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EXTERNAL REACHABILITY (REACHABILITY WITH POLE ASSIGNMENT BY P. D. FEEDBACK) FOR IMPLICIT DESCRIPTIONS

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Even for flat implicit linear systems (i. e. having more state components than state equations), reachability is a well defined concept in terms of the set of state trajectories: it characterizes the property that from any initial state can start a smooth state trajectory which reaches any final state. What can happen, however, for this general class of systems, is that a system with no control input can be completely reachable. We introduce here the notion of "external reachability" which expresses the fact that trajectories can actually be controlled through the input (by proportional and derivative state feedback). Geometric necessary and sufficient conditions are given for external reachability. A new design method is proposed for pole assignment which uses this concept and relies upon right inversion techniques.

1. INTRODUCTION

We shall deal with linear, time-invariant, implicit (E, A, B)-systems, $\Sigma : \mathcal{U} \to \mathcal{Y}$, described by:

$$E\,\dot{x}(t) = A\,x(t) + B\,u(t); \quad t \geq 0, \tag{1.1.a}$$

where $x(t) \in \mathcal{X}$, $\dot{x}(t) \in \mathcal{X}$, $u(t) \in \mathcal{U}$, and E, A, and B are linear maps defined as follows:

$$E : \mathcal{X} \to \underline{\mathcal{X}}; \quad A : \mathcal{X} \to \underline{\mathcal{X}}; \quad B : \mathcal{U} \to \underline{\mathcal{X}}, \tag{1.1.b}$$

where the respective dimensions of the state space $(\mathcal{X})$ and of the state equation space $(\underline{\mathcal{X}})$ are not necessarily equal.

One of the most important structural concepts in System Theory, which has been widely studied for years, is that of reachability, which characterizes all the states which can be controlled. For classical (I, A, B) systems, Wonham [29] has characterized the reachable space of Σ from a geometric point of view, as "the set of states $x \in \mathcal{X}$, say $\mathcal{R}_0$, which are reachable in a finite time from the origin through trajectories of Σ generated by piecewise continuous controls $t \mapsto u(t) \in \mathcal{U}$, defined for $t \geq 0$". It is well known that $\mathcal{R}_0 = \operatorname{Im} B + A \operatorname{Im} B + \cdots A^{n-1} \operatorname{Im} B$, with $\operatorname{Im} B$ denoting the Image of B and n being the dimension of the state space $\mathcal{X}$. Moreover, in this classical case, reachability has been connected with pole placement abilities (through the action of state feedback F, i. e. for the spectrum of $(\lambda I - A - BF)$).

In the case of (E, A, B) systems, with E and A square and $(\lambda E - A)$ regular ($(\lambda E - A)$ invertible), reachability has been studied from various points of view, with tools like for instance: transfer functions [28], distributions [12], geometry [25]...

In the case of (E, A, B) systems, with E and A square but $(\lambda E - A)$ not necessarily regular, Ozçaldiran [26] extended his previous geometric characterization of reachability, given in [25], by means of the supremal reachability subspace, say $\mathcal{R}^*_{\mathcal{X}}$, obtained as follows:

$$\mathcal{R}^*_{\mathcal{X}} = \mathcal{V}^*_{\mathcal{X}} \cap \mathcal{S}^*_{\mathcal{X}} \tag{1.2}$$

where

$$\mathcal{V}^*_{\mathcal{X}} = \mathrm{Sup}\,\{\mathcal{T} \subset \mathcal{X} \text{ such that } A\mathcal{T} \subset E\mathcal{T} + \mathrm{Im}\,B\} \tag{1.3.a}$$

$$\mathcal{S}^*_{\mathcal{X}} = \mathrm{Inf}\,\{\mathcal{T} \subset \mathcal{X} \text{ such that } \mathcal{T} = E^{-1}(A\mathcal{T} + \mathrm{Im}\,B)\}\,. \tag{1.3.b}$$

This way of computing $\mathcal{R}^*_{\mathcal{X}}$ is a nice generalization of the afore-mentioned $\mathcal{R}_0$ reachable space for classical (I, A, B)-systems ([29]). It would then appear quite natural to extend such a characterization to general (E, A, B)-systems, for which E and A are not necessarily square, and thus to consider $\mathcal{R}^*_{\mathcal{X}}$ as the good candidate for the reachable subspace in this more general situation. The truthfulness of this conjecture has been established by Frankowska [13], with arguments derived from Differential Inclusions Technics.

This quite general setting of non square descriptions (E and A not necessarily square) also induces some interesting pathology, namely the fact that an (E, A, B)-system may be reachable though having no input at all! Indeed, let us consider the following illustrative example:

Example 1. Consider the following system

$$[\,1\ 0\,]\ \dot{x}(t) = [\,0\ 1\,]\ x(t) + [\,0\,]\ u(t). \tag{1.4}$$

It is quite easy to check that (see for instance [20], or [23]):

$$\mathcal{V}^*_{\mathcal{X}} = \mathcal{X} \quad \text{and} \quad \mathcal{S}^*_{\mathcal{X}} = \mathcal{X}, \tag{1.5}$$

and thus:

$$\mathcal{R}^*_{\mathcal{X}} = \mathcal{X}. \tag{1.6}$$

This example is thus completely reachable, although having no inputs!

This comes from the precise definition of reachability [13]: (E, A, B) is reachable if and only if any state can be reached in a finite time from the origin following at least one smooth state trajectory generated by the system. This trajectory may or may not be generated by the forcing input $u(t)$. Indeed, for our example, any state say $x^{\mathrm{T}} =: [\,x_1\ x_2\,]$ of (1.4) can be reached in a finite time θ with the help of the following state trajectory: $x^{\mathrm{T}} =: \left[\int_0^\theta f(\tau)\,\mathrm{d}\tau\ f(\theta)\right]$, when $f(\theta) = x_2$ and

$\int_0^\theta f(\tau)\,\mathrm{d}\tau = x_1$, with $0 \le \theta < \infty$. In fact, system (1.4) is driven by its own degree of freedom (see for instance [14], [2] and [7]), i. e. is reachable in a purely autonomous way.

The specificity of this example comes from the fact that all the reachability chains which form $\mathcal{R}^*_{\mathcal{X}}$ (here, there exist only one) start in $\operatorname{Ker} E$ (the Kernel of E) and not in $E^{-1}\operatorname{Im} B \,/ \operatorname{Ker} E$ (here $B = 0$). In the terminology initially proposed in [27], these chains are called non-proper chains, in opposition with the proper ones, which are directly generated by some input (these proper chains are the only existing ones for classical (I, A, B)-systems).

As a direct consequence, purely autonomous systems, though reachable, have the drawback that placement of the dynamics or tracking of an external reference is obviously impossible. The goal of this paper is to give a further geometric interpretation of Frankowska's result [13], in order to give necessary and sufficient conditions for placing at will the dynamics of the system by feedback compensation. On the other hand, we will show how a right inverse of a reachable system can be used in order to track a given trajectory and to assign the corresponding poles by means of Proportional and Derivative state feedback. Indeed, as left invertibility plays an important role in observability problems (see [8]), right invertibility is a nice notion adapted to reachability problems.

2. EXTERNAL REACHABILITY

Let us consider (E, A, B)-realizations described in (1.1) with the additional classical assumption: $\operatorname{Ker} B = \{0\}$, and let us define the following set of smooth trajectories:

$$\mathcal{L} =: \Big\{ f(t) \in \mathcal{X} \text{ such that } N[E\,\dot{f}(t) - A\,f(t)] = 0;\ t \ge 0 \Big\} \tag{2.1}$$

where

$$N : \underline{\mathcal{X}} \to \underline{\mathcal{X}} \,/ \operatorname{Im} B \text{ is the canonical projection.} \tag{2.2}$$

Then we have:

Fact 1. A trajectory $x(t) \in \mathcal{X}$ is a solution of (1.1) only if:

$$x(t) \in \mathcal{L}. \tag{2.3}$$

Indeed, every trajectory $x(t) \in \mathcal{X}$ solution of (1.1) is also solution of the restricted system introduced by Jaffe and Karcanias [16]:

$$N\,E\,\dot{x}(t) = N\,A\,x(t), \quad t \ge 0. \tag{2.4}$$

Furthermore, (2.4) expresses the restriction of the initial system (1.1), which is inherent to its internal structure, and which cannot be altered, whatever be the chosen input $u(t)$. Let us then adopt the following:

Definition 1. A state $x_1 \in \mathcal{X}$ is reachable from $x_0 \in \mathcal{X}$ if and only if there exists some finite time, say θ, $0 \leq \theta < \infty$, and at least one $f(t) \in \mathcal{L}$, solution of (1.1), which satisfy $f(0) = x_0$ and $f(\theta) = x_1$.

Definition 2. The system (1.1) is called reachable if and only if any $x_1 \in \mathcal{X}$ is reachable.

Definition 3. The system (1.1) is called externally reachable if and only if:

i) it is reachable

ii) the spectrum $[\lambda(E - B\,F_D) - (A + B\,F_P)]$ can be fixed at will, by means of a suitable $P - D$ feedback, $(F_P,\, F_D)$.

It has to be noted that Definition 2 is in accordance with the classical one. Definition 3 is a mean to exclude purely autonomous systems.

The following Theorem gives a geometric characterization for external reachability:

Theorem 1. An (E, A, B)-system is externally reachable if and only if:

$$\mathcal{R}^*_{\mathcal{X}} = \mathcal{X}, \tag{2.5}$$

$$\dim\left\{\frac{\operatorname{Im} B}{E\mathcal{V}^*_{\mathcal{X}} \cap \operatorname{Im} B}\right\} \geq \dim(\mathcal{V}^*_{\mathcal{X}} \cap \operatorname{Ker} E)\,. \tag{2.6}$$

2.1. Necessary conditions for external reachability

In order to prove the necessity of (2.5) – (2.6), we will use Frankowska's result [13] and the state uniqueness property after P – D feedback given by Lebret [19], which is similar to the state uniqueness property after P feedback established by Banaszuk, Kociecki and Przyluski [1], in the discrete time case. Remember first the following:

Result 1. **([13])** An (E, A, B)-system is reachable if and only if

$$\mathcal{R}^*_{\mathcal{X}} = \mathcal{X}. \tag{2.7}$$

Result 2. **([19])** There exists a Proportional and Derivative state feedback law, $u(t) = F_D\,\dot{x}(t) + F_P\,x(t) + G\,v(t)$, such that the state of (1.1) compensated with this control law, has the uniqueness property for $v(t)$, if and only if:

$$\dim\left\{\frac{\operatorname{Im} B}{E\mathcal{V}^*_{\mathcal{X}} \cap \operatorname{Im} B}\right\} \geq \dim(\mathcal{V}^*_{\mathcal{X}} \cap \operatorname{Ker} E)\,. \tag{2.8}$$

The state uniqueness property basically characterizes the fact that state trajectories are uniquely fixed by the initial conditions and the external input. Though this is guaranteed for any (I, A, B)-system, Example 1 shows that this is not always the case for general (E, A, B) ones (see also [7] for examples with $B \neq 0$ but with some degrees of freedom).

Necessity of conditions (2.5) – (2.6) in Theorem 1 directly comes from Results 1 and 2. Indeed, if the state trajectory cannot be uniquely characterized by its initial condition and its external input, it is a fortiori impossible to fix the spectrum of the dynamics (see for instance [3]).

2.2. Sufficient conditions for external reachability

The sufficiency of condition (2.5) – (2.6) for Theorem 1 will be established through the following rewriting of (1.1):

$$\left\{\begin{array}{l} E\,\dot{z}(t) = A\,z(t) + B\,u(t), \\ \\ x(t) = \mathbb{I}\,z(t), \end{array}\right. \qquad t \geq 0. \tag{2.9}$$

Indeed, we will show that (2.5) – (2.6) allow for the design of a right inverse of (2.9). More precisely, driving (1.1) with the right inverse of (2.9) and choosing as a reference a signal $\bar{x}(t)$ such that $\bar{x}(\theta) = x_1$, any fixed $x_1 \in \mathcal{X}$ will be reached in the finite time θ.

It is clear that $x(t) \in \mathcal{X}$ is a solution of (2.9) for all $u(t) \in \mathcal{U}$ only if $x(t) \in \mathcal{L}$ (see (2.1)). As a direct consequence of Fact 1 and Theorem 1 of [6], we have:

Fact 2. The system (2.9) is solvable (i. e. possesses at least one state solution for any possible input) if and only if:

$$\operatorname{Im} B \subset \left\{ g(t) \in \underline{\mathcal{X}} \mid \exists\, f(t) \in \mathcal{L} \text{ and } g(t) = E\,\dot{f}(t) - A\,f(t);\ t \geq 0 \right\}. \tag{2.10}$$

To conclude the proof, we shall need the following Lemmas (proved in Appendix A):

Lemma 1. If:

$$\mathcal{V}^*_{\mathcal{X}} = \mathcal{X} \tag{2.11}$$

then, for every state $\bar{x} \in \mathcal{X}$, there exists at least one state trajectory, $f(t) \in \mathcal{X}$, such that $f(\theta) = \bar{x}$, with θ finite.

It is worth pointing out at this level that condition (2.11) corresponds, in Frankowska's terminology [13], to strict systems (since $\mathcal{V}^*_{\mathcal{X}} = \mathcal{X}$ if and only if $\operatorname{Im} A \subset \operatorname{Im} E + \operatorname{Im} B$).

Lemma 2. If the following conditions are satisfied:

$$\mathcal{S}^*_{\mathcal{X}} = \mathcal{X} \tag{2.12}$$

$$\dim \left\{ \frac{\operatorname{Im} B}{E\mathcal{V}^*_{\mathcal{X}} \cap \operatorname{Im} B} \right\} \geq \dim \left(\mathcal{V}^*_{\mathcal{X}} \cap \operatorname{Ker} E \right), \tag{2.13}$$

then there exist:

$$F_P : \mathcal{X} \rightarrow \mathcal{U} \quad \text{and} \quad F_D : \mathcal{X} \rightarrow \mathcal{U}, \tag{2.14}$$

such that the following system has no finite dynamics:

$$(E - B\,F_D)\,\dot{x}(t) = (A + B\,F_P)\,x(t) + B\,r(t). \tag{2.15}$$

We are then in position to prove sufficiency of conditions (2.5)-(2.6).

Proof of the sufficiency part of Theorem 1: Starting from (2.5)-(2.6), and thanks to Lemma 2, we can use a first feedback law (2.15) in order to obtain a compensated system with only algebraic and derivative blocks. On the other hand, if (2.9) is right invertible, then, one of its possible right inverses is the following:

$$\left\{\begin{array}{l} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \dot{\xi}(t) = \begin{bmatrix} A & B \\ \mathbb{I} & 0 \end{bmatrix} \xi(t) + \begin{bmatrix} 0 \\ -\mathbb{I} \end{bmatrix} \bar{x}(t), \\ u(t) = \begin{bmatrix} 0 & \mathbb{I} \end{bmatrix} \xi(t) \end{array}\right. \qquad t \geq 0, \tag{2.16}$$

(see Fact 1 of [6] and also [15]).

Testing right invertibility of (2.9) is the same thing as testing left invertibility of (2.16), which will be done hereafter.

It is easy to check that (2.16) is solvable (see Theorem 1 of [6]), and therefore, there exists at least one linear transformation: $\Psi(\cdot) : \mathcal{U} \to \mathcal{X}$, solution of (2.16) (see Corollary 1 of [6]), i.e.:

$$\left\{\begin{array}{l} E\,\dot{\psi}_1(\bar{x}(t)) = A\,\psi_1(\bar{x}(t)) + B\,\psi_2(\bar{x}(t)), \\ 0 = \psi_1(\bar{x}(t)) - \bar{x}(t) \\ u(t) = \psi_2(\bar{x}(t)) \end{array}\right. \qquad t \geq 0, \tag{2.17}$$

with: $\Psi^{\mathrm{T}}(\cdot) = \left[\psi_1^{\mathrm{T}}(\cdot)\ \psi_2^{\mathrm{T}}(\cdot)\right]$.

Left invertibility of (2.16) is equivalent to (see Lemma 1 of [6]):

$$u(t) = 0 \Longrightarrow \bar{x}(t) = 0; \quad t \geq 0, \tag{2.18}$$

or, equivalently, to the following assertion: the only solution of

$$E\,\dot{\bar{x}}(t) = A\,\bar{x}(t); \quad t \geq 0 \tag{2.19.a}$$

is:

$$\bar{x}(t) = 0; \quad t \geq 0 \tag{2.19.b}$$

This is trivially satisfied since (2.9) (after the first feedback compensation (2.15)) is only formed with algebraic and derivative blocks, and so, there is no transient response. This means that any state $x_1 \in \mathcal{X}$ can be reached by means of system (2.16), with the reference, $\bar{x}(t)$, being any trajectory $f(t) \in \mathcal{L}$, such that $f(\theta) = x_1$ (see Lemma 1).

Finally, thanks to right invertibility of (2.9), it is sufficient to choose:

$$\bar{x}(t) = B_0\,r(t) + x(t) - [E_0\,\dot{x}(t) - A_0\,x(t)] \tag{2.20}$$

in order to freely assign the spectrum (see Figure 1). Indeed, from Figure 1, we can assert that:

$$E_0\,\dot{x}(t) = A_0\,x(t) + B_0\,r(t), \tag{2.21}$$

with the compulsory restriction (recall (2.20), and with $\bar{x}(t) \equiv x(t)$):

$$\operatorname{Ker} N(\lambda E - A) \subset \operatorname{Ker} N_0(\lambda E_0 - A_0), \tag{2.22}$$

where $N_0 : \mathcal{X} \longrightarrow \mathcal{X} \,/\, \operatorname{Im} B_0$ is the canonical projection. □

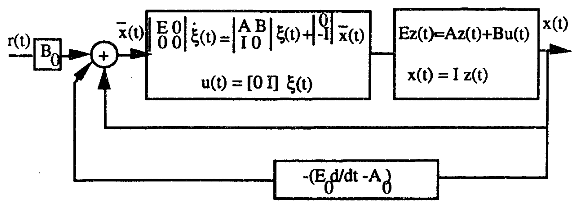

Fig. 1.

The sufficiency proof suggests a procedure for freely assigning the dynamics, say changing (E, A, B) into (E_0, A_0, B_0), as follows:

Procedure for the synthesis:

i) Use a first feedback compensation (like (2.15)) to obtain a system with no integrator,

ii) Synthesize the proposed right inverse (2.16),

iii) In order to achieve (2.22), choose $N_0 E_0$ and $N_0 A_0$ such that:

$$N(\lambda E - A) \equiv N_0(\lambda E_0 - A_0),$$

iv) Choose a B_0 having at least as many independent columns as the number of blocks $L_{\epsilon(i)}$, for the pencil $N(\lambda E - A)$ (see the proof of Lemma 2 in Appendix A), and complete the matrices E_0 and A_0 already partially chosen, in order to have the desired dynamics. This procedure is illustrated in Section 4.

3. CONCLUDING REMARKS

i) We have introduced a reachability concept which enables us to freely modify the system dynamics (external reachability).

ii) We have shown how right inverses can be used to insure some trajectory tracking (for a trajectory satisfying (2.3), of course) and also how to assign the poles,

iii) The geometric condition (2.6), which guarantees the dynamic assignment, can also be expressed in an algebraic way. Indeed, if we apply Lewis's Structure Algorithm [21], [9], to $(E, A, B, 0)$, we obtain that condition (2.6) is equivalent to (see [24]):

$$\mathrm{rank}(B) - \mathrm{rank}(D_{n+1}) \geq \mathrm{rank}(E_{n+1}),$$

where D_{n+1} and E_{n+1} are the limiting matrices extracted from Lewis' Algorithm (see also [8]).

iv) It has to be noted that our proposed right inverse (2.16) is, usually, not minimal (see for instance [17], [18] and [10]), but it can easily be minimized (see for instance the procedure in [9]).

iv) In the case when there exists some degree of freedom, condition (2.6) is never satisfied. In that case, it is better to obtain output dynamics assignment, as introduced in Bonilla, Lebret and Malabre [5].

vi) Implicit systems allow for the description of either proper or derivative systems. This ability to handle with pure derivators is, of course, fundamental in our use of inverse systems, which belong to the same class of models as the systems themselves. We are now working on the manner a non proper system may be approximated (in a stable way) by a sequence of proper ones (see for instance [4] and [11]).

4. ILLUSTRATIVE EXAMPLE

Let us consider the following system:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t); \quad t \geq 0. \tag{4.1}$$

We would like to design a control law so that the system behaves as a first order system. A solution is given by the procedure sketched at the end of Section 2:

i) Apply first the following P – D state feedback law:

$$u(t) = [-1\,0]\, x(t) + [0\,1]\, \dot{x} + v(t). \tag{4.2}$$

The compensated system behaves like a pure derivator:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t); \quad t \geq 0, \tag{4.3}$$

ii) The right inverse system obtained by the rewriting (2.16) and after having applied the minimization algorithm of [10], is:

$$\left\{ \begin{array}{l} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \dot{\xi}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xi(t) + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \bar{x}(t), \\ v(t) = [\,1\,]\,\xi(t) \end{array} \right. \quad t \geq 0, \tag{4.4}$$

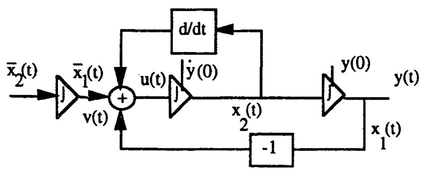

Fig. 2.

In Figure 2, we show the combination of (4.4), (4.2) and (4.1). From Figure 2, we directly see that:

$$\bar{x}_1(t) \equiv x_1(t) \quad \text{and} \quad \bar{x}_2(t) \equiv x_2(t), \quad t > 0, \tag{4.5}$$

whatever be the initial conditions on $x(t)$!

iii) Choosing:

$$N_0E_0 = [\,1\ 0\,]\,; \quad N_0A_0 = [\,0\ 1\,]\,; \quad \text{and} \quad N_0B_0 = [\,0\,], \tag{4.6}$$

we satisfy (2.22), since:

$$\mathcal{L} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{X} \text{ such that } \dot{x}_1(t) - x_2(t) = 0;\, t \geq 0 \right\} \tag{4.7}$$

iv) Complete (E_0, A_0, B_0) in order to only have one pole at -1:

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \quad A_0 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}; \quad B_0 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}. \tag{4.8}$$

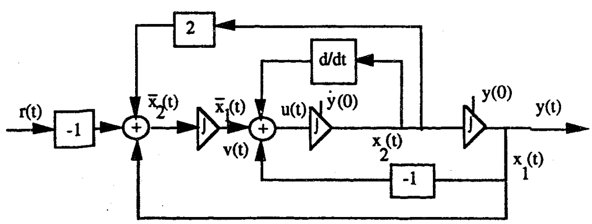

Fig. 3.

In Figure 3, we show the final implementation. From that, we easily see that:

$$\begin{cases} \dot{x}_1(t) + x_1(t) = r(t) \\ \dot{x}_2(t) + x_2(t) = \dot{r}(t) \end{cases} \tag{4.9}$$

APPENDIX A

A.1. Proof of Lemma 1

In view of (2.11), we have

$$\operatorname{Im} A \subset \operatorname{Im} E + \operatorname{Im} B, \tag{A.1}$$

and thus, we can decompose the state equation space, $\underline{\mathcal{X}}$, and the state space, $\mathcal{X}$, as follows:

$$\mathcal{X} = \mathcal{X}_1 \oplus E^{-1}\operatorname{Im} B, \tag{A.2}$$

$$\underline{\mathcal{X}} = \operatorname{Im} B \oplus E\,\mathcal{X}_1 \oplus \underline{\tilde{\mathcal{X}}}_0. \tag{A.3}$$

In suitable bases respecting these decompositions, (1.1) becomes:

$$\begin{bmatrix} 0 & E_2 \\ \mathbb{I} & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} \star & \star \\ A_1 & A_2 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} \mathbb{I} \\ 0 \\ 0 \end{bmatrix} u(t) \tag{A.4}$$

where $\star$ is not precised.

From (A.4), we can easily assert that the allowed trajectory set satisfies:

$$\mathcal{L} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{X} \text{ such that } \dot{x}_1(t) = A_1 x_1(t) + A_2 x_2(t);\ t \geq 0 \right\}. \tag{A.5}$$

This means that we always will be able to draw a trajectory $f^{\mathrm{T}}(t) = \left[f_1^{\mathrm{T}}(t)\ f_2^{\mathrm{T}}(t) \right] \in \mathcal{L}$ passing through any state $\bar{x}^{\mathrm{T}} = \left[\bar{x}_1^{\mathrm{T}}\ \bar{x}_2^{\mathrm{T}} \right] \in \mathcal{X}$, in a finite time θ; such a trajectory must satisfy $f(\theta) = \bar{x}$, with $\dot{f}_1(\theta) = A_1\bar{x}_1 + A_2\bar{x}_2$. □

A.2. Proof of Lemma 2

In order to prove this Lemma, we will express (1.1) in the Proportional and Derivative Canonical Form of [22], which is quickly recalled in Appendix B.

From (B1), we can easily assert that, if there exist as many inputs as blocks of the type $L_{\varepsilon(i)}$, we will be able, after having applied some ad-hoc feedback, to obtain a compensated system with only infinite elementary divisors. Indeed, assumption (2.12) imply that only blocks of the type $L_{n(i)}$ and $L_{\varepsilon(i)}$ are present. Now, since blocks $L_{n(i)}$ are already infinite elementary divisors, it is then sufficient, with the help of a proportional state feedback, to place a "1" below each block $L_{\varepsilon(i)}$. This is possible since condition (2.13) insures that there are as many independent inputs as blocks $L_{\varepsilon(i)}$.

APPENDIX B

The Proportional and Derivative Canonical Form of [22] may be describes as follows (s denoting the Laplace variable):

$$\left[\begin{array}{c} s\,E_K - A_K \\ 0 \end{array}\right] x(s) = \left[\begin{array}{c} 0 \\ \mathbb{I} \end{array}\right] u(s), \tag{B.1}$$

where $s\,E_K - A_K$ is the Kronecker Canonical Form of the restricted pencil (recall [16]) $N(sE - A)$, with $N : \underline{\mathcal{X}} \to \underline{\mathcal{X}}\,/\,\mathrm{Im}\,B$ the canonical projection.

Blocks of the type $L_{n(i)}$ and $L_{\varepsilon(i)}$ respectively correspond to the so-called "infinite elementary divisors" and "column minimal indices".

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