothesis									
$H_0 : \delta_1 \delta_2 = \delta_3 = 0$ in (5.7)	0.30	0.091	0.098	0.075	0.13	0.34	0.53	0.42	0.18
$H_{03} : \delta_3 = 0$		0.42	0.33	0.29					
$H_{02} : \delta_2 = 0 \mid \delta_3 = 0$		0.040	0.041	0.068					
$H_{01} : \delta_1 = 0 \mid \delta_2 = \delta_3 = 0$		0.44	0.57	0.32					

$$\begin{aligned}
x_t = {} & \underset{(0.080)}{1.55}\, x_{t-1} - \underset{(0.15)}{0.82}\, x_{t-2} + \underset{(0.040)}{0.27}\, x_{t-7} \\
& + (\underset{(0.79)}{2.58} - \underset{(0.11)}{0.69}\, x_{t-1} + \underset{(0.15)}{0.82}\, x_{t-2} - \underset{(0.038)}{0.31}\, x_{t-3} - \underset{(0.040)}{0.27}\, x_{t-7} - \underset{(0.064)}{0.12}\, x_{t-8} \\
& + \underset{(0.084)}{0.15}\, x_{t-9} + \underset{(0.059)}{0.16}\, x_{t-11})(1 + \exp\{-\underset{(2.2)}{4.7} \times 0.178(x_{t-2} - \underset{(0.69)}{7.8})\})^{-1} + \hat{u}_t
\end{aligned} \tag{6.1}$$

$$s = 1.91, \quad \sum_{t=1}^{T} \hat{u}_t^2 / T = 3.47.$$

Restrictions $\varphi_2 = -\theta_2$ and $\varphi_7 = -\theta_7$ have been suggested by the data and imposed. The figures below the parameter estimates are asymptotic standard deviations based on the Hessian matrix, and s is the standard error of residuals. The inverse of the sample standard deviation of x_t appearing in (6.1) ($1/\hat{\sigma}(x) = 0.178$) is a scale factor with the purpose of making γ scale independent.

The dynamic properties of the estimated model are best seen through the roots of the characteristic polynomial $q(z) = z^p - \sum_{j=1}^{p} (\varphi_j + \theta_j F)\, z^{p-j}$ at different values of F. Two interesting values are $F = 0$ and $F = 1$, respectively, because they represent the two extreme regimes. In turns out that for $F = 1$, the dominant as well as all the other roots lie inside the unit circle indicating that the model is locally stationary for moderate and high values of transition variable y_{t-2}. In fact, its local dynamics are close to those of the linear $AR(9)$ model. On the other hand, for $F = 0$ there exists a complex pair of explosive roots which describe the behaviour of the process at the troughs and shortly thereafter. The number of sunspots after a trough seems to increase at a faster rate than it decreased, see Figure 1, and the local nonstationarity is there to characterize that phenomenon.

At the evaluation stage the estimated model is subjected to a number of tests; see Teräsvirta [25]. As this paper is focusing on parametric linearity tests, the results of the test of no remaining nonlinearity are of interest. They are found in Table 1. It is seen that (6.1) has captured most of the nonlinearity. There may be some ESTAR-type nonlinearity left as the p-values at delays 2, 3 and 4 are between 0.05 and 0.1, and each test sequence points at ESTAR. The p-values are not very small, however, so that tentatively accepting (6.1) is not an unreasonable thing to do.

7. CONCLUSIONS

This paper illustrates the role of parametric linearity tests in nonlinear model building. They can be used not only for testing linearity against parametric nonlinear alternatives but also as helpful tools in attempts of modelling the nonlinearity that may be discovered in the data when these tests are applied. To do the latter, however, the family of nonlinear models under consideration has to be rather restricted. Otherwise the data-based specification techniques may easily fail or be inefficient. This may be considered a drawback, but another advantage of the Lagrange-multiplier type tests discussed here is that they usually work well already in short time series that are frequently encountered for instance in econometric applications. Tests against nonspecified alternatives such as the BDS or bispectrum test may have power against a wide range of alternatives, but the power tends to be less satisfactory in short series.

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