EXTERIOR ALGEBRA AND INVARIANT SPACES
OF IMPLICIT SYSTEMS:
THE GRASSMANN REPRESENTATIVE APPROACH[1]

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The matrix pencil algebraic characterisation of the families of invariant subspaces of an implicit system $S(F,G) : F\dot{z} = Gz$ F, $G \in \mathbb{R}^{m \times n}$, is further developed by using tools from Exterior Algebra and in particular the Grassmann Representative $g(\mathcal{V})$ of the subspace $\mathcal{V}$ of the domain of (F,G). Two different approaches are considered: The first is based on the compound of the pencil $C_d(sF - G)$, which is a polynomial matrix and the second on the compound pencil $sC_d(F) - C_d(G)$, $d = \dim \mathcal{V}$. For the family of proper spaces of the domain of (F,G), $m \geq d$, new characterisations of the invariant spaces $\mathcal{V}$ are given in terms of the properties of $g(\mathcal{V})$ as generalised eigenvectors, or invariance conditions for the spaces $\Lambda^p \mathcal{V}$, $p = 1, 2, \ldots, d$.

1. INTRODUCTION

The natural operator associated with Generalized Autonomous Dynamic System $S(F,G) : F\dot{z} = Gz$, F, $G \in \mathbb{R}^{m \times n}$ is the matrix pencil $sF - G$. Singular systems $S_e : E\dot{x} = Ax + Bu$ ($|E| = 0$), as well as proper systems ($|E| \neq 0$), are special cases of the $S(F,G)$ descriptions. Matrix pencil theory [3] provides a unifying framework for the study of algebraic, geometric, dynamic and computational aspects of $S(F,G)$ systems [12],[7]. For proper systems, a matrix pencil approach for the study and classification of the fundamental invariant subspaces [24],[23] has been developed in [8],[6]. Extending the standard geometric theory concepts and results [24],[23] to the case of Singular systems, has been an active research area [12],[14],[7],[2],[15],[16],[17],[20] and references therein.

The different types of invariants subspaces of a given dimension associated with a linear system may be seen as elements of a certain Grassmannian and the characterisation of such families of invariant spaces in terms of possible dimensions and related parameterisation issues is an important topic. Exterior algebra tools [4],[18],[9] are central in providing a representation of the Plücker embedding of a Grassmannian

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into a projective space [5]; these tools have been deployed by Karcanias and Gian-nakopoulos [10] to introduce new sets of invariants for families of linear systems. The new invariants introduced through the exterior algebra machinery are the Plücker matrices and the associated canonical Grassmann representatives and provide complete sets of invariants for the corresponding families of systems [10]. The character-isation of invariant spaces in terms of Strict Equivalence (SE)-invariants of matrix pencils, provides the means for the characterisation of invariant subspaces also in terms of the tools provided by Exterior Algebra. The aim of this paper is to provide the framework and the tools for the Exterior algebra characterisation of families of invariant subspaces of singular systems. The results provide alternative algebraic characterisations of such families, which is closely related to parameterisation issues.

The starting point of our investigation is that for the implicit system $S(F, G)$: $F\dot{z} = Gz$, F, $G \in \mathbb{R}^{m \times n}$ the subspaces $\mathcal{V}$, $\dim \mathcal{V} = d$, of the domain of (F, G) may be classified by the invariants of the restriction pencil $(F, G)/\mathcal{V} = sFV - GV$ [8], [12], which in turn lead to the definition of different notions of invariant spaces based on the relationship of the images of FV, GV, [14]. The algebraic characterisation, based on $sFV - GV$ pencil and the geometric based on the properties of $\mathcal{V}$ with respect to (F, G) lead in a natural way to two different approaches for the exterior algebra characterisation of the subspaces of the domain of (F, G). The first is based on properties of Grassmann Representative $g(\mathcal{V})$ with respect to the compound of the $sF - G$ pencil [19], $C_d(sF - G)$, whereas the second relies on the properties of $g(\mathcal{V})$, as well as the exterior space $\Lambda^p \mathcal{V}$, $p = 1, 2, \ldots, d$ with respect to the compound pair $(C_d(F), C_d(G))$. The essence of both characterisations is that they characterise the different families of invariant spaces as eigenvectors of the new compound operators, or as invariant exterior spaces with respect to the $(C_d(F), C_d(G))$ pair. An implicit assumption behind the present characterisations is that we consider spaces $\mathcal{V}$ for which $d = \dim \mathcal{V} \le m$ and such spaces are referred to as proper. The study of nonproper spaces, $d > m$, requires alternative means and not the Grassmann Representative $g(\mathcal{V})$ and it is not considered here. The framework established by the Exterior Algebra characterisation provides tools for studying issues, such as the parameterisation of families of invariant spaces; this new framework has also the advantage that it is closer to the spirit of the determinantal assignment problems [10].

Throughout this paper we adopt the following notations. If $\mathcal{V}$ is a vector space, then V denotes a basis matrix of $\mathcal{V}$. If $T(s) = sF - G$ is a matrix pencil [3], $\Psi_T = \{\mathcal{D}_{\lambda T}, \mathcal{D}_{\infty T} \mathcal{H}_{0T}, \mathcal{H}_{rT}\}$ denotes the set of SE-invariants of $T(s)$ and Φ_T denotes the root range of $T(s)$, (i.e., set of distinct eigenvalues). In particular, $\mathcal{D}_{\lambda T} = \{(s - \lambda_i)^{\tau_i} \mid i \in \tilde{r}\}$, $\mathcal{D}_{\infty T} = \{\hat{s}^{q_i} \mid i \in \tilde{\mu}\}$ denote the set of all finite, ∞ elementary divisors (fed, ied) and $\mathcal{H}_{cT} = \{\varepsilon_i \mid i \in \tilde{p}, \varepsilon_1 = \cdots = \varepsilon_g = 0, 1 \le \varepsilon_{g+1} \le \cdots \le \varepsilon_p\}$, $\mathcal{H}_{rT} = \{\eta_i = \cdots = \eta_h = 0, 1 \le \eta_{h+1} \le \cdots \le \eta_t\}$ denote the sets of column, row minimal indices (cmi, rmi), respectively of $T(s)$ and $i \in \tilde{p}$, is used for $i = 1, 2, \ldots, p$. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}(s)$ denote the field of real, complex numbers and rational functions, respectively and $\mathbb{R}[s]$ denotes the ring of polynomials over $\mathbb{R}$. $\mathbb{R}^n$, $\mathbb{C}^n$ and $\mathbb{R}^n(s)$ denote the n-dimensional vector space over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{R}(s)$ respectively. If $\mathcal{F}$ is a field, or ring, then $\mathcal{F}^{m \times n}$ denotes the set of $m \times n$ matrices with ele-

ments from $\mathcal{F}$. $\mathbb{R}^{m \times n}[s]$ denotes the set of $m \times n$ matrices with elements from $\mathbb{R}[s]$. $Q_{k,n}$ denotes the set of lexicographically ordered, strictly increasing sequences of k integers from $1, 2, \ldots, n$. If $\{x_{i1}, \ldots, x_{ik}\}$ is a set of vectors of a vector space $\mathcal{V}$, $\omega = (i_1, \ldots, i_k) \in Q_{k,n}$, then $x_{i1} \Lambda \cdots \Lambda x_{ik} = x_{\omega} \Lambda$ denotes the exterior product of these vectors [19]. If $\mathcal{V}$ is a vector space, then $\Lambda^r \mathcal{V}$ denotes its rth exterior power [4],[18]. If $H \in F^{m \times n}$ and $r \leq \min\{m, n\}$, then by $C_r[H]$ denotes the rth compound matrix $\mathcal{H}$. $\rho[H]$ denotes the rank of $\mathcal{H}$ and $\mathcal{N}_r(H)$, $\mathcal{N}_\ell(\mathcal{H})$, $\mathcal{R}(H)$ denote the right, left null space and range space, correspondingly. If a set is either o, or consists only of zero numbers, it is denoted by $\langle 0 \rangle$.

2. PROBLEM STATEMENT AND PRELIMINARY RESULTS

The aim of this section is to provide a short introduction to the main definitions and properties of the algebraic (matrix pencil) and geometric characterization of subspaces of a Singular system, which are essential for studying the parameterization of the various families of subspaces we intend to study. Consider the Singular system described by

$$S_e : E\dot{x} = Ax + Bu \tag{2.1}$$

where $E, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times \ell}$, $\ell \leq n$ and $\rho(B) = \ell$. Now, let $\ell < n$. If $(N, B^\dagger)$ is a pair of a left annihilator, inverse of B (i.e., $N \in \mathbb{R}^{(n-\ell) \times n}$, $\rho[N] = n - \ell$, $NB = 0$, $B^\dagger B = I_\ell$). It is readily shown [8],[6] that S_e is equivalent to

$$NE\dot{x} = NAx \tag{2.2}$$

$$u = B^\dagger\{E\dot{x} - Ax\}. \tag{2.3}$$

If $\mathcal{V} \in \mathbb{R}^n$ is a vector space and V is a basis matrix of $\mathcal{V}$, then the solution of S_e, which are restricted in $\mathcal{V}$, are defined as the solutions of the $\mathcal{V}$-restricted differential system

$$NEV\dot{v} = NAVv, \quad x = Vv \tag{2.4}$$

whereas the control input that generates $x(t) \in \mathcal{V}$ is defined by the equation (2.4). The nature of strict equivalence (SE) invariants [3] of the $\mathcal{V}$-restriction pencil $R_v(s) = sNEV - NAV$ characterizes the properties of the solution space of (2.4). The descriptions (2.2) and (2.4) are Generalized Autonomous Dynamic systems of the type

$$S(F, G) : F\dot{z} = Gz, \quad F, G \in \mathbb{R}^{m \times n} \tag{2.5}$$

Throughout this paper, we denote $F = NE$, $G = NA$, $m = n - \ell$, and $T(s) = sF - G$, the pencil associated with this system. The pencils $sF - G$ is said to be *Strict equivalent* to the pencil $sF' - G'$, iff $P(sF - G)Q = sF' - G'$, where P and Q are square nonsingular matrices over a field $\mathcal{F}$. Thus, the class of SE-pencils is characterized by a uniquely defined element which has the canonical form, known as the Kronecker canonical form (KCF), i.e.

$$P(sF - G)Q = sF_k - G_k = \tag{2.6}$$

$$= \operatorname{diag}\left\{O_{h,g}, L_{\epsilon(g+1)}, \ldots, L_{\epsilon(p)}, L^t_{\eta(h+1)}, \ldots, L^t_{\eta(t)}, D_{q(1)}, \ldots, D_{q(\mu)}, sI - J\right\},$$

where P and Q are square nonsingular matrices over $\mathcal{F}$, $L_\xi = s[I_\xi; \underline{0}] - [\underline{0}, I_\xi] \in \mathbb{R}^{\xi \times (\xi+1)}[s]$, $\xi = \varepsilon_i$, or η_j, $D_{q(i)} = s\,E_{q(i)} - I$, $i \in \tilde{\mu}$ corresponds to ied of $sF - G$, $E_{q(i)}$ is a matrix of order q_i, whose elements in the first super diagonal are 1, whereas the remaining are zero; $sI - J$ is the normal Jordan Form, which is uniquely defined by the set of fed $(s - \lambda_i)^{\tau i}$, $i \in \tilde{r}$. If $\mathcal{V} \in \mathbb{R}^n$ is a vector space, then for the triple $(F, G; \mathcal{V})$, the set of SE-invariants and root range of the $\mathcal{V}$-restriction pencil $\mathbb{R}_V = sFV - GV$ are denoted by $\Phi_V = \{\mathcal{D}_{\lambda V}, D_{\infty V}, \mathcal{H}_{rV}, \mathcal{H}_{cV}\}$ and Φ_V, respectively. If the restriction pencil $R_V(s)$ is transformed by the transformation P_k and Q_k to the Kronecker canonical form (2.6), then it is written as

$$P_k(sF - G)\,V\,Q_k = (sF - G)\,[V_\varepsilon, \, V_\eta, \, V_\infty, \, V_\lambda] \tag{2.7}$$

where $(sF - G)\,V_\varepsilon$, $(sF - G)\,V_\eta$, $(sF - G)\,V_\infty$ and $(sF - G)\,V_\lambda$ correspond to the cmi, rmi, ied and fed partitions, respectively. Thus, $\mathcal{V}$ can be decomposed as

$$\mathcal{V} = \mathcal{V}_\varepsilon \oplus \mathcal{V}_\eta \oplus \mathcal{V}_\infty \oplus \mathcal{V}_\lambda \tag{2.8}$$

where $\mathcal{V}_\varepsilon$, $\mathcal{V}_\eta$, $\mathcal{V}_\infty$ and $\mathcal{V}_\lambda$ are the subspaces spanned by V_ε, V_η, V_∞ and V_λ, respectively [8] and these subspaces are called $((F, G)$-*column minimal indices subspace*, $((F, G)$-cmis), (F, G)-*row minimal indices subspace* $((F, G)$-rmis), *infinite-*(F, G) *elementary divisors subspace* $(\infty - F, G)$-eds), (F, G)-*finite elementary divisors subspace* $((F, G)$-fed), respectively where (F, G)-fed may be decomposed to o and finite nonzero eds V_0 and V_α, $\alpha \neq 0$ respectively. Finally and (F, G)-eds $\mathcal{V}$ for which $\Phi_V = \{\lambda, \lambda^*\}$, $\lambda, \lambda^* \in \mathbb{C}$, or $\Phi_V = \{\lambda, \varepsilon R \cup \{\infty\}\}$ is called $\{\lambda, \lambda^*\}$, or (λ)-*simple* (F, G)-*eds*.

If $\mathcal{V} \in R^n$ is a subspace of the domain of (F, G), then the fundamental properties of $\mathcal{V}$ with respect to (F, G) may be expressed as relationships between the images FV, GV. Then the following families of spaces can be defined [12], [14]: $\mathcal{V}$ is called and (F, G)-*invariant subspace* $((F, G)$-is, if $FV \subseteq GV$, a (G, F)-*invariant subspace* (G, F)-is if $GV \subseteq FV$ and a *complete* (F, G)-*is* $(c - (F, G)$-is) if $FV = GV$. The relationships between the different families of invariant subspaces are summarised by the following two results [14].

Theorem 2.1. Let $\mathcal{V} \in \mathbb{R}^n$ be a subspace of the domain of (F, G). Then,

(i) $\mathcal{V}$ is an $\infty - (F, G)$-eds, iff $\mathcal{F}V \subset GV$, $\mathcal{N}_r(G) \cap \mathcal{V} = 0$ and there is no proper subspace $\mathcal{V}' \subset \mathcal{V}$ for which $FV' = GV'$.

(ii) $\mathcal{V}$ is a $o - (F, G)$-eds, iff $GV \subset FV$, $\mathcal{N}_r(F) \cap \mathcal{V} = 0$ and there is no proper subspace $\mathcal{V}' \subset \mathcal{V}$ for which $FV' = GV'$.

(iii) $\mathcal{V}$ is a nonzero $-\{F, G\}$-eds, iff $FV = GV$, $\mathcal{N}_r(F) \cap \mathcal{V} = 0$ and $\mathcal{N}_r(G) \cap \mathcal{V} = 0$.

Theorem 2.2. Let $\mathcal{V} \in \mathbb{R}^n$ be a subspace of the domain of (F, G). The following statements are equivalent:

(i) $\mathcal{V}$ is an (F, G)-cmis

(ii) $\mathcal{V}$ is an (G, F)-cmis

(iii) If $\dim \mathcal{V} = d$, there exists a polynomial vector $x(s) = [x_0, x_1, \ldots, x_{d-1}] \, e_{d-1}(s) = X_d e_{d-1}(s)$, $e_{d-1}(s) = [1, s, \ldots, s^{d-1}]^t$ of degree $d - 1$ such that X_d is a basis, matrix for $\mathcal{V}$ and $(sF - G) \, x(s) = 0$.

(iv) There exists a basis $[x_0, x_1, \ldots, x_{d-1}]$ of $\mathcal{V}$ such that

$$Fx_{d-1} = 0, \quad Fx_{d-2} = Gx_{d-1}, \ldots, Fx_0 = Gx_1, \quad 0 = Gx_0$$

(v) $G\mathcal{V} = F\mathcal{V}$ and there exists a basis matrix V_c such that the restriction maps J_0, J_0^t defined by

$$GV_c = FV_c J_0, \quad FV_c = GV_c J_0^t$$

are simple Jordan blocks associated with the zero eigenvalue.

A vector $X(s) = X_d e_{d-1}(s)$ is called a *right annihilating* vector, X_d is a cyclic basis, and $\mathcal{R}\{X_d\}$ is the *Supporting space* of $x(s)$. Note that the family of (F, G)-cmis is a subfamily of the complete $- (F, G)$-is family. A characterisation of the (F, G)-is, (G, F)-is, $c - (F, G)$-is families in terms of invariants is given in [14].

Remark 2.1. The theorems of rmis is dual to those for cmis. This duality is of the transposed type i.e., $\beta^t(s) \, (sF - G) = 0$ is equivalent to $(sF^t - G^t) \, \beta(s) = 0$.

Some further families of invariant subspaces characterised by the above general notions of invariants spaces are introduced below [14].

Corollary 2.1. Let $\mathcal{V}$ be a subspace of the domain of (F, G). The following properties hold true:

(i) $\mathcal{V}$ is a finite $- (F, G)$-eds $((F, G)$-feds i.e., $\mathcal{V} = \mathcal{V}_\alpha \oplus \mathcal{V}_0$, iff $G\mathcal{V} \subseteq F\mathcal{V}$ and $\mathcal{N}_r(F) \cap \mathcal{V} = 0$.

(ii) $\mathcal{V}$ is a nonzero $- (F, G)$-eds $((F, G)$-nzeds) i.e., $\mathcal{V} = \mathcal{V}_\alpha \oplus \mathcal{V}_\infty$, iff $F\mathcal{V} \subseteq G\mathcal{V}$ and $\mathcal{N}_r(G) \cap \mathcal{V} = 0$,

Definition 2.1. [7], [14], [21], [17], [16], [15]. Let $F, G \in \mathbb{R}^{m \times n}$. We may define the following sequences:

$$\mathcal{P}(G, F) := \left\{ \mathcal{T}^0 = \mathbb{R}^n, \, \mathcal{T}^{k+1} = G^{-1}(F\mathcal{T}^k), \, k \geq 0 \right\}$$
$$\mathcal{P}(F, G) := \left\{ \mathcal{S}^0 = \mathbb{R}^n, \, \mathcal{S}^{k+1} = F^{-1}(G\mathcal{S}^k), \, k \geq 0 \right\}$$
$$\mathcal{L}(F, G) := \left\{ \mathcal{K}^0 = 0, \, \mathcal{K}^{k+1} = F^{-1}(G\mathcal{K}^k), \, k \geq 0 \right\}$$
$$\mathcal{L}(G, F) := \left\{ \mathcal{L}^0 = 0, \, \mathcal{L}^{k+1} = G^{-1}(F\mathcal{L}^k), \, k \geq 0 \right\}.$$

$\mathcal{P}(G, F)$, $\mathcal{P}(F, G)$ are nonincreasing and converge to subspace $\mathcal{T}^*$, $\mathcal{S}^*$, respectively, in at most n steps and the $\mathcal{T}^*$ and $\mathcal{S}^*$ are the supremal (G, F)-is, (F, G)-is in $\mathbb{R}^n$, respectively. $\mathcal{L}(G, F)$ and $\mathcal{L}(F, G)$ are nondecreasing and converge to subspaces $\mathcal{K}^*$, $\mathcal{L}^*$ respectively in at most n steps and $\mathcal{K}^*$, $\mathcal{L}^*$ are the supremal (F, G)-is and (G, F)-is, in $\mathbb{R}^n$, respectively.

Theorem 2.2. [14]. Let $\mathbb{R}^n = \mathcal{V}_\epsilon \oplus \mathcal{V}_\eta \oplus \mathcal{V}_\infty \oplus \mathcal{V}_\lambda$ be an invariant decomposition of $\mathbb{R}^n$ with respect to (F, G). Then

  (i) $\mathcal{T}^* = \mathcal{V}_\epsilon \oplus \mathcal{V}_\lambda$

 (ii) $\mathcal{S}^* = \mathcal{V}_\epsilon \oplus \mathcal{V}_\infty \oplus \mathcal{V}_\lambda$

(iii) $\mathcal{W}^* = \mathcal{T}^* \cap \mathcal{S}^* = \mathcal{V}_\epsilon \oplus \mathcal{V}_\lambda$ is the supremal $c - (F, G)$-is of $\mathbb{R}^n$

 (iv) $\mathcal{K}^* = \mathcal{V}_\epsilon \oplus \mathcal{V}_\infty$

  (v) $\mathcal{L}^* = \mathcal{V}_\epsilon \oplus \mathcal{V}_0$, where $\mathcal{V}_0$ is the $o - (F, G)$ eds

 (vi) $\mathcal{R}^* = \mathcal{V}_\epsilon = \mathcal{T}^* \cap \mathcal{K}^* = \mathcal{S}^* \cap \mathcal{L}^* = \mathcal{K}^* \cap \mathcal{L}^*$.

A subspace $\mathcal{V} \in \mathbb{R}^n$ of the domain of (F, G) for which the restriction pencil does not have nonzero rmi is called a *partitioned-(F, G)-is $(p - (F, G)$-is)* [14],[7]. Clearly, all invariant subspaces of the type considered above are elements of the general family of $p - (F, G)$-is.

Consider the spaces $\mathcal{T}^*$, $\mathcal{S}^*$, $\mathcal{K}^*$ and $\mathcal{R}^*$ given in the Definition 2.1 and Theorem 2.2. If we put NE and NA instead of F and G, respectively in their related sequences, we will obtain the following sequences,

$$\mathcal{T}^0 = \mathbb{R}^n, \ \mathcal{T}^{k+1} = (NA)^{-1}NE\mathcal{T}^k = A^{-1}(E\mathcal{T}^k + \mathcal{R}(B))$$
$$\mathcal{S}^0 = \mathbb{R}^n, \ \mathcal{S}^{k+1} = (NE)^{-1}NA\mathcal{S}^k = E^{-1}(E\mathcal{S}^k + \mathcal{R}(B))$$
$$\mathcal{K}^0 = 0, \ \mathcal{K}^{k+1} = (NE)^{-1}NA\mathcal{K}^k = E^{-1}(A\mathcal{K}^k + \mathcal{R}(B))$$
$$\mathcal{L}^0 = 0, \ \mathcal{L}^{k+1} = (NA)^{-1}NE\mathcal{L}^k = A^{-1}(E\mathcal{L}^k + \mathcal{R}(B))$$

it is clear that the subspaces $\mathcal{T}^*$, $\mathcal{S}^*$, $\mathcal{K}^*$ and $\mathcal{R}^*$ define supremal (A, E, B) is, supremal almost (E, A, B)-is, supremal almost reachability subspace and supremal reachability subspace for Singular systems, respectively [20],[15],[17].

Problem Statement. The problem in this study is to provide a characterization, based on Exterior Algebra, of the various types of feedback invariant subspaces, i.e. $\mathcal{T}^*$, $\mathcal{S}^*$, $\mathcal{K}^*$, and $\mathcal{R}^*$ and their subspaces for the case of $S(F, G)$; the results then may be specialised to the case of singular systems.

3. BACKGROUND FROM EXTERIOR ALGEBRA

The present approach uses the notions of a Compound matrix, decomposability of multivectors and the geometry of the so-called Grassmann variety [5] in an essential way. Some background notation and standard results [18],[19] related to the above mentioned notions are summarized below.

Compound matrices. [18]. Let $A \in \mathcal{F}^{m \times n}$ and $1 \leq p \leq \min\{m, n\}$, then the pth compound matrix of A is the $\binom{m}{p} \times \binom{n}{p}$ matrix whose entries are the minors $\det\{A[\alpha|\beta]\}$, defined for the $\alpha \in Q_{p,m}$, $\beta \in Q_{p,n}$, sets of rows, columns which are arranged lexicographically in α and β. This matrix will be denoted by $C_p(A)$. Some of the basic properties of compound matrices are:

a) If $A \in \mathcal{F}^{n \times m}$, $1 \leq p \leq n$ and also A is nonsingular, then
   (i) $[C_p(A)]^{-1} = C_p(A^{-1})$,   (ii) $C_p(A^t) = [C_p(A)]^t$,   (iii) $C_p(A)$ is also nonsingular.

b) *Binet–Cauchy Theorem*: If $A \in \mathcal{F}^{p \times n}$, $B \in \mathcal{F}^{n \times k}$ and $1 \leq p \leq \min\{m, n, k\}$, then $C_p(AB) = C_p(A) \, C_p(B)$

c) If $A \in \mathcal{F}^{p \times n}$ and the p rows of A are denoted by $a_1^t, \ldots, a_p^t$ in succession $(1 \leq p \leq n)$, then $C_p(A)$ is an $\binom{n}{p}$-tuple and it is called the *Grassmann product* or *skew-symmetric product* of the vectors $a_1^t, \ldots, a_p^t$. The usual notation for this $\binom{n}{p}$-tuple of subdeterminants of A is $a_1^t \wedge \cdots \wedge a_p^t$ and it denotes a row vector. Similar definition can be given for the exterior product of the columns of A.

Definition 3.1. [4],[18]. Let $\mathcal{V}$ be a linear vector space over a field $\mathcal{F}$ and let $\dim \mathcal{V} = p$. A vector $z \in \Lambda^r$, $r \leq p$ is called *decomposable*, if there exist vectors $v_i \in \mathcal{V}$, $i \in \tilde{r}$ such that $z = v_1 \wedge \cdots \wedge v_r$.

Let $\mathcal{U}$ be a linear vector space over a field $\mathcal{F}$ and let $\dim \mathcal{U} = n > m$. The Grassmannian $G(m, \mathcal{U})$ is defined as the set of m-dimensional subspaces $\mathcal{V}$ of $\mathcal{U}$; $G(m, \mathcal{U})$ actually admits the structure of an analytic manifold which is known as the *Grassmann manifold*. The following important results hold true [18].

Theorem 3.1. Let $x\Lambda = x_1 \wedge \cdots \wedge x_m$, $z\Lambda = z_1 \wedge \cdots \wedge z_m$ be two non-zero decomposable vectors of $\mathcal{U}$ and let $\mathcal{V}_x$, $\mathcal{V}_z \in G(m, \mathcal{U})$ be the associated subspaces. Necessary and sufficient condition, for $\mathcal{V}_x = \mathcal{V}_z$ is that $x\Lambda = az\Lambda$, where $a \in \mathcal{F} \setminus \{0\}$.

Corollary 3.1. A nonzero decomposable vector $c \in \Lambda^m \mathcal{U}$, may be written as $c = v\Lambda = v_1 \wedge \cdots \wedge v_m$ and uniquely defines $\mathcal{V} = \mathrm{sp}\{v_i \mid i \in \tilde{m}\} \in G(m, \mathcal{U})$.

Let $\mathcal{V} \in G(m, \mathcal{U})$, then any non-zero decomposable element $x_1 \wedge \cdots \wedge x_m$, $x_i \in \mathcal{V}$, $i \in \tilde{m}$ is called a *Grassmann representative* (GR) of $\mathcal{V}$. The Grassmann representatives all differ only by non zero scalar factors so that we shall denote anyone of them simply by $g(\mathcal{V})$. Note that although the vector space $\Lambda^r \mathcal{V}$ is generated by decomposable vectors, not every vector in this space is decomposable. If B^V is a basis for $\mathcal{V}$, $\Lambda^r B_V$, is the induced basis of Λ^r and $\{\xi_\omega\}$, $\omega \in Q_{r,p}$ are the coordinates of $z \in \Lambda^r \mathcal{U}$ with respect to B_V, then z is decomposable, if any only if $\{\xi_\omega\}$ satisfy a set of quadratic equations known as Quadratic Plücker Relationships (QPR) [5],[18]. An alternative test for decomposability, which also allows the reconstruction of the subspace, when the multivector is decomposable, has been given in [11].

4. EXTERIOR ALGEBRA CHARACTERIZATION OF MATRIX PENCIL INVARIANTS

In this section some general results on the characterization of matrix pencil invariants using exterior algebra are derived, which provide the necessary background of the present characterization. The analysis in this section follows that developed in [10]

for general rational vector space and polynomial bases. Let $A(s) \in \mathbb{R}^{m \times n}[s]$, $m \geq n$ and $\rho = \text{rank}_{\mathbb{R}(s)}(A(s))$. We may define

$$\mathcal{X}_a = \text{col sp}_{\mathbb{R}(s)}\{A(s)\} \quad \text{and} \quad C_\rho(A(s)) = [\ldots p_i(s) \ldots] \tag{4.1}$$

Proposition 4.1. [10]. Every nonzero $p_i(s)$ is a GR of $\mathcal{X}_a$. Furthermore, if $p_i(s)$, $p_j(s) \neq 0$ are two GRs, then $p_i(s) = \lambda_{ij}(s) p_j(s)$, $\lambda_{ij}(s) \in \mathbb{R}(s) \setminus \{0\}$.

Proposition 4.2. Let $\phi(s)$ be the gcd of all the entries of $C_\rho(A(s))$. Then, there exists a coprime vector $p^*(s)$ such that any nonzero vector $p_i(s)$ may be expressed by

$$p_i(s) = \lambda_i(s) \phi(s) p^*(s), \quad \lambda_i(s) \in \mathbb{R}[s] \setminus \{0\}. \tag{4.2}$$

Now, let $A(s) = sF - G$, $\rho = \text{rank}_{\mathbb{R}(s)}(sF - G)$ and define

$$\mathcal{X}_r = \text{row sp}_{\mathbb{R}(s)}\{sF - G\}, \quad \mathcal{X}_c =: \text{col sp}_{\mathbb{R}(s)}\{sF - G\} \tag{4.3}$$

Calculate $C_\rho(sF - G)$ and define

$$C_\rho(sF - G) = \begin{bmatrix} \vdots \\ \vdots \\ v_i^t(s) \\ \vdots \\ \vdots \end{bmatrix} =: [\ldots, u_i(s), \ldots] \tag{4.4}$$

By Proposition 4.1, every nonzero $v_i^t(s)$ is a GR for $\mathcal{X}_r$ and every nonzero $u_i(s)$ is a GR for $\mathcal{X}_c$, and thus, for some $\alpha_i(s)$, $\beta_i(s) \in \mathbb{R}\{s\} \setminus \{0\}$ we have that if $\phi(s)$ is the gcd of all entries of $C_\rho(sF - G)$, then

$$v_i^t(s) = \alpha_i(s) \phi(s) v^*(s)^t, \qquad u_i(s) = \beta_i(s) \phi(s) u^*(s) \tag{4.5}$$

where $v^*(s)^t$, $u^*(s)$ are coprime vectors. For the (4.6) descriptions we have

Proposition 4.3. For the pencil $sF - G$ we have:
  (i) The zeros of $\phi(s)$ are the roots of fed's of $sF - G$.
  (ii) If $u^*(s)$ $(v^*(s)^t)$ is constant vector $\Longrightarrow \nexists$ nonzero rmi (cmi)
  (iii) If $u^*(s)$ $(v^*(s)^t)$ is nonconstant polynomial vector $\Longrightarrow \exists$ rmi (cmi) which are nonzero, and

$$\deg\{v^*(s)^t\} = \sum_{i=1}^{p} \varepsilon_i, \qquad \deg\{u^*(s)^t\} = \sum_{i=1}^{t} \eta_i.$$

Furthermore, since the number of cmi $= n - \rho =: \pi_c$ and the number of rmi $= m - \rho =: \pi_r$ we have:

$$\pi_c + \sum_{i=1}^{p} \varepsilon_i = \sum_{i=1}^{t}(\varepsilon_i + 1), \qquad \pi_r + \sum_{i=1}^{t} \eta_i = \sum_{i=1}^{t}(\eta_i + 1).$$

Proposition 4.4. Let $D_d(s, \hat{s})$ be the greatest common divisors (gcd) of all the minors of order d of the matrix $sF - \hat{s}G$ and let $\rho = \text{rank}_{\mathbb{R}(s)}\{sF - \hat{s}G\}$. There exists a homogeneous matrix $A_d(s, \hat{s})$ and a polynomial $\sigma_d(s, \hat{s})$ such that

(i) $C_d(sF - \hat{s}G) = D_d(s, \hat{s}) A_d(s, \hat{s})$;

(ii) $D_d(s, \hat{s}) = \hat{s}^{p(d)} \sigma_d(s, \hat{s})$, where $\hat{s}^{p(d)}$ is the gcd of all the entries of $C_d(sF - \hat{s}G)$ with respect to $\hat{s}$.

(iii) $p_d = \sum_{i=1}^{d} t_i$ where $p_i - p_{i-1} = t_i$ and $\{t_d \,|\, d = 1, 2, \ldots, \rho, \, t_d \neq 0\} = \{q_i \,|\, i \in \tilde{\mu}\}$, $D_\infty = \{\hat{s}^{q_i} \,|\, i \in \tilde{\mu}\}$.

(iv) $D_d(s, 1)$ is the product of all fcd's obtained until the dth step and $D_\rho(s, 1) = \phi(s)$. Thus,

$$D_\rho(s, 1) = \prod_{i=1}^{\rho}(s - \lambda_i)^{\tau i}.$$

Remark 4.1. By considering the conditions of Proposition 4.3 and Proposition 4.4 it is clear that we can find the SE-invariants by the compound matrices for $d = 1, 2, \ldots$ of $sF - \hat{s}G$.

We can demonstrate the above properties in terms of an example.

Example 4.1. Consider the pencil

$$sF - \hat{s}G = \begin{bmatrix} s & -\hat{s} & 0 & 0 & 0 \\ 0 & 0 & \hat{s} & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{s} \\ 0 & 0 & 0 & \hat{s} & s \end{bmatrix}$$

Then

$$C_2(sF - \hat{s}G) = \begin{bmatrix} 0 & s\hat{s} & 0 & 0 & -s\hat{s} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s\hat{s} & 0 & 0 & \hat{s}^2 & 0 & 0 & 0 \\ 0 & 0 & s\hat{s} & s^2 & 0 & -\hat{s}^2 & -s\hat{s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{s}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{s}^2 & s\hat{s} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{s}^2 \end{bmatrix}$$

$$C_3(sF - \hat{s}G) = \begin{bmatrix} 0 & 0 & 0 & 0 & s\hat{s}^2 & 0 & 0 & \hat{s}^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & s\hat{s}^2 & s^2\hat{s} & 0 & -\hat{s}^3 & -s\hat{s}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{s}^3 \\ 0 & 0 & 0 & 0 & 0 & s\hat{s}^2 & 0 & 0 & -\hat{s}^3 & 0 \end{bmatrix}$$

$$C_4(sF - G) = \begin{bmatrix} 0 & 0 & 0 & -s & 1 \end{bmatrix}$$

$p_1 = 0$, $p_2 = 0$, $p_3 = 1$, $p_4 = 3$ and thus $p_2 - p_1 = 0$, $p_3 - p_2 = 1 = q_1$, $p_4 - p_3 = 2 = q_2$, $\deg[C_4(sF - G)] = 1$, sum of the degrees of cmi is $n - p = 5 - 4 = 1 = \pi_c$. Thus, $\varepsilon_1 = 1$.

5. EXTERIOR ALGEBRA BASED CHARACTERIZATION OF INVARIANT SUBSPACES

In this part the invariant subspaces of (F, G) domain, $F, G \in \mathbb{R}^{m \times n}$ described in Section 2, are characterized by using exterior algebra. We adopt two different approaches; the first is based on the properties of the compound of the pencil $sF - G$ and the second on the properties of the compound of the matrices F and G. In both of the approaches considered here we examine the case where the invariant subspaces $\mathcal{V}$ have dimension $d \leq m$. The reason for this is that under this assumption the GR of $\mathcal{V}$ may be used for the characterisation of the nature of $\mathcal{V}$; in the case where $d > m$, the GR characterisation of $\mathcal{V}$ is not a suitable representation and alternative tools are needed. All spaces of the domain of (F, G) for which $\dim \mathcal{V} \leq m$, will be called *proper spaces* of (F, G). The different elementary invariant families of subspaces may be characterised as follows:

Proposition 5.1. If $\mathcal{V}$ is a $p - (F, G)$-is with $\dim \mathcal{V} = d$, $d \leq m$, which contains a proper (F, G)-cmis, $\mathcal{V}'$, there exists a constant nonzero vector $\alpha \in \mathbb{R}^k$, $k = \binom{n}{d}$ such that

(i) α is decomposable

(ii) $C_d(sF - G)\alpha = 0$.

Remark 5:1. If condition (i) and (ii) of Proposition 5.1 hold true, then $\mathcal{V}$ always contains a proper (F, G)-cmis $\mathcal{V}'$, but $\mathcal{V}$ is not necessarily a $p - (F, G)$-is.

Theorem 5.1. There exists an (F, G)-rmis $\mathcal{V}$ with dimension d, $d \leq m$ iff there exists a constant nonzero vector $\alpha \in \mathbb{R}^k$, $k = \binom{n}{d}$ such that

(i) α is decomposable

(ii) $C_d(sF - G)\alpha = \beta(s)$ where $\beta(s) = Be_d(s)$, B is a full column rank, constant matrix and $e_d(s) = [s^d, s^{d-1}, \ldots, 1]^t$.

Theorem 5.2. There exists an (F, G)-ieds $\mathcal{V}$ with dimension d, $d \leq m$, iff there exists a constant nonzero vector $\alpha \in \mathbb{R}^k$, $k = \binom{n}{d}$, such that

(i) α is decomposable

(ii) $C_d(sF - G)\alpha = \beta \neq 0$, β constant decomposable vector over $\mathbb{R}$.

Theorem 5.3. There exists an (F, G)-feds $\mathcal{V}$ with dimension d, $d \leq m$ iff there exist constant, nonzero vectors α and β such that

(i) α is decomposable

(ii) $C_d(sF - G)\alpha = \beta\phi(s)$, $\beta \neq 0$ decomposable, $\deg \phi(s) = d$.

Remark 5.2. A result similar to that described by Theorem 5.3 may be given for subspaces expressed as direct sums of (F, G)-feds and (F, G)-ieds. The only difference is that in part (ii), of Theorem 5.3 we substitute the condition $\deg \phi(s) = d$ with $\deg \phi(s)) \leq d$.

The above results provide an exterior algebra based characterisation for the different families of elementary invariant spaces based on the properties of the GR $\alpha = g(\mathcal{V})$ of the space $\mathcal{V}$ and the properties of the $C_d(sF - G)$ matrix. In the following we examine some alternative characterisation where $\alpha = g(\mathcal{V})$ is used in relationship to $(C_d(F), C_d(G))$ pair of exterior maps. Those two approaches are not equivalent since in the first case we deal with the polynomial matrix $C_d(sF - G)$, the *pencil compound*, whereas in the second case, with the *compound pencil* is $sC_d(F) - C_d(G)$. The alternative characterisations based on the compound pencil are considered next.

Theorem 5.4. Let $\mathcal{V} \in \mathbb{R}^n$ be a subspace of the domain of (F, G), $d = \dim \mathcal{V} \leq m$ and let $\Lambda^d \mathcal{V} = g(\mathcal{V})$ be a GR of $\mathcal{V}$. $\mathcal{V}$ is an $\infty - (F, G)$-eds, iff there exists an integer ν, $1 \leq \nu < d$, such that the following conditions hold true:

 (i) $C_\nu(F) \Lambda^\nu \mathcal{V} \neq 0$, and $C_i(F) \Lambda^i \mathcal{V} = 0$ for $i = \nu + 1, \ldots, d$.
 (ii) $C_d(G) g(\mathcal{V}) \neq 0$ and thus also $C_j(G) \Lambda^j \mathcal{V} \neq 0$ for all $j = 1, 2, \ldots, d - 1$.
 (iii) $\Lambda^\nu \mathcal{V}$ is strongly $(C_\nu(F), C_\nu(G))$-invariant i.e.,

$$C_\nu(F) \Lambda^\nu \mathcal{V} \subseteq C_\nu(G) \Lambda^\nu \mathcal{V}. \tag{5.1}$$

 and every decomposable vector of $\Lambda^\nu \mathcal{V}$ is mapped to a decomposable vector of the images of $C_\nu(F)$ and $C_\nu(G)$.
 (iv) There exists no integer μ, $1 \leq \mu < d$ and no decomposable vector $a \in \Lambda^\mu \mathcal{V}$ which is an eigenvector of the $(C_\mu(F), C_\mu(G))$ pair for a nonzero, finite eigenvalue.

The dual statement of the above results characterises the $0 - (F, G)$-eds. Thus we have:

Theorem 5.5. Let $\mathcal{V} \in \mathbb{R}^n$ be a subspace for the domain of (F, G), $d = \dim \mathcal{V} \leq m$ and let $\Lambda^d \mathcal{V} = g(\mathcal{V})$ be a GR of $\mathcal{V}$. $\mathcal{V}$ is a $0 - (F, G)$-eds, iff there exists an integer ν, $1 \leq \nu < d$, such that the following conditions hold true:

 (i) $C_\nu(G) \Lambda^\nu \mathcal{V} \neq 0$ and $C_i(G) \Lambda^i \mathcal{V} = 0$ for $i = \nu + 1, \ldots, d$.
 (ii) $C_d(F) g(\mathcal{V}) \neq 0$ and thus also $C_j(F) \Lambda^j \mathcal{V} \neq 0$ for all $j = 1, 2, \ldots, d - 1$.
 (iii) $\Lambda^\nu \mathcal{V}$ is strongly $(C_\nu(G), C_\nu(F))$-invariant, i.e.,

$$C_\nu(G) \Lambda^\nu \mathcal{V} \subseteq C_\nu(F) \Lambda^\nu \mathcal{V} \tag{5.2}$$

 and every decomposable vector for $\Lambda^\nu \mathcal{V}$ is mapped to a decomposable vector of the images of $C_\nu(F)$ and $C_\nu(G)$.
 (iv) There exists no integer μ, $1 \leq \mu < d$ and no decomposable vector $a \in \Lambda^\nu \mathcal{V}$ which is an eigenvector of the $(C_\mu(G), C_\mu(F))$ pair for a nonzero finite eigenvalue.

Remark 5.3. Note that conditions (5.1) and (5.2) imply that if V is a basis matrix of $\mathcal{V}$ then

$$(5.1) \iff C_\nu(F) C_\nu(V) = C_\nu(G) C_\nu(V) \underline{A}_\nu \tag{5.3}$$

$$(5.2) \iff C_\nu(G) C_\nu(V) = C_\nu(F) C_\nu(V) \overline{A}_\nu \tag{5.4}$$

The property that every decomposable vector of the image of $C_\nu(F)$, $C_\nu(V)$ is mapped to a decomposable vector of the image $C_\nu(G)$, $C_\nu(V)$ (case of (5.1)) is necessary and sufficient for the *decomposability* of the $\underline{A}_\nu$ map [22]; this notion of decomposability of $\underline{A}_\nu$ ($\overline{A}_\nu$) means that there exists $\underline{A} \in \mathbb{R}^{d \times d}$ ($\overline{A} \in \mathbb{R}^{d \times d}$) such that

$$\underline{A}_\nu = C_\nu(\underline{A}) \tag{5.5}$$

$$\overline{A}_\nu = C_\nu(\overline{A}). \tag{5.6}$$

The decomposability of $\underline{A}_\nu$, $\overline{A}_\nu$ maps is not always true and this is the reason for the strong invariance notion used in Theorem 5.4 and 5.5.

For the case of nonzero-(F, G)-feds (fixed, finite non zero spectrum spaces) we have the following result.

Theorem 5.6. Let $V \in \mathbb{R}^n$ be a subspace of the domain of (F, G), $d = \dim V \leq m$ and let $\Lambda^d \mathcal{V} = g(\mathcal{V})$ be a GR of $\mathcal{V}$. $\mathcal{V}$ is a nonzero-(F, G)-feds iff the following two conditions hold true:

  (i) $C_d(F) g(\mathcal{V}) \neq \underline{0}$ and $C_d(G) g(\mathcal{V}) \neq \underline{0}$.
  (ii) $g(\mathcal{V})$ is an eigenvector for the pair $(C_d(F), C_d(G))$, that is there exists λ, $\lambda \neq 0$ such that

$$C_d(F) g(\mathcal{V}) = \lambda C_d(G) g(\mathcal{V}). \tag{5.7}$$

Corollary 5.1. Let $\mathcal{V} \in \mathbb{R}^n$ be a subspace of the domain of (F, G), $d = \dim \mathcal{V} \leq m$ and let $\Lambda^d \mathcal{V} = g(\mathcal{V})$ be a GR of $\mathcal{V}$. The following properties hold true:

  (a) $\mathcal{V}$ is an (F,G)-feds, iff $C_d(F) g(\mathcal{V}) \neq \underline{0}$ and

$$C_d(G) g(\mathcal{V}) = \lambda C_d(F) g(\mathcal{V}), \quad \lambda \in \mathbb{R} \tag{5.8}$$

  (b) $\mathcal{V}$ is an (F, G)-nzeds, iff $C_d(G) g(\mathcal{V}) \neq \underline{0}$ and

$$C_d(F) g(\mathcal{V}) = \lambda C_d(G) g(\mathcal{V}), \quad \lambda \in \mathbb{R}. \tag{5.9}$$

Remark 5.4. By the properties of the Bilinear strict equivalence of matrix pencils [13], that is invariance of minimal indices and covariance of elementary divisors, it follows that if for a pair (F, G), $\mathcal{V}$ is an elementary divisor subspace, $((F, G)$-eds), that is $\mathcal{V} = \mathcal{V}_\infty \oplus \mathcal{V}_\alpha \oplus \mathcal{V}_0$, there exists a bilinear transformation $\mathbf{b} = \{a, b, c, d\}$ $(ad - bc \neq 0)$ such that

$$\mathbf{b} \circ (F, G) = (F', G') = (aF - cG, bF - dG) \tag{5.10}$$

with $\mathcal{V}$ being a nonzero (F', G')-feds. In fact, this property holds for any generic bilinear transformation.

From the above remark and Theorem 5.6 we have:

Corollary 5.2. $\mathcal{V} \in \mathbb{R}^n$, $d = \dim \mathcal{V} \leq m$ is an (F,G)-eds of the domain of (F,G), iff for some generic bilinear transformation $\mathbf{b} = \{a,b,c,d\}$, $g(\mathcal{V}) = \Lambda^d \mathcal{V}$ is a eigenvector associated with a nonzero eigenvalue of the $(C_d(F'), C_d(G'))$ pair, where (F',G') is defined as in (5.10).

The results so far indicate the following parametrization.

Remark 5.5. The family of elementary divisor type of subspaces $\mathcal{V}$ of the domain of (F,G), $F, G \in \mathbb{R}^{m \times n}$, which have dimension d is defined by the family of decomposable vectors $a \in \mathbb{R}^\mu$, $\mu = \binom{n}{d}$ which satisfy the following properties:

(i) At least one of the conditions $C_d(F)\,a \neq 0$, $C_d(G)\,a \neq 0$ is satisfied

(ii) a is an eigenvector of $(C_d(F), C_d(G))$ or of $(C_d(F'))$, $(C_d(G'))$ pairs, where $(F',G') = \mathbf{b} \circ (F,G)$ as defined by (5.10) and $\mathbf{b}$ generic.

For the family of (F,G)-cmis we have:

Theorem 5.7. Let $\mathcal{V} \in \mathbb{R}^n$ be a subspace of the domain of (F,G), $d = \dim \mathcal{V} \leq m$ and let $g(\mathcal{V}) = \Lambda^d \mathcal{V}$ be a GR of $\mathcal{V}$. $\mathcal{V}$ is an (F,G)-cmis, iff the following conditions hold true:

(i) $C_d(F)\,g(\mathcal{V}) = \underline{0}$ and $C_d(G)\,g(\mathcal{V}) = \underline{0}$.

(ii) $\Lambda^i \mathcal{V}$ is a strongly complete $(C_i(F), C_i(G))$-is for all $i = 1, 2, \ldots, d-1$; furthermore, the restriction maps $\underline{A}_{(i)}$, $\overline{A}_{(i)}$ defined on any basis V of $\mathcal{V}$ by

$$C_i(F)\,C_i(V) = C_i(G)\,C_i(V)\,\underline{A}_{(i)}$$
$$C_i(G)\,C_i(V) = C_i(F)\,C_i(V)\,\overline{A}_{(i)}$$

are nillpotent for all $i = 1, 2, \ldots, d-1$ and for $i = d-1$ they are of rank one.

Remark 5.6. From the above result it follows that if $d \leq m$, then any (F,G)-cmis has a GR $g(\mathcal{V})$ which is a constant vector in $\mathcal{N}_r\{sC_d(F) - C_d(G)\}$. Every vector satisfying $C_d(F)\,g(\mathcal{V}) = 0$, $C_d(G)\,g(\mathcal{V}) = 0$, however, does not necessarily – characterise an (F,G)-cmis.

We may close this section by giving a final result relating the pencil invariants to the properties of the dimension of the various invariant families discussed in Section 2.

Corollary 5.3. Consider the singular system S_e and the autonomous dynamic system $S(F,G)$ related to S_e as shown by (2.5) and let $\mathcal{V} \in R^n$, $\dim \mathcal{V} = \rho$. If we define

$$C_\rho(sF - \hat{s}G)\,V = \begin{bmatrix} \vdots \\ \vdots \\ p_i^t(s,\hat{s}) \\ \vdots \\ \vdots \end{bmatrix} \qquad (5.11)$$

then
$$p_i^t(s, \hat{s}) = \alpha_i(s, \hat{s}) \, D_\rho(s, \hat{s}) \, p^*(s, \hat{s})^t \tag{5.12}$$

where $\alpha_i(s, \hat{s}) \in \mathbb{R}[s, \hat{s}] \setminus [0]$, $D_\rho(s, \hat{s})$ is the gcd of all the entries of $C_\rho(sF - \hat{s}G)$ and

(i) If $\mathcal{V} = \mathcal{T}^*$ is the supremal (A, E, B)-is, then $\dim(\mathcal{T}^*) = \deg\{D_\rho(s, 1)\} + \deg\{p^*(s, 1)\} + n - \rho$.

(ii) If $\mathcal{V} = \mathcal{S}^*$, is the supremal almost (E, A, B)-is, then $\deg\{D_\rho(s, \hat{s})\} - \deg\{D_\rho(s, 1)\} > 0$ and $\dim\{\mathcal{S}^*\} = \deg\{D_\rho(s, \hat{s})\} + \deg\{p^*(s, 1)\} + n - \rho$.

(iii) If $\mathcal{V} = \mathcal{K}^*$, is the supremal almost reachability subspace, then $\dim\{\mathcal{K}^*\} = \deg\{D_\rho(s, \hat{s})\} - \deg\{D_\rho(s, 1)\} + \deg\{p^*(s, 1)\} + n - \rho$.

(iv) If $\mathcal{V} = \mathcal{R}^*$, is the supremal reachability subspace, then $\dim(\mathcal{R}^*) = \deg\{p^*(s, 1)\} + n - \rho$.

The proof of this result readily follows by the Propositions 4.2 and Theorem 4.4. The approach suggested here may also be used to characterize the dimensions of the above supremal invariant subspaces which are contained in a given space of $\mathbb{R}^n$.

6. CONCLUSIONS

A new characterisation of the different basic families of invariant subspaces of the domain of the paid (F, G), or subspaces associated with a matrix pencil $sF - G$ has been given using tools from Exterior Algebra. This new characterisation is not completely equivalent, in its present form, to the algebraic characterisation based on matrix pencils, since it makes no distinction between individual values of cmi, or partitioning of the greatest common divisors into Segrè characteristics associated with a zero; this however, is not necessarily a disadvantage, because very frequently we are just interested in the nature of the invariant subspace, rather than the exact partitioning of the different invariants associated with the space. The two approaches considered here, based on the compound of the pencil $C_d(sF - G)$ and the pencil compound $sC_d(F) - C_d(G)$ provide new characterisations for families of invariant subspaces based on the properties of the Grassmann representative $g(\mathcal{V})$ with respect to these two operations. A limitation of the present approach is that it deals only with the proper subspaces of (F, G), i.e. those for which $d \leq m$. The extension of the results to the case $d > m$ is well as the use of the new characterisation into parameterisation problems is currently under investigation.

APPENDIX: PROOF OF RESULTS

P r o o f of Proposition 4.1. Let $U(s)$ be a unimodular matrix $\mathbb{R}[s]$ such that $A(s) \, U(s) = [\overline{A}(s) : 0]$, where $\overline{A}(s)$ is of full column rank ρ and calculate $C_\rho(A(s) \, U(s)) = C_\rho[\overline{A}(s) : 0]$. By Binet–Cauchy Theorem, [18], $C_\rho[A(s) \, U(s)] = C_\rho(A(s)) \, C_\rho(U(s))$. Since $U(s)$ is nonsingular, $C_\rho(U(s))$ is nonsingular and

$$C_\rho(A(s)) = [p(s) : 0] \, C_\rho(U(s))^{-1}$$

where $p(s) = C_\rho(\overline{A}(s))$. If we denote the first row of $C_\rho(U(s))^{-1}$ by $(\lambda_{11}(s), \ldots, \lambda_{1\tau}(s))$, where $\tau = \binom{n}{p}$, we have

$$C_\rho(A(s)) = (\lambda_{11}(s)\, p(s), \lambda_{1\tau}(s)\, p(s))). \qquad (a.1)$$

Thus, since $p(s)$ is GR of $\mathcal{X}_a$, every nonzero $p_i(s)$ are GR of $\mathcal{X}_a$. Now, let two different nonzero columns $p_i(s) = \lambda_{1i}(s)\, p(s)$, $p_j(s) = \lambda_{1j}(s)\, p(s)$ of $C_\rho(A(s))$. By setting $\lambda_{ij}(s) = \lambda_{1i}(s)/\lambda_{1j}(s)$, the result follows. □

P r o o f of Proposition 4.2. According to the equation (a.1) $p_i(s) = \lambda_{1i}(s)\, p(s)$. If $\phi(s)$ is the gcd of all vectors $p_i(s) \neq \{0\}$, there exists a nonzero polynomial $\lambda_i(s)$ and a coprime vector $p^*(s)$ such that $p_i(s) = \lambda_i(s)\, \phi(s)\, p^*(s)$. □

P r o o f of Proposition 4.3. Consider the equation (2.6) and factorise it as

$$\begin{aligned} P(sF - G)Q &= \mathrm{diag}\{L_\eta(s),\, .L_\varepsilon(s),\, sH - K,\, 0\} = \\ &= \mathrm{diag}\{I,\, I,\, sH - K,\, I\}\,\mathrm{diag}\{L_\eta(s),\, L_\varepsilon(s),\, I,\, 0\} \end{aligned}$$

where $L_\eta(s)$ and $L_\varepsilon(s)$ have rmi and cmi blocks, respectively and $sH - K$ consists of fed and ied blocks. By the definition of $L_\varepsilon(s)$, there exists an unimodular matrix $U(s)$ such that

$$\mathrm{diag}\{L_\eta(s),\, L_\varepsilon(s),\, I,\, 0\}\, U(s) = \mathrm{diag}\{L_\eta(s),\, I,\, 0\}.$$

We must note that $\mathrm{diag}\{L_\eta(s),\, I\}$ is of full column rank. Then the ρth compound of $P(sF - G)\,QU(s)$ is

$$C_\rho[P(sF - G)\,QU(s) = \det(sH - K)\,[1(s),\, 0]$$

where $0 \neq 1(s)\, C_\rho[\mathrm{diag}\{L_\eta(s)\,(s),\, I\}]$. By Binet–Cauchy theorem and the properties a, b and c of the compound matrices we have:

$$C_\rho[(sF - G)] = \det(sH - K)\, C_\rho^{-1}(P)\,[1(s);\, 0]\, C_\rho[QU(s)]^{-1}.$$

If we define $\det(sH - K) =: \phi(s)$, the first row of $C_\rho[QU(s)]^{-1}$ by $(\beta_1(s), \ldots, \beta_t(s))$ and $C_\rho[P]^{-1}\, 1(s) = u^*(s)$, we will obtain $u_i(s) = \beta_i(s)\, \phi(s)\, \underline{u}^*(s)$. It is clear that $u^*(s)$ is coprime, or constant by the definition of $1(s)$. Since $QU(s)$ is unimodular, $C_\rho[QU(s)]^{-1}$ is also unimodular and the gcd of $(\beta_1(s), \ldots, \beta_t(s))$ is 1. This implies that the gcd of all the entries of $C_\rho[sF - G]$ is $\phi(s)$. Consequently (i) and (ii), or (iii) hold by the definitions of $\phi(s)$ and $u^*(s)\,(v^*(s)^t)$ respectively. (iv) follows from (ii) and (iii). □

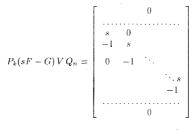P r o o f of Proposition 4.4. The proof follows from the definitions of fed and ied [3]. □

P r o o f of Proposition 5.1. If $\mathcal{V}$ is a partitioned $-(F, G)$-is, which contains a proper (F, G)-cmis, $\mathcal{V}'$, then $sFV - GV$ has $m \times d$ dimensions, $m \leq d$ and there

exists $\nu(s) \in \mathbb{R}^d[s]$ such that $(sFV - GV)\,\nu(s) = 0$; thus, $\rho_{\mathbb{R}(s)}\{sFV - GV\} < d$ and $C_d[sFV - GV] = 0$. By the Binet-Cauchy Theorem we have

$$C_d[(sF - G)\,V] = C_d[sF - G]\,C_d(V) = 0$$

Thus, if $\alpha = C_d(V)$, part (i) and (ii) follow. Reversely the decomposability of α and the above condition implies that $C_d[(sFV - GV)] = 0$ and thus $\rho_{\mathbb{R}(s)}(sFV - GV) < d$; the last condition also implies that $\mathcal{N}_\ell\{sFV - GV\} \neq \{0\}$ and thus we also have rmi, which are not necessarily zero. The implies that $\mathcal{V}$ contain a $\mathcal{V}'(F, G)$-cmis. $\square$

Proof of Theorem 5.1. (Necessity) Let V be a basis of an (F, G)-cmis $\mathcal{V}$. By equations (2.6) and (2.8), we obtain

$$P_k(sF - G)\,V\,Q_n = \begin{bmatrix} & & 0 & \\ \cdots\cdots\cdots\cdots\cdots \\ s & 0 & \\ -1 & s & \\ 0 & -1 & \ddots \\ & & & \ddots\, s \\ & & & -1 \\ \cdots\cdots\cdots\cdots\cdots \\ & & 0 & \end{bmatrix}$$

Thus,

$$C_d(P_k(sF - G)\,V\,Q_k) = [0, \ldots, s^d, -s^{d-1}, \ldots, (-1)^{d-1}s, (-1)^d, \ldots]^t = A\,e_d(s)$$

where $e_d(s) = [s^d s^{d-1}, \ldots, 1]^t$. By Binet–Cauchy Theorem. $C_d(P_k(sF - G)\,V_\eta) = C_d(P_k)\,(C_d(sF - G)\,C_d(VQ_k)$. Since P_k is nonsingular $C_d(P_k)^{-1}$ exists and $C_d(sF - G)\,C_d(V_k) = C_d(P_k)^{-1}Ae_d(s)$. Since VQ_k and A are of full column rank, we can define $C_d(VQ_k) = \alpha$, $C_d(P_k)^{-1}A = B$ and also $Be_d(s) = \beta(s)$. Thus, part (i) and (ii) follow.

(Sufficiency) Since α is decomposable, there exists a constant matrix V such that $\alpha = C_d(V)$. By (ii) and the Binet–Cauchy Theorem $C_d(sF - G)\,C_d(VQ_k) = C_d((sF - G)\,V) = \beta(s)$ and thus consequently $\beta(s)$ is decomposable. Thus, there exists a polynomial vector $\gamma(s)$ of minimal degree d such that $\gamma^t(s)\,(sF - G)\,V = 0$ or $(V^t(sF^t - G^t)\,\gamma(s) = 0$. Thus, there exists a (F^t, G^t)-cmis $\mathcal{V}^t$ with basis V^t which is a dual of (F, G)-rmis $\mathcal{V}$ and this completes the proof. $\square$

Proof of Theorem 5.2. (Necessity) Let $\mathcal{V}$ be a basis of the (F, G)-ieds. By the

equations (2.6) and (2.8), we have

$$
P_k(sF - G)\,V\,Q_k = \begin{bmatrix}
& & & 0 & & & \\
\hdotsfor{7} \\
-1 & s & 0 & \ldots\ldots & & & 0 \\
0 & -1 & s & 0 & \ldots & & 0 \\
\vdots & \vdots & \vdots & & & & \vdots \\
0 & 0 & 0 & \ldots\ldots & & & s \\
0 & 0 & \ldots\ldots\ldots & & & & -1 \\
\hdotsfor{7} \\
& & & 0 & & &
\end{bmatrix}.
$$

Thus, $C_d[P_k(sF-G)\,VQ_k] = [0,\ldots,0,(-1)^d,\ldots,0]^t$. By the Binet–Cauchy theorem $C_d[P_k(sF-G)\,VQ_k] = C_d(P_k)\,C_d(sF-G)\,C_d(VQ_k)$. Since P_k is nonsingular, $C_d(P_k)$ is nonsingular, and thus, $C_d(sF-G)\,C_d(VQ_k) = C_d(P_k)^{-1}[0,\ldots,0,(-1)^d,\ldots,0]^t$. Now, let us define $C_d(VQ_k) = \alpha$, $C_d(P_k)^{-1}[0,\ldots,0(-1)^d,\ldots,0]^t =: \beta$. It is obvious that α is decomposable and β is constant; thus, necessity is proved.

(Sufficiency) Assume (i) and (ii) hold true. By the decomposability of α, there exists a full column rank, $n \times d$, constant matrix V such that $\alpha = C_d(V)$. Then, by the Binet–Cauchy Theorem, $C_d(sF-G)\,C_d(V) = C_d[(sF-G)\,V]$ and by (ii)

$$
C_d[(sF-G)\,V] = \beta \neq 0. \tag{a.2}
$$

This equation implies

(a) $\rho_{\mathbb{R}(s)}[(sF-G)\,V] = d$ and

(b) all the nonzero minors of $(sF-G)\,V$ are unimodular.

Thus, there exists a nonsingular matrix U, such that

$$
U(sF-G)\,V = \begin{bmatrix} M(s) \\ 0 \end{bmatrix}. \tag{a.3}
$$

By the Binet–Cauchy Theorem $C_d[U(sF-G)\,V] = C_d(U)\,C_d[(sF-G)\,V]$. Since $|U| \neq 0$, $|C_d(U)| \neq 0$ (a.3) yields

$$
C_d[(sF-G)\,V] = c_d(U)^{-1}\left(\begin{bmatrix} M(s) \\ 0 \end{bmatrix}\right) = \beta. \tag{a.4}
$$

Let $\alpha(s)$ be the first column of $C_d(U)^{-1}$. Then, the equations (a.2) and (a.4), lead to $C_d[(sF-G)\,V] = \alpha(s)\,|M(s)| = \beta \neq 0$. Since β is constant $\alpha(s)$ and $|M(s)|$ are constant and consequently $M(s)$ is unimodular. Since the left hand side of the equation (a.3) is of the first degree polynomial matrix and $M(s)$ is unimodular, we can write $M(s)$ as $M(s) = M + sM_1$, where M is nonsingular and M_1 is singular. Thus, we may write

$$
(sF-G)\,V = U^{-1}\begin{bmatrix} M + sM_1 \\ 0 \end{bmatrix} \tag{a.5}
$$

and $\qquad GV = -U^{-1} \begin{bmatrix} M \\ 0 \end{bmatrix}, \quad FV = -GVM^{-1}M_1$ $\qquad$ (a.6)

thus, GV is of full column rank and there exists a singular matrix $\underline{A}$ such that $FV = GV\underline{A}$. By the equation in (2.9), sp $V \overset{\triangle}{=} \mathcal{V}$ and it is an (F, G)-ieds.

P r o o f of Theorem 5.3. (Necessity) Let V be a basis of the (F, G)-feds. By the equations (2.6) and (2.8), we obtain

$$P_k(sF - G)VQ_k = \begin{bmatrix} sI - A \\ \dots \\ 0 \end{bmatrix}, \quad |sI - A| = \phi(s).$$

Thus, $C_d[P_k(sF - G)VQ_k] = [\phi(s), 0, \dots, 0]^t$. By the Binet–Cauchy Theorem $C_d[P_k(sF - G)VQ_k] = C_d(P_k)C_d(sF - G)C_d(VQ_k)$. Since P_k is nonsingular, $C_d(P_k)$ is also nonsingular and $C_d(sF - G)C_d(VQ_k) = C_d(P_k)^{-1}[(\phi(s), \dots, 0]^t$. Define $C_d(VQ_k) = \alpha$, $C_d(P_k)^{-1}[\phi(s), \dots, 0] = \beta\phi(s)$ and thus, $C_d(sF - G)\alpha = \beta\phi(s)$ i. e. the necessity is proved.

For the sufficiency, we assume that (i) and (ii) hold. By the decomposability of α, there exists a full column rank matrix V such that $\alpha = C_d(V)$. Then, by the Binet–Cauchy Theorem and the decomposability of β we have

$$C_d\{(sF - G)\, C_d(V)\, \phi(s) = C_d(U_1)\, \phi(s)$$

where $\qquad \beta = C_d(V_1), \quad \text{and} \quad \phi(s) = |M(s)|.$

The above implies

$$C_d\{(sF - G)\, V\} = C_d\{U_1 M(s)\}$$

or equivalently

$$(sF - G)\, V = U_1\, M(s)\, Q = U_1\, M'(s)$$

where $Q \in \mathbb{R}^{d \times d}$, $|Q| \neq 0$. Since U_1 is constant, $M'(s)$ should be written as $M'(s) = sM_0 + M_1$; since $|M'(s)| = \phi(s)$ and deg $\phi(s) = d$, $|M_0| \neq 0$ and thus $M'(s) = U_1 R(sI - \underline{A}) = U_1'(sI - A)$ with $|sI - A| = \phi(s)$. Thus, $(sFV - GV)$ is characterised only by zero rmi and fed, defined as those of $sI - A$. $\qquad \square$

P r o o f of Theorem 5.4. The conditions of the theorem are based on part (i) of Theorem 2.1 which provides a characterisation of the $\infty - (F, G)$-eds $\mathcal{V}$. In fact, $F\mathcal{V} \subset G\mathcal{V}$ and $\mathcal{N}_r(G)\mathcal{V} = \{0\}$ imply that for any basis V of $\mathcal{V}$, then there exists $\underline{A} \in \mathbb{R}^{d \times}$ such that

$$FV = GV\underline{A}, \quad \rho(FV) < d, \quad \rho(GV) = d \qquad \text{(a.7)}$$

Clearly, $\rho(FV) = \rho(\underline{A}) = \nu < d$. By the Binet–Cauchy Theorem for $i = 1, 2, \dots, d$ we have

$$C_i(FV) = C_i(F)\, C_i(G) = C_i(GV\underline{A}) = C_i(G)\, C_i(V)\, C_i(\underline{A}) \qquad \text{(a.8)}$$

and thus since $\rho(FV) = \rho(\underline{A}) = \nu$ we have

$$C_\nu(F)\, C_\nu(V) = C_\nu(G)\, C_\nu(V)\, C_i(\underline{A}) \neq 0 \qquad \text{(a.9)}$$

which proves part (i) and (iii). Note that for $i \geq \nu + 1$, $C_i(\underline{A}) = 0$ and this implies that $C_i(F) \Lambda^i \mathcal{V} = \{0\}$. Part (ii) is a mere restatement of the fact that $\rho(G\mathcal{V}) = d$. Note that $F\mathcal{V} \subset G\mathcal{V}$ and $\mathcal{N}_r(G) \cap \mathcal{V} = \{0\}$ implies that $\mathcal{V}$ may contain an (F, G)-feds subspace V' with non zero frequencies in its spectrum; for this space

$$FV' = GV' \underline{A}', \underline{A}' \in \mathbb{R}^{\mu \times \mu}, \quad |\underline{A}'| \neq 0, \ \mu = \dim \mathcal{V}' \tag{a.10}$$

and thus for $i = \mu$, by the Binet Cauchy Theorem we have

$$C_\mu(F) \, g(\mathcal{V}') = |\underline{A}'| \, C_\mu(G) \, g(\mathcal{V}') \tag{a.11}$$

which proves part (iv) and completes the proof of the necessity of the above conditions. The sufficiency is established by a mere reversion of the arguments. In fact condition (iii) implies that (a.8) holds true for some $\underline{A}$ map and thus $FV = GV\underline{A}$, or that $F\mathcal{V} \subseteq G\mathcal{V}$. Part (ii) implies $\rho(G\mathcal{V}) = d$ and thus from part (i) and the previous expressions $\rho(F\mathcal{V}) = \rho(A) = \nu$ and $\mathcal{N}_r(F) \cap \mathcal{V} \neq \{0\}$ which shows that $F\mathcal{V} \subset G\mathcal{V}$. Part (iv) implies that there is no proper subspace $\mathcal{V}'$ which is c-(F, G)- is and this completes the sufficiency. $\qquad \square$

Proof of Theorem 5.5. This proof is dual to that of Theorem 5.4 and in fact follows by a reversion of the roles of F and G. $\qquad \square$

Proof of Theorem 5.6. By part (iii) of Theorem 2.1 we have that if V is a basis of $\mathcal{V}$ then

$$FV = GV\underline{A} \tag{a.12}$$

and $\rho(FV) = \rho(GV) = d$. The latter clearly implies and $C_d(F) \, C_d(V) \neq \underline{0}$ and $C_d(G) \, C_d(V) \neq 0$ and this proves part (i). By using the Binet–Cauchy Theorem (a.12) implies

$$C_d(F) \, C_d(V) = C_d(G) \, C_d(V) \, C_d(\underline{A}) = |\underline{A}| \, C_d(G) \, C_d(V) \tag{a.13}$$

and from (i), $|\underline{A}| \neq 0$ and this shows that $g(\mathcal{V}) = C_d(V)$ is an eigenvector for a nonzero eigenvalue. Note that $F\mathcal{V} = G\mathcal{V}$ also implies that

$$GV = FV\overline{A} \tag{a.14}$$

and thus

$$C_d(G) \, C_d(V) = |\overline{A}| \, C_d(F) \, C_d(V) \tag{a.15}$$

with $|\overline{A}| \neq 0$ and thus $g(\mathcal{V})$ is an eigenvector of the dual eigenvector problem, which proves the necessity. The sufficiency follows from the decomposability of $g(\mathcal{V})$ and from that (a.13) and (a.15) always imply (a.12) and (a.14) respectively since $|\underline{A}|$, $|\overline{A}|$ as scalars are always decomposable. Thus $F\mathcal{V} = G\mathcal{V}$ and since $C_d(F) \, g(\mathcal{V}) \neq 0$, $C_d(G) \, g(\mathcal{V}) \neq \underline{0}$, $\mathcal{V}$ cannot contain an (F, G)-cmis. $\qquad \square$

Proof of Corollary 5.1. From Corollary (2.1) and Theorem 5.6, both parts of the result readily follow. $\qquad \square$

Proof of Theorem 5.7. By part (v) of Theorem 2.2 we have that if $\mathcal{V}$ is an (F, G)-cmis, then there exists a basis V_c such that

$$FV_c = GV_c \, \underline{J}_0, \quad GV_c = FV_c \overline{J}_0 \tag{a.16}$$

where

$$\overline{J}_0 = \begin{bmatrix} 0 & .. & | & 0 \\ \cdots\cdots & | & \cdots \\ & & \vdots & \vdots \\ I_{d-1} & & | & \\ & & | & 0 \end{bmatrix}, \quad \underline{J}_0 = \begin{bmatrix} 0 & | & \cdots\cdots & 0 \\ \cdots & | & \cdots\cdots \\ \vdots & | & \\ \vdots & | & I_{d-1} \\ 0 & | & \end{bmatrix} \tag{a.17}$$

For any basis V of $\mathcal{V}$, $V = V_c Q$, $Q \in \mathbb{R}^{d \times d}$, $|Q| \neq 0$, the above conditions imply that

$$FV = GV \, \underline{A}_0, \quad GV = FV \overline{A}_0 \tag{a.18}$$

where

$$\overline{A}_0 = Q^{-1} \overline{J}_0 \, Q, \quad \underline{A}_0 = Q^t \, \underline{J}_0 \, Q^{t-1} \tag{a.19}$$

By applying the Binet–Cauchy Theorem on (a.18) and taking into account the properties of $\overline{A}_0$, $\underline{A}_0$ we have

$$C_d(F) \, C_d(V) = C_d(G) \, C_d(V) |\underline{A}_0| = 0 \tag{a.20}$$

$$C_d(G) \, C_d(V) = C_d(F) \, C_d(V) \, |\overline{A}_0| = 0 \tag{a.21}$$

and this proves part (i). For all $i < d$ we have:

$$C_i(F) \, C_i(V) = C_i(G) \, C_i(V) \, C_i(\underline{A}_0) \tag{a.22}$$

$$C_i(G) \, C_i(V) = C_i(F) \, C_i(V) \, C_i(\overline{A}_0) \tag{a.23}$$

which proves the complete $-(C_i(F), C_i(G))$-invariance of $\Lambda^i \mathcal{V}$. Since $\underline{A}_0$, $(\overline{A}_0)$ is nillpotent matrix then there exists k such that $\underline{A}_0^k = 0$ and

$$C_i(\underline{A}_0)^k = 0 \tag{a.24}$$

Thus any compound is also nillpotent with an index $\mu \leq k$. It is not difficult to see that for $i = d - 1$, then

$$C_{d-1}(A_0) = C_{d-1}(Q^t) \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 1 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} C_{d-1}(Q^{t-1}) = q \, p^t$$

which is a dyad and this proves the necessity. By reversing the steps and using the decomposability it follows that $FV = GV$ and that $\mathcal{N}_r(F) \cap \mathcal{V} \neq \{0\}$, $\mathcal{N}_r(G) \cap \mathcal{V} \leq \leq \{0\}$. From the nillpotency of the restriction maps for all $i = 1, \ldots, d$ it follows that $\mathcal{V}$ cannot be expressed as $\mathcal{V} = \mathcal{V}_\epsilon \oplus \mathcal{V}_\alpha$, since then, some of these restrictions will have at least a compound which is not nillpotent; this completes the sufficiency. $\square$

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