KYBERNETIKA — VOLUME 29 (1993), NUMBER 6, PAGES 617–627

ON THE DERIVATION OF A STATE-SPACE MODEL OF A PERIODIC PROCESS DESCRIBED BY RECURRENT EQUATIONS[1]

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Here the problem is considered of obtaining a periodic description in state-space form of a linear process which can be modelled by linear recurrent equations with periodic coefficients. A polynomial time-invariant description of such a model is used, in order to characterize the order of the model and to introduce an equivalence relation between two models.

1. INTRODUCTION

For processes which can be modelled by linear recurrent (or differential) equations with constant coefficients, Rosenbrock [1] introduced the polynomial matrix description in form of the following pair of vector equations

$$T(\delta)\,\xi = U(\delta)\,u, \tag{1.1}$$

$$y = V(\delta)\,\xi + W(\delta)\,u, \tag{1.2}$$

where $T(\delta)$, $U(\delta)$, $V(\delta)$ and $W(\delta)$ are polynomial matrices in the indeterminate δ, which for recurrent equations can have the meaning of the one-step forward-shift operator. He showed that under the polynomial transformations on (1.1), (1.2) that he called strict system equivalence, if $T(\delta)$ is square, $\det T(\delta) \neq 0$ (and has a degree equal to the dimension of $T(\delta)$) and the input-output transfer matrix corresponding to (1.1), (1.2) is proper, then it is possible to obtain a description of the same process in state-space form, i.e., in the case of recurrent equations, of the type

$$x(k+1) = A\,x(k) + B\,u(k), \tag{1.3}$$

$$y(k) = C\,x(k) + D\,u(k). \tag{1.4}$$

Since then, several authors studied the polynomial matrix description (1.1), (1.2) and the procedures for the computation of a state-space realization (1.3), (1.4) strict system equivalent to (1.1), (1.2) (see, e.g. [2]–[8]).

[1] This work has been supported by funds of Ministero dell'Università e della Ricerca Scientifica e Tecnologica e Ministero della Pubblica Istruzione.

In this paper the same kind of problem is faced for processes which can be modeled by linear recurrent equations with periodic coefficients (whose period will be denoted by ω) of the following form:

$$\sum_{i=0}^{r} T_i(k)\xi(k+i) = \sum_{i=0}^{r} U_i(k)u(k+i), \tag{1.5}$$

$$y(k) = \sum_{i=0}^{r} V_i(k)\xi(k+i) + \sum_{i=0}^{r} W_i(k)u(k+i), \tag{1.6}$$

for some integer $r > 0$, where the matrices $T_i(k)$, $U_i(k)$, $V_i(k)$ and $W_i(k)$ ($i = 0,\dots,r$) are real periodic matrices with period ω (briefly ω-periodic), and $T_i(k)$ ($i = 0,\dots,r$) are possibly square. That is, the problem considered here is that of obtaining a description of such a process in state-space form. The case $r = 0$ is not considered, in order to rule out the case when (1.5), (1.6) are not recurrent equations.

For the sake of brevity, all the proofs are omitted.

2. PRELIMINARIES AND NOTATIONS

For a given matrix F, its element of position (i,j) will be denoted by $(F)_{i,j}$, its ith row (column) by $[F]^i$ ($[F]_i$). The identity matrix of dimension ν will be denoted either by I_ν, or simply by I if confusion does not arise.

Hereafter δ will denote the one-step forward-shift operator, δ^{-1} its inverse, $\Delta := \delta^\omega$ the ω-steps forward-shift operator, and Δ^{-1} its inverse. Let $R_\nu(\Delta)$, with $\nu \in \mathbf{Z}^+$, denote the operator represented by the following $(\omega\nu) \times (\omega\nu)$ matrix:

$$R_\nu(\Delta) := \begin{bmatrix} 0 & I_{(\omega-1)\nu} \\ \Delta I_\nu & 0 \end{bmatrix}. \tag{2.1}$$

Let a vector function $z(t) \in \mathbf{R}^\nu$ be given, with $t \in \mathbf{Z}$. Then, for any $k \in \mathbf{Z}$, the *ω-stacked form of $z(t)$ at (the initial) time k* is defined as the following $(\omega\,\nu)$-dimensional vector function:

$$z_k(h) := \begin{bmatrix} z(k+h\omega) \\ z(k+h\omega+1) \\ \vdots \\ z(k+h\omega+\omega-1) \end{bmatrix}, \quad \forall\, h \in \mathbf{Z}. \tag{2.2}$$

The vector $z_k(h)$ can be considered a function either of k or of h. In the following, the operator Δ will have the meaning of an ω-steps forward-shift in the k variable, whenever the operator $R_\nu(\Delta)$ will be applied to $z_k(h)$.

Let an ω-periodic matrix $F(t) \in \mathbf{R}^{\nu\times\mu}$ be given, with $t \in \mathbf{Z}$. Let the vector functions $z(t) \in \mathbf{R}^\nu$, and $w(t) \in \mathbf{R}^\mu$ be related by the linear map represented by $F(t)$, i.e. $z(t) = F(t)\,w(t)$, $\forall\, t \in \mathbf{Z}$. Then, for any $k \in \mathbf{Z}$, the *ω-stacked* (or, simply, *stacked) form of $F(t)$ at (the initial) time k*, which is defined as the following matrix of dimension $(\omega\,\nu) \times (\omega\,\mu)$:

$$\mathcal{F}_k := \begin{bmatrix} F(k) & 0 & \cdots & 0 \\ 0 & F(k+1) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & F(k+\omega-1) \end{bmatrix},$$

represents the induced linear map between the ω-stacked forms at time k of the vector functions $z(t)$ and $w(t)$, i.e. $z_k(h) = \mathcal{F}_k w_k(h)$, $\forall h \in \mathbf{Z}$.

Lemma 1. [9, 10] For any vector function $z(t) \in \mathbf{R}^\nu$ and ω-periodic matrix $F(t) \in \mathbf{R}^{\nu\times\mu}$ $(t \in \mathbf{Z})$, and for any $k \in \mathbf{Z}$, the following identities hold for all i, $j \in \mathbf{Z}$:

$$R_\nu^{j\omega+i}(\Delta)\, z_k(h) = z_{k+j\omega+i}(h) = z_{k+i}(h+j), \tag{2.3}$$

$$R_\nu^{j\omega+i}(\Delta)\, \mathcal{F}_k R_\mu^{-(j\omega+i)}(\Delta) = \mathcal{F}_{k+j\omega+i} = \mathcal{F}_{k+i}. \tag{2.4}$$

Corollary 1. [10] If in (2.1) the operator Δ is substituted by a scalar complex variable s, then identity (2.4) still holds with Δ replaced by s.

Notice that, by (2.3), we have

$$R_\nu^{\omega}(\Delta)\, z_k(h) = z_{k+\omega}(h) = z_k(h+1) = \Delta z_k(h), \tag{2.5}$$

in accordance with the identity $R_\nu^\omega(\Delta) = \Delta I$.

3. A TIME-INVARIANT CHARACTERIZATION OF ω-PERIODIC SYSTEMS AND MODELS

Consider a linear ω-periodic system described by:

$$x(k+1) = A(k)\,x(k) + B(k)\,u(k), \tag{3.1}$$

$$y(k) = C(k)\,x(k) + D(k)\,u(k), \tag{3.2}$$

where $k \in \mathbf{Z}, x(k) \in \mathbf{R}^n =: \mathcal{X}$ is the state, $u(k) \in \mathbf{R}^p =: \mathcal{U}$ is the input, $y(k) \in \mathbf{R}^q =: \mathcal{Y}$ is the output, and $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$, are real ω-periodic matrices. When $\omega = 1$, equations (3.1), (3.2) reduce to the state-space model (1.3), (1.4) of a linear time-invariant system.

It is convenient to consider also the more general way of describing a periodic physical process, by means of (1.5), (1.6), since, if the equations of the periodic process are written down, they may not initially be in the state-space form (3.1), (3.2). In (1.5), (1.6), where $T_i(k)$, $U_i(k)$, $V_i(k)$ and $W_i(k)$ $(i = 0, \dots, r)$ are ω-periodic real matrices, u and y have the same meaning as in (3.1), (3.2), while $\xi(k+i) \in \mathbf{R}^m =: \Xi$ is the vector of the internal variables needed for writing the equations describing the behaviour of the physical process in the form (1.5), (1.6). Vector ξ is termed the *pseudo-state*, while equations (1.5), (1.6) are termed the *model* of the physical process under consideration. Equations (3.1), (3.2) are called the *state-space model* of the corresponding physical process, or, briefly, the *system* describing such a process. When $\omega = 1$, model (1.5), (1.6) reduces to the polynomial matrix description (1.1), (1.2), with the positions: $T(\delta) := \sum_{i=0}^{r} T_i\delta^i, U(\delta) := \sum_{i=0}^{r} U_i\delta^i, V(\delta) := \sum_{i=0}^{r} V_i\delta^i$, and $W(\delta) := \sum_{i=0}^{r} W_i\delta^i$.

For any $k_0 \in \mathbf{Z}$, model (1.5), (1.6) can be rewritten for $k = k_0 + h\omega + \ell$, $h \in \mathbf{Z}$, $\ell = 0, \ldots, \omega - 1$, as follows:

$$\sum_{i=0}^{r} T_i(k_0+\ell)\,\xi(k_0+h\omega+\ell+i) = \sum_{i=0}^{r} U_i(k_0+\ell)\,u(k_0+h\omega+\ell+i), \quad \ell = 0, \ldots, \omega-1, \tag{3.3}$$

$$y(k_0+h\omega+\ell) = \sum_{i=0}^{r} V_i(k_0+\ell)\,\xi(k_0+h\omega+\ell+i) + \sum_{i=0}^{r} W_i(k_0+\ell)\,u(k_0+h\omega+\ell+i), \ell = 0, \ldots, \omega-1. \tag{3.4}$$

By introducing the ω-stacked forms at time k_0 of vectors $\xi(k)$, $u(k)$, $y(k)$ and those of matrices $T_i(k)$, $U_i(k)$, $V_i(k)$, $W_i(k)$, $i = 0, \ldots, r$, equations (3.3), (3.4) can be rewritten in the following compact form:

$$\sum_{i=0}^{r} \mathcal{T}_{i,k_0}\xi_{k_0+i}(h) = \sum_{i=0}^{r} \mathcal{U}_{i,k_0}u_{k_0+i}(h), \tag{3.5}$$

$$y_{k_0}(h) = \sum_{i=0}^{r} \mathcal{V}_{i,k_0}\xi_{k_0+i}(h) + \sum_{i=0}^{r} \mathcal{W}_{i,k_0}u_{k_0+i}(h), \tag{3.6}$$

where $\xi_{k_0}(h)$, $u_{k_0}(h)$, and $y_{k_0}(h)$ will be called briefly, respectively, the ω-stacked pseudo-state, the ω-stacked input, and the ω-stacked output. Taking (2.3) into account, (3.5), (3.6) take the form:

$$\sum_{i=0}^{r} \mathcal{T}_{i,k_0}R_m^i(\Delta)\,\xi_{k_0}(h) = \sum_{i=0}^{r} \mathcal{U}_{i,k_0}R_p^i(\Delta)\,u_{k_0}(h), \tag{3.7}$$

$$y_{k_0}(h) = \sum_{i=0}^{r} \mathcal{V}_{i,k_0}R_m^i(\Delta)\,\xi_{k_0}(h) + \sum_{i=0}^{r} \mathcal{W}_{i,k_0}R_p^i(\Delta)\,u_{k_0}(h). \tag{3.8}$$

Equations (3.7), (3.8) are termed the *ω-stacked* (or, simply, *stacked*) *form at* (*the initial*) *time* k_0 *of model* (1.5), (1.6) or, briefly, the *stacked model at time* k_0, and can be rewritten in the following form:

$$\mathcal{T}_{k_0}(\Delta)\,\xi_{k_0}(h) = \mathcal{U}_{k_0}(\Delta)\,u_{k_0}(h), \tag{3.9}$$

$$y_{k_0}(h) = \mathcal{V}_{k_0}(\Delta)\,\xi_{k_0}(h) + \mathcal{W}_{k_0}(\Delta)\,u_{k_0}(h), \tag{3.10}$$

where:

$$\mathcal{T}_{k_0}(\Delta) := \sum_{i=0}^{r} \mathcal{T}_{i,k_0} R_m^i(\Delta), \quad \mathcal{U}_{k_0}(\Delta) := \sum_{i=0}^{r} \mathcal{U}_{i,k_0} R_p^i(\Delta),$$

$$\mathcal{V}_{k_0}(\Delta) := \sum_{i=0}^{r} \mathcal{V}_{i,k_0} R_m^i(\Delta), \quad \mathcal{W}_{k_0}(\Delta) := \sum_{i=0}^{r} \mathcal{W}_{i,k_0} R_p^i(\Delta).$$

In the special case of the state-space model (3.1), (3.2), since $r = 1$, $m = n$, $\xi(k) = x(k)$, equations (3.9), (3.10) reduce to

$$R_n(\Delta)\, x_{k_0}(h) = \mathcal{A}_{k_0}\, x_{k_0}(h) + \mathcal{B}_{k_0}\, u_{k_0}(h), \tag{3.11}$$

$$y_{k_0}(h) = \mathcal{C}_{k_0}\, x_{k_0}(h) + \mathcal{D}_{k_0}\, u_{k_0}(h), \tag{3.12}$$

which are termed the *ω-stacked* (or, simply, *stacked*) *form at* (*the initial*) *time* k_0 *of the state-space model* (3.1), (3.2), or, briefly, the *stacked system at time* k_0. The ω-stacked forms $\mathcal{A}_{k_0}$, $\mathcal{B}_{k_0}$, $\mathcal{C}_{k_0}$, $\mathcal{D}_{k_0}$ at time k_0 of $A(k)$, $B(k)$, $C(k)$, $D(k)$ were introduced in previous papers (see, e. g. [11, 12]), and allow the notions of invariant zeros, transmission zeros, input (output) decoupling zeros to be studied in a direct way for the ω-periodic system (3.1), (3.2) [11]. This is obtained, as far as invariant zeros and decoupling zeros are concerned, through the matrix obtained by substituting the operator Δ by the complex variable z in

$$S_{k_0}^S(\Delta) := \begin{bmatrix} \mathcal{A}_{k_0} - R_n(\Delta) & \mathcal{B}_{k_0} \\ \mathcal{C}_{k_0} & \mathcal{D}_{k_0} \end{bmatrix}, \tag{3.13}$$

which will be called the *ω-stacked* (or, simply, *stacked*) *system matrix at* (*the initial*) *time* k_0 of system (3.1), (3.2). For $\omega = 1$ it reduces to Rosenbrock's system matrix [1], and, in general, it plays a similar role for the ω-periodic system (3.1), (3.2) [11, 13].

Similarly, we can define in general the *ω-stacked* (or, simply, *stacked*) *system matrix at* (*the initial*) *time* k_0 of model (1.5), (1.6) in the following way:

$$S_{k_0}^M(\Delta) := \begin{bmatrix} -\mathcal{T}_{k_0}(\Delta) & \mathcal{U}_{k_0}(\Delta) \\ \mathcal{V}_{k_0}(\Delta) & \mathcal{W}_{k_0}(\Delta) \end{bmatrix}. \tag{3.14}$$

Notice that, for a fixed k_0, equations (3.11), (3.12) can be considered as a time-invariant description of the ω-periodic system (3.1), (3.2), since they can be seen to be similar to equations (1.1), (1.2) by formally substituting δ, ξ, u, y, respectively, by Δ, $x_{k_0}(h)$, $u_{k_0}(h)$, $y_{k_0}(h)$, and by setting $T(\Delta) := R_n(\Delta) - \mathcal{A}_{k_0}$, $U(\Delta) := \mathcal{B}_{k_0}$, $V(\Delta) := \mathcal{C}_{k_0}$, $W(\Delta) := \mathcal{D}_{k_0}$. The first n components $x(k_0)$ of $x_{k_0}(0)$ play the role of initial conditions at time k_0 for computing the solution of (3.11), (3.12) as in [9, 10].

More generally, for a fixed k_0, equations (3.9), (3.10) can be viewed as a time-invariant description of the ω-periodic model (1.5), (1.6), since they are formally similar to (1.1), (1.2). The pseudo-state and output responses $\xi(k)$ and $y(k)$ of (1.5), (1.6), for any $k \geq k_0$, can be uniquely computed in ω-stacked form through (3.9), (3.10), under the assumption that $\mathcal{T}_{k_0}(\Delta)$ is square and nonsingular (as a polynomial matrix), from the input function $u(\cdot)$ and from some initial conditions, as stated by the following extension of well-known time-invariant results [1].

Proposition 1. [9, 10] If the polynomial matrix $\mathcal{T}_{k_0}(\Delta)$ is square and nonsingular, then for each input function $u(\cdot)$ there exist solutions $\xi_{k_0}(\cdot)$ of equation (3.9), and they depend on arbitrary independent initial conditions whose number is equal to the degree of $\det \mathcal{T}_{k_0}(\Delta)$.

If $\mathcal{T}_{k_0}(\Delta)$ is square and nonsingular, the degree of $\det \mathcal{T}_{k_0}(\Delta)$ is called the *order of the model* (1.5), (1.6) *at (the initial) time* k_0. For the determination of the initial conditions needed for the solution of (3.9), and for its effective computation, see [9, 10].

Proposition 2. [9, 10] If $\mathcal{T}_{k_0}(\Delta)$ is not square and nonsingular, then one (or more) of the following situations occurs:

(α) one of the rows of (3.9) can be reduced to the trivial identity $0 = 0$ by a finite sequence of elementary operations of the types (i), (ii), and (iii) that will be defined in Section 4;

(β) there exists an ω-stacked input function $u_{k_0}(\cdot)$ for which (3.9) admits no solution;

(γ) there exist solutions of (3.9) for any $u_{k_0}(\cdot)$, but they depend on an infinite number of independent initial conditions.

Now, notice that in (3.9), (3.10) the time k_0 is arbitrary, whence similar relations hold for k_0+1 instead of k_0, involving $\xi_{k_0+1}(h)$, $u_{k_0+1}(h)$, $y_{k_0+1}(h)$, $\mathcal{T}_{k_0+1}(\Delta)$, $\mathcal{U}_{k_0+1}(\Delta)$, $\mathcal{V}_{k_0+1}(\Delta)$ and $\mathcal{W}_{k_0+1}(\Delta)$. Notice that (2.3) can relate $\xi_{k_0+1}(h)$, $u_{k_0+1}(h)$, and $y_{k_0+1}(h)$ with $\xi_{k_0}(h)$, $u_{k_0}(h)$, and $y_{k_0}(h)$, respectively. Relation (2.4) allows the following lemma to be proved, which exhibits relations between $\mathcal{T}_{k_0+1}(\Delta)$, $\mathcal{U}_{k_0+1}(\Delta)$, $\mathcal{V}_{k_0+1}(\Delta)$, $\mathcal{W}_{k_0+1}(\Delta)$ and, respectively, $\mathcal{T}_{k_0}(\Delta)$, $\mathcal{U}_{k_0}(\Delta)$, $\mathcal{V}_{k_0}(\Delta)$, $\mathcal{W}_{k_0}(\Delta)$, similar to (2.4) written with $j = 0$ and $i = 1$.

Lemma 2. If $\mathcal{T}_{k_0}(\Delta)$ is square, then the following identities hold:

$$\mathcal{T}_{k_0+1}(\Delta) = R_m(\Delta)\mathcal{T}_{k_0}(\Delta)R_m^{-1}(\Delta), \tag{3.15}$$

$$\mathcal{U}_{k_0+1}(\Delta) = R_m(\Delta)\mathcal{U}_{k_0}(\Delta)R_p^{-1}(\Delta), \tag{3.16}$$

$$\mathcal{V}_{k_0+1}(\Delta) = R_q(\Delta)\mathcal{V}_{k_0}(\Delta)R_m^{-1}(\Delta), \tag{3.17}$$

$$\mathcal{W}_{k_0+1}(\Delta) = R_q(\Delta)\mathcal{W}_{k_0}(\Delta)R_p^{-1}(\Delta), \tag{3.18}$$

$$\det \mathcal{T}_{k_0+1}(\Delta) = \det \mathcal{T}_{k_0}(\Delta). \tag{3.19}$$

If $\mathcal{T}_{k_0}(\Delta)$ is square and nonsingular for $k_0 = \bar{k}_0$, then it is for all $k_0 \in \mathbf{Z}$, and the order of the model (1.5), (1.6) is independent of the initial time k_0.

The following corollary of Lemma 2 follows from Corollary 1.

Corollary 2. If in (2.1) the operator Δ is substituted by a scalar complex variable z, then the identities (3.15)–(3.19) still hold with Δ replaced by z.

Hereafter, by virtue of Proposition 2, $\mathcal{T}_{k_0}(\Delta)$ will be assumed to be square and nonsingular. By virtue of Lemma 2, for an arbitrary $k_0 \in \mathbf{Z}$ the degree of $\det \mathcal{T}_{k_0}(\Delta)$ will be simply called the *order of the model* (1.5), (1.6).

In the special case of the stacked system (3.11), (3.12), relation (2.4) of Lemma 1 yields directly relations similar to (3.16)-(3.18) for $\mathcal{B}_{k_0+1}$, $\mathcal{C}_{k_0+1}$, $\mathcal{D}_{k_0+1}$, and relation (3.15) reduces to:

$$R_n(\Delta) - \mathcal{A}_{k_0+1} = R_n(\Delta)\,[R_n(\Delta) - \mathcal{A}_{k_0}]\,R_n^{-1}(\Delta).$$

4. STRICT SYSTEM EQUIVALENCE

In the time-invariant case, a class of elementary operations on the system matrix corresponding to equations (1.1), (1.2) was considered by Rosenbrock [1] in order to derive a state-space description (1.3), (1.4) from the given model (1.1), (1.2). In a similar way, it seems natural to introduce the following six types of admissible elementary operations on the stacked system matrix (3.14) obtained from the given ω-periodic model (1.5), (1.6).

(i) Multiply any one of the first $(\omega\, m)$ rows of (3.14) by a non-zero real constant α.

(ii) Interchange any two among the first $(\omega\, m)$ rows of (3.14).

(iii) Add a multiple, by a polynomial $\beta(\Delta)$ in Δ with real coefficients, of any one of the first $(\omega\, m)$ rows of (3.14) to any other row.

(iv) Multiply any one of the first $(\omega\, m)$ columns of (3.14) by a non-zero real constant α.

(v) Interchange any two among the first $(\omega\, m)$ columns of (3.14).

(vi) Add a multiple, by a polynomial $\beta(\Delta)$ in Δ with real coefficients, of any one of the first $(\omega\, m)$ columns of (3.14) to any other column.

The operations (i), (ii), and (iii) can be generated by left multiplying (3.14) by a polynomial matrix in Δ, with real coefficients, of the following form:

$$\begin{bmatrix} M(\Delta) & 0 \\ Y(\Delta) & I_{q\omega} \end{bmatrix}, \tag{4.1}$$

with $M(\Delta)$ square and unimodular, thus yielding:

$$\hat{S}^M_{k_0}(\Delta) := \begin{bmatrix} M(\Delta) & 0 \\ Y(\Delta) & I_{q\omega} \end{bmatrix} S^M_{k_0}(\Delta). \tag{4.2}$$

Matrix $M(\Delta)$ being unimodular, and taking into account that:

$$\begin{bmatrix} M(\Delta) & 0 \\ Y(\Delta) & I_{q\omega} \end{bmatrix} \begin{bmatrix} 0 \\ y_{k_0}(h) \end{bmatrix} = \begin{bmatrix} 0 \\ y_{k_0}(h) \end{bmatrix}, \tag{4.3}$$

we have that the solutions $\xi_{k_0}(h), y_{k_0}(h)$ of

$$\hat{S}^M_{k_0}(\Delta) \begin{bmatrix} \xi_{k_0}(h) \\ u_{k_0}(h) \end{bmatrix} = \begin{bmatrix} 0 \\ y_{k_0}(h) \end{bmatrix} \tag{4.4}$$

are exactly the same of (3.9), (3.10).

The operations (iv), (v), and (vi) can be generated by right multiplying (3.14) by a polynomial matrix in Δ, with real coefficients, of the following form:

$$\begin{bmatrix} N(\Delta) & X(\Delta) \\ 0 & I_{p\omega} \end{bmatrix}, \tag{4.5}$$

with $N(\Delta)$ square and unimodular, thus yielding

$$\tilde{S}^M_{k_0}(\Delta) := S^M_{k_0}(\Delta) \begin{bmatrix} N(\Delta) & X(\Delta) \\ 0 & I_{p\omega} \end{bmatrix}. \tag{4.6}$$

It is easy to check that, matrix $N(\Delta)$ being unimodular, the solutions $\tilde{\xi}_{k_0}(h)$, $y_{k_0}(h)$ of

$$\tilde{S}^M_{k_0}(\Delta) \begin{bmatrix} \tilde{\xi}_{k_0}(h) \\ u_{k_0}(h) \end{bmatrix} = \begin{bmatrix} 0 \\ y_{k_0}(h) \end{bmatrix} \tag{4.7}$$

and the solutions $\xi_{k_0}(h)$, $y_{k_0}(h)$ of (3.9), (3.10) are biuniquely related in the vectors $\xi_{k_0}(h)$ and $\tilde{\xi}_{k_0}(h)$ by means of the following functional relationship:

$$\xi_{k_0}(h) = N(\Delta)\tilde{\xi}_{k_0}(h) + X(\Delta)u_{k_0}(h), \tag{4.8}$$

and are exactly the same in the ω-stacked output.

It is stressed that, matrix $N(\Delta)$ being unimodular, the inverse of (4.8)

$$\tilde{\xi}_{k_0}(h) = N^{-1}(\Delta)\xi_{k_0}(h) - N^{-1}(\Delta)X(\Delta)u_{k_0}(h) \tag{4.9}$$

is of the same type of (4.8).

Obviously for the following system matrix:

$$\overline{S}^M_{k_0}(\Delta) := \begin{bmatrix} M(\Delta) & 0 \\ Y(\Delta) & I_{q\omega} \end{bmatrix} S^M_{k_0}(\Delta) \begin{bmatrix} N(\Delta) & X(\Delta) \\ 0 & I_{p\omega} \end{bmatrix}, \tag{4.10}$$

with $M(\Delta)$ and $N(\Delta)$ square and unimodular, the solutions $\tilde{\xi}_{k_0}(h)$, $y_{k_0}(h)$ of

$$\overline{S}^M_{k_0}(\Delta) \begin{bmatrix} \tilde{\xi}_{k_0}(h) \\ u_{k_0}(h) \end{bmatrix} = \begin{bmatrix} 0 \\ y_{k_0}(h) \end{bmatrix} \tag{4.11}$$

and the solutions $\xi_{k_0}(h), y_{k_0}(h)$ of (3.9), (3.10) are biuniquely related in the vectors $\xi_{k_0}(h)$ and $\tilde{\xi}_{k_0}(h)$ by (4.8) and (4.9), and coincide in the ω-stacked output. The following definition is formally the same as the one introduced by Rosenbrock [1] in the time-invariant case.

Definition 1. Two $(m\omega + q\omega) \times (m\omega + p\omega)$ polynomial system matrices $S^1(\Delta)$ and $S^2(\Delta)$ with real coefficients of the following form:

$$S^i(\Delta) = \begin{bmatrix} -\mathcal{T}_i(\Delta) & \mathcal{U}_i(\Delta) \\ \mathcal{V}_i(\Delta) & \mathcal{W}_i(\Delta) \end{bmatrix}, \quad i = 1, 2, \tag{4.12}$$

with $\mathcal{T}_i(\Delta)$ of dimensions $m\omega \times m\omega$, are said to be *strict system equivalent* if a relation of the following form holds:

$$S^2(\Delta) = \begin{bmatrix} M(\Delta) & 0 \\ Y(\Delta) & I_{q\omega} \end{bmatrix} S^1(\Delta) \begin{bmatrix} N(\Delta) & X(\Delta) \\ 0 & I_{p\omega} \end{bmatrix}, \tag{4.13}$$

where $M(\Delta)$, $N(\Delta)$, $X(\Delta)$, $Y(\Delta)$ are polynomial in Δ with real coefficients, and $M(\Delta)$, $N(\Delta)$ are unimodular.

Therefore system matrices $S^M_{k_0}(\Delta)$ and $\overline{S}^M_{k_0}(\Delta)$ defined by (3.14) and (4.10) are strict system equivalent.

In addition to the elementary operations (i) – (vi), the following extra operations can be considered on the ω-stacked form $\xi_{k_0}(h)$ at time k_0 of the pseudo-state $\xi(k)$ of the ω-periodic model (1.5), (1.6).

(vii) Add to each $\xi(k_0+h\omega+\ell)$, $\ell = 0,\dots,\omega-1$, which are the vector components of $\xi_{k_0}(h)$, ν scalar components, $\nu \geq 0$, which are defined to be equal to zero for each $h \geq 0$.

(viii) Remove, if they exist, from each $\xi(k_0 + h\omega + \ell)$, $\ell = 0,\dots,\omega-1$, which are the vector components of $\xi_{k_0}(h)$, ν scalar components, $0 \leq \nu \leq m$, which are equal to zero for each $h \geq 0$.

Obviously operation (vii) [resp. (viii)] is equivalent to add [resp. to remove] ν scalar equations to (1.5) [resp. from (1.5)] of the following form

$$\xi_{j_i}(k) = 0, \tag{4.14}$$

with $j_i \in \{m+1,\dots,m+\nu\}$ [resp. $j_i \in \{1,\dots,m\}$], $i = 1,\dots,\nu$. The system matrix obtained from (3.14) after the elementary operation of type (vii) have been carried out, is strict system equivalent (through the operations of the type (ii) and (v)) to the following one:

$$S^{ME}_{k_0}(\Delta) = \begin{bmatrix} I_{\nu\omega} & 0 & 0 \\ 0 & -\mathcal{T}_{k_0}(\Delta) & \mathcal{U}_{k_0}(\Delta) \\ 0 & \mathcal{V}_{k_0}(\Delta) & \mathcal{W}_{k_0}(\Delta) \end{bmatrix}. \tag{4.15}$$

A similar characterization of operation (viii) holds.

Definition 2. Two ω-periodic models of the type (1.5), (1.6) having inputs and outputs of the same dimensions p and q, respectively, pseudo-states of dimensions m_i, $i = 1,2$, and corresponding ω-stacked models $\mathcal{M}^i_{k_0}$, of the form (3.9), (3.10), $i = 1,2$, at the same initial time k_0, are said to be *system equivalent at time* k_0 if there exists an operation of type (vii) or (viii) to be carried out on $\mathcal{M}^1_{k_0}$ and an operation of type (vii) or (viii) to be carried out on $\mathcal{M}^2_{k_0}$ such that the ω-stacked system matrices at time k_0 of the resulting ω-stacked models at time k_0 are strict system equivalent.

Proposition 3. The relation between two ω-periodic models of the type (1.5), (1.6) introduced by Definition 2 is an equivalence relation.

Remark 1. By means of Definition 2 and of Proposition 3, whose proof is omitted, given the ω-periodic model (1.5), (1.6), an equivalence class of ω-periodic models, containing model (1.5), (1.6), is introduced. The solutions of each pair of ω-periodic models in such a class are biuniquely related in the pseudo-state, and are exactly the same in the the output.

The following proposition follows from well-known time invariant results [1].

Proposition 4. Given two ω-periodic models $\mathcal{M}_1$ and $\mathcal{M}_2$ of the type (1.5), (1.6), having inputs and outputs of the same dimensions p and q, respectively, pseudo-states of dimensions m_i, $i = 1, 2$, and the following ω-stacked system matrices at time k_0:

$$S^M_{k_0,i}(\Delta) = \begin{bmatrix} -\mathcal{T}_{k_0,i}(\Delta) & \mathcal{U}_{k_0,i}(\Delta) \\ \mathcal{V}_{k_0,i}(\Delta) & \mathcal{W}_{k_0,i}(\Delta) \end{bmatrix}, \quad i = 1, 2,$$

if $\mathcal{M}_1$ and $\mathcal{M}_2$ are system equivalent at time k_0, then the following pairs of matrices:

$$\begin{array}{ll} S^M_{k_0,1}(\Delta), & S^M_{k_0,2}(\Delta), \\ \mathcal{T}_{k_0,1}(\Delta), & \mathcal{T}_{k_0,2}(\Delta), \\ [-\mathcal{T}_{k_0,1}(\Delta) \;\; \mathcal{U}_{k_0,1}(\Delta)], & [-\mathcal{T}_{k_0,2}(\Delta) \;\; \mathcal{U}_{k_0,2}(\Delta)], \\ \begin{bmatrix} -\mathcal{T}_{k_0,1}(\Delta) \\ \mathcal{V}_{k_0,1}(\Delta) \end{bmatrix}, & \begin{bmatrix} -\mathcal{T}_{k_0,2}(\Delta) \\ \mathcal{V}_{k_0,2}(\Delta) \end{bmatrix}, \end{array}$$

have the same Smith forms, apart from some unit invariant polynomials, equal in number to $\omega|m_1 - m_2|$.

5. CONCLUDING REMARKS

It is worth to mention that, for a given ω-periodic model (1.5), (1.6), it is possible to introduce, in addition to the algebraic machinery here presented, the transfer function matrix $G_{k_0}(z)$ from the z-transform $u_{k_0}(z)$ of $u_{k_0}(h)$ to the z-transform $y_{k_0}(z)$ of $y_{k_0}(h)$, and to show that such a model is causal only if $G_{k_0}(z)$ is proper and its upper-block-triangular part is strictly proper for all $k_0 \in \mathbf{Z}$ [9, 10]; in addition, this is guaranteed if such conditions are satisfied for an arbitrary $k_0 \in \mathbf{Z}$.

In a subsequent paper it will be shown that: (i) such a transfer matrix is invariant under system equivalence, so that the above mentioned causality conditions are necessary for the existence of an ω-periodic system that is system equivalent to the given model; (ii) under some additional assumptions, such conditions become also sufficient.

(Received January 21, 1993.)

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