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ON VARIOUS INTERPRETATIONS OF THE ROSENBROCK THEOREM

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The Rosenbrock theorem concerning the problem of eigenstructure assignment by state feedback in linear systems is reconsidered and its various interpretations are shown. Some relations to other problems of control theory are hinted, too.

1. INTRODUCTION

During the last three decades of developing the modern control theory we can observe a strong tendency, in some sense justifying the term modern, consisting in exploiting some "structural" information (as controllability indices, infinite zero orders, invariant polynomials, etc.) to establish conditions for the existence of solutions to various control problems.

As these quantities form a set of invariants with respect to some groups of system transformations and can be obtained in many different ways, this tendency represents in fact an approach frequently termed as a structural approach nowadays.

Such a point of view enables us to exploit plenty of well-developed methods and techniques (geometric algorithms, polynomials equations,...) to achieve solutions to long-termed and unresolved problems of linear control theory [2, 3, 4].
Studying the history of this approach, we can find that its background is mainly formed by the problem of pole placement. No wonder, the pole placement problem is closely related to basic concepts of linear systems theory like, for instance, system zeros and poles, controllability etc. This problem can also be found, in more or less disguised form, in the back of many other control problems [1] and various branches of applied algebra.

The main aim of this paper is to reconsider again the famous result of Rosenbrock, which is also a nice specimen of the above mentioned structural approach, and provide its different reformulations from which it should be clear the basic importance of this result for control theory.

Throughout the paper, $\mathbb{R}$ will denote the field of real numbers, $\mathbb{R}[s]$ will stand for the ring of polynomials over $\mathbb{R}$ while $\mathbb{R}(s)$ will stand for the field of rational functions. We denote $\mathbb{R}^{m\times n}$, $\mathbb{R}^{m\times n}[s]$ and $\mathbb{R}^{m\times n}(s)$ the sets of $m \times n$ matrices having elements in the corresponding fields or ring.

2. THE ROSENBROCK THEOREM

To begin with, we shall consider linear, time-invariant and controllable system

$$\dot{x} = Ax + Bu \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ with $\operatorname{rank} B = m$.
Let $c_1 \geq c_2 \geq \ldots \geq c_m$ denote the controllability indices of (1) and let $\phi_1(s) \geq \phi_2(s) \geq \ldots \geq \phi_n(s)$, where $\phi_i(s) \geq \phi_{i+1}(s)$ denotes $\phi_{i+1}(s)$ divides $\phi_i(s)$, be the invariant factors of the matrix $sI_n - A$. We know that the polynomials $\phi_i(s)$'s completely describe the (finite) pole structure of (1). More precisely, this structure is given by the (finite) elementary divisors of $sI_n - A$. Then the fundamental problem of linear control theory reads as follows:

(FP): *Does there exist a state feedback of the form*

$$u = Fx + v, \quad F \in \mathbb{R}^{m\times n} \tag{2}$$

such that the closed-loop system

$$\dot{x} = (A + BF)x + Bv \tag{3}$$

will have its (finite) pole structure given by monic polynomials $\psi_1(s) \geq \psi_2(s) \geq \ldots \geq \psi_n(s)$?

An answer is given by the following theorem, which is referred to as the fundamental theorem of state feedback elsewhere.

Theorem 1. [5] There exists a solution to the problem (FP) if and only if

$$\sum_{i=j}^{n} \deg \psi_i(s) \leq \sum_{i=j}^{n} c_i, \quad j = 1, 2, \ldots, n, \tag{4}$$

where equalities hold for $j \geq m$ and $c_i := 0$ for $i > m$.

The inequalities (4) imply that $\psi_j(s) = 1$ for $j > m$ and equality for $j = m$ enables us to rearange (4) into the form

$$\sum_{i=1}^{j} \deg \psi_i(s) \geq \sum_{i=1}^{j} c_i, \quad j = 1, 2, \ldots, m \tag{5}$$

These inequalities correspond to the original result of Rosenbrock. In fact, Theorem 1 does not directly tackle the problem of pole assignment. The poles can be moved to any finite locations of complex plane at will. The only limitations concern the sizes of the cyclic subspaces of the controllable space of (1). The dimensions of these subspaces are equal to $\deg \psi_i(s), i = 1, 2, ..., m$.

3. VARIOUS FORMULATIONS OF THE ROSENBROCK THEOREM

The above formulation of the Rosenbrock theorem emphasizes mainly the control theoretical point of view. But there are also mathematical aspects of the problem. To introduce them, we shall need some concepts of the theory of polynomial matrices.

Let $N(s)$ and $D(s)$ be polynomial matrices over $\mathbb{R}[s]$ of respective sizes $n \times m$ and $m \times m$ such that

$$[sI_n - A, \quad -B] \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = 0 \tag{6}$$

Then we shall say that $N(s)$ and $D(s)$ form *a (right) normal external description* (NED) of (1) [6] if

- $\begin{bmatrix} N(s) \\ D(s) \end{bmatrix}$ is minimal polynomial and nonincreasingly column-degree ordered basis of $\operatorname{Ker}[sI_n - A, \quad -B]$.
- $N(s)$ is a minimal polynomial basis of $\operatorname{Ker} P(sI_n - A)$ where P represents the maximal annihilator of B.

It should be noted that the matrices $N(s)$ and $D(s)$ are not unique. They can be postmultiplied by a unimodular matrix that does not destroy their above stated properties. As follows from the definitions of the matrices $D(s)$ and $N(s)$, the matrix $D(s)$ is column reduced with column degrees $c_1 \geq c_2 \geq ... \geq c_m$ and the matrix $N(s)$ has column degrees $c_i - 1, \ i = 1, 2, ..., m$. Moreover,

$$N(s) = K \operatorname{block\,diag} \left(\begin{bmatrix} 1 \\ s \\ \vdots \\ s^{c_1 - 1} \end{bmatrix}, ..., \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{c_m - 1} \end{bmatrix} \right)$$

where $K \in \mathbb{R}^{n \times n}$ is nonsingular and $\sum c_i = n$.

Due to these special properties, the matrix $N(s)$ is said to form *a polynomial basis* of $\mathbb{R}^n$ [4].

Using NED of (1) and on the basis of (6), we can see that the problem (FP) is equivalent to the problem of finding a matrix F such that the matrix

$$D_F(s) = D(s) - FN(s) \tag{7}$$

has prescribed invariant factors $\psi_i(s),\ i = 1, 2, ..., m$.
The relationship (7) then describes the class of all the matrices $D_F(s)$ obtainable by state feedback (2).

Lemma 1. [4] Let $N(s)$ and $D(s)$ be as above and $C(s)$ be an $m \times m$ polynomial matrix. Then there exist matrices $X \in \mathbb{R}^{m\times m}$ and $Y \in \mathbb{R}^{m\times n}$, X nonsingular, such that

$$XD(s) + YN(s) = C(s) \tag{8}$$

if and only if $C(s)$ is column reduced with the same column degrees as $D(s)$.

Theorem 2. [7] There exists an $m \times m$ polynomial and column reduced matrix $D_F(s)$ with column degrees $c_1, c_2, ..., c_m$ having $\psi_1(s) \geq \psi_2(s) \geq ... \geq \psi_n(s)$ as its invariant polynomials if and only if the inequalities (4) hold.

Theorem 2 states a general property of square, polynomial and column reduced matrices, which gives, together with Lemma 1, an efficient tool for computing the matrix F. The next theorem, Theorem 3, can be useful when considering the realizability of a precompensator by static state feedback.

Theorem 3. Let $D(s)$ be an $m \times m$, polynomial and column reduced matrix with column degrees $c_1 \geq c_2 \geq ... \geq c_m$. Then there exists a biproper matrix $B(s)$ such that the matrix $M(s) := B(s)D(s)$ is polynomial and has the invariant factors $\psi_1(s) \geq \psi_2(s) \geq ... \geq \psi_m(s)$ if and only if the inequalities (4) hold.

The Rosenbrock theorem can also be stated in the terms of matrix pencils and their Kronecker indices. This time, the role of the matrix B will be stressed.

Theorem 4. [2] Let $A \in \mathbb{R}^{n\times n}$ and let $\psi_1(s) \geq \psi_2(s) \geq ... \geq \psi_n(s)$ be the invariant polynomials of $sI_n - A$. Then there exists a matrix $B \in \mathbb{R}^{n\times m}$ with $\operatorname{rank} B = m$ such that $[sI_n - A, B]$ is irreducible, i.e. $\operatorname{rank}[zI_n - A, B] = n$ for every complex number z, with the Kronecker indices $c_1 \geq c_2 \geq ... \geq c_m$, $\sum c_i = n$ if and only if the inequalities (4) hold.

All the above theorems are equivalent and, in different ways, describe the same thing. It is quite natural to put a question whether the inequalities like (4) appear when we try to find solutions to another problems of control or, in other words, if it is possible to express the solutions of these problems in a similar way.

The answer appears to be affirmative and the reader can find [2,4] various generalizations of Theorem 1 to the realm of implicit and uncontrollable systems.

Another interesting cases are mentioned below, in the form of remarks.

Remark 1. [8] Let $D(s)$ and $N(s)$ be as above and let $\begin{bmatrix} N_r(s) \\ D_r(s) \end{bmatrix}$ denotes a selection of $r, r \leq m$, columns of $\begin{bmatrix} N(s) \\ D(s) \end{bmatrix}$. Let

$$D_{rF}(s) := D_r(s) + FN_r(s), \quad F \in \mathbb{R}^{m\times n}.$$

We shall be interested in characterizing all the possible invariant polynomials of $D_{rF}(s)$ assignable by F.

As $D(s)$ is column reduced, so do $D_r(s)$ and $D_{rF}(s)$ and the problem resembles that solved by Theorem 1. Indeed, it is easy to find necessary and sufficient conditions for $\psi_1(s) \geq \psi_2(s) \geq ... \geq \psi_r(s)$ to be the invariant polynomials of $D_{rF}(s)$. We obtain

$$\sum_{i=j}^{r} \deg \psi_i(s) \leq \sum_{i=j}^{r} c_i, \quad j = 1, 2, ..., r \tag{9}$$

where $c_1 \geq c_2 \geq ... \geq c_r$ are the column degrees of $D_r(s)$ and hence, those of $D_{rF}(s)$. What is surprising here is that no equality holds for $j = 1$. For instance, $\psi_i(s) = 1$, $i = 1, 2, ..., r$ are assignable. It is to be noted that the inequalities (9) got some attention in the studies concerning the problem of invariant factors assignment in implicit systems.

Remark 2. The inequalities like (4) and (9) are not only related to static state feedback. Suppose now the dynamic state feedback

$$u = F(s)x + v$$

is used for pole assignment instead of (2). Clearly, the number of poles of the closed-loop system is greater than or equal to the the number of poles of (1) and the problem is equivalent, in the light of Lemma 1, to finding a solution $X(s)$ and $Y(s)$, with $X(s)$ invertible and $X^{-1}(s)Y(s)$ proper, to the equation

$$X(s)D(s) + Y(s)N(s) = C(s) \tag{10}$$

where $C(s)$ reflects the desired pole structure of the closed-loop system.

Such a solution exists [9] if and only if invariant polynomials $\psi_1(s) \geq \psi_2(s) \geq ... \geq \psi_m(s)$ of $C(s)$ satisfy the inequalities

$$\sum_{i=1}^{k} \deg \psi_i(s) \geq \sum_{i=1}^{k} c_i, \quad j = 1, 2, ..., m$$

where $c_1 \geq c_2 \geq ... \geq c_m$ denote again the controllability indices of (1). Of course, to get all the invariant polynomials of the closed-loop system, we must add $\psi_{m+1}(s) = ... = \psi_n(s) = 1$ to those mentioned above.

Remark 3. When considering a contant output feedback, i.e. the feedback of the form

$$u = Ky + v, \quad K \in \mathbb{R}^{m \times p}, \tag{11}$$

where p is the number of outputs defined by the equation

$$y = Cx, \quad C \in \mathbb{R}^{p \times n}, \text{ rank } C = p,$$

then the inequalities (4) are only necessary for $\psi_i(s)$'s to be the invariant polynomials of $sI_n - A - BKC$.
There are many other problems concerning the feedback (11) and we only note that the problem of finding a necessary and sufficient condition for $\psi_i(s)$'s to be assignable by (11) is one of the most challenging problems of linear control theory.

Remark 4. It might be expected that the problem is easier when applying the dynamic output feedback

$$u = K(s)y + v,$$

where $K(s)$ is a proper transfer function, to the system (1) with the output equation $y = Cx$, as above. But this is only partially true.
Suppose the system is controllable and observable and gives rise to the transfer function

$$T(s) = \bar{D}^{-1}\bar{N}(s) = N(s)D^{-1}(s) \in \mathbb{R}^{p \times m}(s)$$

where $\bar{D}(s), \bar{N}(s)$ and $D(s), N(s)$ form left and right normalized matrix fraction descriptions of $T(s)$, respectively. By the term a right normalized matrix fraction description is meant a factorization of $T(s)$ such that the matrices $N(s)$ and $D(s)$ are right coprime with $D(s)$ being column reduced.

A left normalized matrix fraction description is defined in a similar way. Then the column degrees $c_1 \geq c_2 \geq ... \geq c_m$ of $D(s)$ are the controllability indices and the row degrees $\nu_1 \geq \nu_2 \geq ... \geq \nu_p$ of $\bar{D}(s)$ are the observability indices of the system.
A simple analysis shows that we must again solve the equation (11) where $N(s), D(s)$ are as above and $C(s) \in \mathbb{R}^{m \times m}[s]$ is a column reduced matrix with the invariant polynomials $\psi_1(s) \geq \psi_2(s) \geq ... \geq \psi_m(s)$ characterizing the pole structure of the closed-loop system.

Then, see [10, 11] for more details, the sufficient conditions for a proper compensator $K(s) = X^{-1}(s)Y(s)$ to exist are of the form

$$\sum_{i=1}^{j} \deg \psi_i(s) \geq \sum_{i=1}^{j}(c_i + k - 1), \quad j = 1, 2, ..., m \tag{12}$$

where k is any integer, $k \geq \nu_1$, and $c_i + k - 1$ are the column degrees of $C(s)$. Similarly, using the dual version of the equation (10), i.e.

$$\bar{D}(s)\bar{X}(s) + \bar{N}(s)\bar{Y}(s) = \bar{C}(s),$$

where $\bar{C}(s) \in \mathbb{R}^{p\times p}[s]$ is row reduced with the invariant factors $\psi_1(s) \geq \psi_2(s) \geq ... \geq \psi_p(s)$, which gives the desired pole structure of the closed-loop system, we obtain another set of sufficient conditions, under which a proper compensator $K(s) = \bar{Y}(s)\bar{X}^{-1}(s)$ will exist, in the form

$$\sum_{i=1}^{j} \deg \psi_i(s) \geq \sum_{i=1}^{j}(\nu_i + l - 1), \quad j = 1, 2, ..., p \tag{13}$$

where l is any integer, $l \geq c_1$, and $\nu_i + l - 1$, $i = 1, 2, ..., p$ are the row degrees of $\bar{C}(s)$.
A direct consequence of (12) and (13) is that the number of nonunit invariant factors $\psi_i(s)$ is less than or equal to $\min(m, p)$ and the minimal order of the compensator $K(s)$ is less than or equal to $\min[p(c_1 - 1),\ m(\nu_1 - 1)]$.
The necessary conditions are a little bit more involved and the reader is referred to [10, 11] for more details. Let us only note that no necessary and sufficient conditions for $\psi_i(s)$'s to be assignable are known in this case.

4. MODEL MATCHING

To demonstrate the power of ideas expressed via the Rosenbrock theorem, we shall consider the problem of exact model matching [1, 12].
In fact there are no explicit conditions of solvability of this famous problem up to now and, moreover, many questions regarding this problem are still open.
To begin with, let $T(s), T_m(s) \in R^{p\times m}(s)$ be transfer functions of linear controllable and observable systems and let the former system be called a plant and the later one a model. We are interested in finding a static state feedback of the form

$$u = Fx + Gv, \quad G \text{ invertible} \tag{14}$$

such that the plant together with (14) around exactly matches the model.

In other words, since the effect of (14) upon the plant can be described as a postmultiplication of $T(s)$ by some biproper matrix, say $B(s)$, we get the following equation

$$T_m(s) = T(s)B(s) = T_{FG}(s) \tag{15}$$

where $T_{FG}(s)$ denotes the transfer function of the closed-loop system.

Writing all the transfer functions in the form of right normalized matrix fraction descriptions, we have

$$\begin{array}{rcl} T_m(s) &=& N_m(s)D_m^{-1}(s) \\ T(s) &=& N(s)D^{-1}(s) \\ T_{FG}(s) &=& N(s)D_{FG}^{-1}(s) \end{array} \tag{16}$$

It follows from (15) and (16), see [14], that

$$\begin{bmatrix} N(s) \\ D_{FG}(s) \end{bmatrix} = \begin{bmatrix} N_m(s) \\ D_m(s) \end{bmatrix} P(s) \tag{17}$$

where $P(s)$ is some polynomial matrix.

Let $\phi_1(s) \geq \phi_2(s) \geq \dots \geq \phi_m(s)$ and $\psi_1(s) \geq \psi_2(s) \geq \dots \geq \psi_m(s)$ be the invariant polynomials of $D_m(s)$ and $D_{FG}(s)$, respectively, and let $c_1 \geq c_2 \geq \dots \geq c_m$ be the controllability indices of the plant. Then the relationship (17) implies

$$\sum_{i=j}^{m} \deg \phi_i(s) \leq \sum_{i=j}^{m} \deg \psi_i(s) \leq \sum_{i=j}^{m} c_i, \quad j = 1, 2, \dots, m \tag{18}$$

and

$$\sum_{i=1}^{m} \deg \psi_i(s) = \sum_{i=1}^{m} c_i.$$

This is an immediate consequence of Theorem 2 and the fact that $\psi_i(s) \geq \phi_i(s)$ for $i = 1, 2, \dots, m$. However,such equality may not hold between $\sum_{i=1}^{m} \deg \phi_i(s)$ and $\sum_{i=1}^{m} c_i$, which implies that $N(s)$ and $D_{FG}(s)$ may not be coprime.
It can be readily seen that another conditions are implied by (17). But a complete set of such conditions,which would be at the same time also sufficient, is a challenge for some future work. Even the complete characterization of $\phi_i(s)$'s in (17) is an open problem.

5. CONCLUSIONS

It has been our aim to emphasize the importance of the inequalities like (4), (5) in control theory. They characteristize what we call a structural approach and they give undoubtedly more insight into many problems of control.
Naturally, there arises a question why just the inequalities like (4), (5), (18), ... play such an important role in control theory where we usually deal with groups of transformations (feedback group, for instance) that are rather characterized by some equalities than inequalities.
To elucidate this point (see also [13]), consider again the action of state feedback (2) upon the system (1). We can write this effect also in the form

$$[sI_n - A, \quad -B]\begin{bmatrix} I_n \\ F \end{bmatrix} = sI_n - A - BF \tag{19}$$

where the matrix $\begin{bmatrix} I_n \\ F \end{bmatrix}$ is of course non-invertible. Hence, we can view the modification of the pencil $[sI_n - A, \quad -B]$ from two points. The first one is given by the non-invertible transformation (19) and the second one is described by an invertible transformation

$$[sI_n - A, \quad -B]\begin{bmatrix} I_n & 0 \\ F & I_m \end{bmatrix} = [sI_n - A - BF, \quad -B]. \tag{20}$$

In (20), the two pencils, i.e. $[sI_n - A, \quad B]$ and $[sI_n - A - BF, -B]$, have the same Kronecker invariants, which are the controllability indices of (1). This is the case in which the feedback F is considered as an element of feedback group.
In the first case, the feedback F is considered as an element of *a feedback monoid*, i.e. as a non-invertible transformation and the pencils in (19) have different Kronecker invariants. While the pencil $[sI_n - A, \quad -B]$ is characterized by the controllability indices, the other one, which is equivalent to $[sI_n - A - BF, 0]$, is described by its invariant polynomials.
Then the inequalities (5) tell nothing else than the feedback monoid induces some order relation in the set of matrix pencils.
To be more precise, let $\mathcal{S}$ be a set of elements $s, t, \dots$ and let $\mathcal{T}$ be a monoid of transformations $S, T, \dots$ acting on $\mathcal{S}$ along the rule

$$s = St, \quad s, t \in \mathcal{S}, \quad S \in \mathcal{T}$$

We define a relation $\approx$ on $\mathcal{S}$ by

$$s \approx t \iff s = St \text{ and } t = Ts \text{ for some } S, T \in \mathcal{T}.$$

Clearly, $\approx$ is an equivalence relation.

Let $\mathcal{S}/(\approx)$ denote the quotient set defined by the equivalence $\approx$ and let $\bar{s}, \bar{t}, \ldots$ denote the classes of equivalence of the elements $s, t, \ldots$ Then we can define a relation $\leq$ on $\mathcal{S}/(\approx)$ by

$$\bar{s} \leq \bar{t} \Longleftrightarrow t = Ts \quad \text{for some} \quad T \in \mathcal{T}$$

i.e. for some $s \in \bar{s}$ and $t \in \bar{t}$ there exists $T \in \mathcal{T}$ such that $t = Ts$. It can be readily verified that $\leq$ is an order relation.
Hence, when $\mathcal{T}$ is the feedback monoid acting on the set $\mathcal{S}$ of matrix pencils, the relation $\approx$ comes to the Kronecker equivalence and for the controllable systems the relation $\leq$ is explicitly described by the Rosenbrock inequalities.

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REFERENCES

[1] V. Kučera: Analysis and Design of Discrete Linear Control Systems. Copublished by Academia and Prentice Hall, Prague 1992.

[2] I. Zaballa: Interlacing inequalities and control theory. Linear Algebra Appl. *87* (1987), 113–146.

[3] J. J. Loiseau: Sur la modification de la structure à l'infini par un retour d'état statique. SIAM J. Control Optim. *26* (1988), 2, 251–273.

[4] P. Zagalak and J. J. Loiseau: Invariant factors assignment in linear systems. In: Proc. SINS'92, Texas 1992, pp. 197–204.

[5] H. H. Rosenbrock: State-space and Multivariable Theory. Wiley, New York 1970.

[6] M. Malabre, V. Kučera and P. Zagalak: Reachability and controllability indices for linear descriptor systems. Systems Control Lett. *15* (1990), 119–123.

[7] P. Zagalak and V. Kučera: Eigenstructure assignment in linear descriptor systems. In: Proc. 1st Europ. Control Conference., Grenoble 1991, pp. 409–414.

[8] P. Zagalak and J. J. Loiseau: On the problem of pole assignment by feedback in linear systems. Part 1: Constant feedback. Linear Algebra Appl., submitted.

[9] P. Zagalak, J. F. Lafay and A. Herrera: The row-by-row decoupling problem via state feedback: A polynomial approach. Automatica, to appear.

[10] H. H. Rosenbrock and G. E. Hayton: The general problem of pole assignment. Internat. J. Control *27* (1978), 837–852.

[11] P. Zagalak and V. Kučera: The problem of pole assignment: a polynomial approach. IEEE Trans. Automat. Control *AC-30* (1986), 286–289.

[12] W. A. Wolovich: The use of state feedback for exact model matching. SIAM J. Control Optim. *10* (1972), 512–523.

[13] J. J. Loiseau: Pole placement and related problems. Kybernetika *28* (1992), 90–99.

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