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AN APPROACH TO THE MORGAN PROBLEM

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The Morgan problem is reconsidered and new and explicit necessary conditions are established for there to exist a solution to this problem.

1. INTRODUCTION

We shall consider a linear and time invariant system (C, A, B) of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx, \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ *and* $C = \mathbb{R}^{p\times n}$ with $\mathbb{R}$ being the field of real numbers. The system is supposed to be right invertible and controllable with $\operatorname{rank} B = m$. In what follows we shall be interested in finding the state feedback

$$u = Fx + Gv, \quad F \in \mathbb{R}^{m\times n}, \ G \in \mathbb{R}^{m\times p} \ \textit{with} \ \operatorname{rank} G = p \tag{2}$$

such that the transfer function of the closed–loop system $(C, A + BF, BG)$,

$$\begin{aligned} \dot{x} &= (A + BF)x + BGv \\ y &= Cx, \end{aligned}$$

will be of the form

$$T_{FG}(s) := C(sI - A - BF)^{-1}BG = \Lambda(s) := \operatorname{diag}\{\lambda_i(s)\} \tag{3}$$

where $\lambda_i(s) = s^{-r_i}$, $i = 1, 2, ..., p$ and r_i's are some positive integers called *the decouplability indices.*

This problem is also known as the Morgan problem, Morgan [13], or the problem of the row–by–row and integrator decoupling.

Many authors have tried to solve this famous problem. Recall for instance the work of Falb and Wolovich [7] where necessary and sufficient conditions were given for the decoupling of square systems by *regular state feedback (2)* (i.e. G invertible), Morse and Wonham [14] considered the decoupling of (1) by regular state feedback too, and the decoupling of "shifted" systems were dealt with by Descusse, Lafay and Malabre [3]. The cited last ones derived necessary and sufficient conditions, which are called structural elsewhere, for decoupling in terms of infinite zero orders.

As far as the problem of decoupling by dynamic feedback is concerned, the reader is referred to Dion and Commault [6], Kučera [10] and Eldem [16] for more details.

There are many other results concerning the Morgan problem but the question whether there exist explicit necessary and sufficient conditions under which the problem has a solution is still unresolved. For instance, Zagalak, Lafay and Herrera [15] have recently established necessary and sufficient conditions for the existence of a solution, however their conditions are stated in an implicit form, which means in fact that the existence of a solution cannot be directly verified using some quantities related to the original system (1).

Hence, it is still challenging to find another conditions having an explicit form. Such conditions would not only be more purified and conclusive from the theoretical point of view but they could be of interest when studying the problems like stability, robustness, etc. This paper is an attempt in this direction.

The following notation will be used throughout the paper. $\mathbb{R}$ denotes the ring of real numbers, $\mathbb{R}[\cdot]$ stands for the ring of polynomials over $\mathbb{R}$ while $\mathbb{R}(\cdot)$ denotes the field of rational functions.

2. PROBLEM FORMULATION

We introduce first the concept of a normal external description (Malabre, Kučera and Zagalak [12]) of the system (1), which turns out to be useful when studying the effect of (2) upon the system (1).

Let $N_1(s)$ and $D(s)$ be polynomial matrices of respective sizes $n \times m$ and $m \times m$ such that

$$\begin{bmatrix} sI_n - A & -B \end{bmatrix} \begin{bmatrix} N_1(s) \\ D(s) \end{bmatrix} = 0.$$

Then the matrices $N_1(s)$ and $D(s)$ are said to form *a (right) normal external description* of the system (1) if

(i) $\begin{bmatrix} N_1(s) \\ D(s) \end{bmatrix}$ is a minimal polynomial basis of $\operatorname{Ker} \begin{bmatrix} sI_n - A & -B \end{bmatrix}$,

(ii) $N_1(s)$ is a minimal polynomial basis of $\operatorname{Ker} \Pi(sI_n - A)$ where Π is a matrix representation of the maximal annihilator of B, i.e. $\Pi B = 0$.

Moreover,

$$N_1(s) = K \text{ block diag} \left(\begin{bmatrix} 1 \\ s \\ \vdots \\ s^{c_1 - 1} \end{bmatrix}, \ldots, \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{c_m - 1} \end{bmatrix} \right)$$

where $K \in \mathbb{R}^{n \times n}$ is nonsingular and $\sum c_i = n$. Due to these special properties, the matrix $N_1(s)$ is said to form *a polynomial basis* of $\mathbb{R}^n$ [4]; see [17].

Remark 1. It is to be noted that the matrices $N_1(s)$ and $D(s)$ are far from that to be unique. Any normal external description of (1) is given by

$$\begin{bmatrix} N_1(s) \\ D(s) \end{bmatrix} = \begin{bmatrix} \bar{N}_1(s) \\ \bar{D}(s) \end{bmatrix} U(s)$$

where $\bar{N}_1(s)$ and $\bar{D}(s)$ is a particular normal external description of (1) and $U(s)$ is a unimodular matrix that keeps the properties (i) and (ii) of normal external description.

Suppose that $c_i := \deg_{c_i} D(s), c_1 \geq c_2 \geq \ldots \geq c_m$ are the column degrees of $D(s)$. As $D(s)$ is column reduced, the indices $c_1, c_2, \ldots, c_m$ are the controllability indices of (1); see Malabre, Kučera and Zagalak [12]. Let further $T(s)$ denote the transfer function of (1). This transfer function can be written in the form

$$T(s) = N(s) D^{-1}(s) \tag{4}$$

where $N(s) = CN_1(s)$ and $N_1(s), D(s)$ form a normal external description of (1).

Since the Morgan problem is mainly a matter of the infinite zero structure of (1), it will be convenient to apply first the conformal mapping

$$s = \frac{1 + aw}{w}$$

to (4), where $a \neq 0$ is any complex number that is neither a pole nor a zero of $T(s)$. This mapping sends the point $s = a$ to $w = \infty$ and $s = \infty$ to $w = 0$. Hence, both the finite and infinite poles and zeros of $T(s)$ will be at finite positions, which enables us to handle these two structures in a uniform way.

We define

$$\begin{bmatrix} N(w) \\ D(w) \end{bmatrix} := \begin{bmatrix} N\left(\frac{1+aw}{w}\right) \\ D\left(\frac{1+aw}{w}\right) \end{bmatrix} \operatorname{diag}\left[w^{c_1}, \ldots, w^{c_m}\right] \tag{5}$$

It can be readily verified that the matrix $\begin{bmatrix} N(w) \\ D(w) \end{bmatrix}$ is polynomial over $\mathbb{R}[w]$, column reduced, and the column degrees of (5) are $c_1, c_2, \ldots, c_m$. The same holds for $D(w)$ and moreover $D(w)$ is invertible at $w = 0$.

In the same way, we can write $T_{FG}(s)$ in the form

$$T_{FG}(w) = N(w) D_F^{-1}(w) G \tag{6}$$

where $D_F(w) := D(w) - FN_1(w)$ has the same properties as $D(w)$.

Now, since $N(w)$ is right invertible and supposing, without any loss of generality, $G = \begin{bmatrix} I_p \\ 0 \end{bmatrix}$, we can rewrite (6) into the form

$$\begin{bmatrix} Q(w) & 0 \end{bmatrix} U^{-1}(w) D_F^{-1}(w) \begin{bmatrix} I_p \\ 0 \end{bmatrix} = \Lambda(w) \tag{7}$$

where $N(w) = [Q(w) \;\; 0] U^{-1}(w)$ with $Q(w)$ invertible and $U(w)$ unimodular.

The equation (7) then implies that the matrix $\bar{D}(w) := U^{-1}(w) D_F^{-1}(w)$ is of the form

$$\bar{D}(w) = \begin{bmatrix} Q^{-1}(w)\Lambda(w) & X(w) \\ Y(w) & Z(w) \end{bmatrix} \tag{8}$$

where $X(w)$, $Y(w)$ and $Z(w)$ are some matrices over $\mathbb{R}(w)$ of respective sizes $p \times (m-p)$, $(m-p) \times p$ and $(m-p) \times (m-p)$.

Hence, on the basis of the above considerations, we can establish the following necessary and sufficient conditions for a solution to the Morgan problem.

Proposition 1. There exists a solution to the Morgan problem if and only if there exist a polynomial, column reduced matrix $D_F(w)$ and rational matrices $X(w)$, $Y(w)$ and $Z(w)$ such that

(i) $\deg_{c_i} D_F(w) = c_i \quad i = 1, 2, ..., m$;

(ii) the relationship (8) holds;

(iii) $D_F(w)$ is invertible at $w = 0$.

3. EXPLICIT NECESSARY CONDITIONS FOR DECOUPLING

Proposition 1 is of course nothing else than another formulation of Morgan's problem. To bring more insight into the structure of (8), we shall use the following two results that can be found elsewhere; Loiseau and Zagalak [9] and Baragana [1].

Lemma 1. Let $P(s)$ be an $m \times m$, polynomial and column reduced matrix over $\mathbb{R}[s]$ with column degrees $k_1 \geq k_2 \geq ... \geq k_m$. Let $\psi_1(s) \geq \psi_2(s) \geq ... \geq \psi_m(s)$ be its invariant polynomials ($\psi(s) \geq \phi(s)$ means $\phi(s)$ divides $\psi(s)$.) Then

$$\sum_{i=j}^{m} \deg \psi_i(s) \leq \sum_{i=j}^{m} c_i, \quad \text{for } j = 1, 2, ..., m.$$

with equality for $i = 1$.

The second lemma comes from the fact that for every $m \times n$, rational matrix $H(s)$ there exist unimodular matrices, say $U(s)$ and $V(s)$, such that the matrix $U(s)\ H(s)\ V(s)$ is in the Smith–McMillan form, i.e.

$$U(s)\ H(s)\ V(s) = \begin{bmatrix} \operatorname{diag}\left\{\frac{\alpha_i(s)}{\beta_i(s)}\right\}_{i=1}^{r} & 0 \\ 0 & 0 \end{bmatrix}$$

where $\beta_1(s) \geq \beta_2(s) \geq \ldots \geq \beta_r(s), \alpha_r(s) \geq \alpha_{r-1}(s) \geq \ldots \geq \alpha_1(s)$, and $r = \operatorname{rank} H(s)$. The rational functions $\alpha_i(s)/\beta_i(s),\quad i = 1, 2, ..., r$ are called *the Smith–McMillan invariants of* $H(s)$.

Lemma 2. Let $H(s)$ be an $m \times n$ rational matrix with its Smith–McMillan invariants $\frac{\alpha_1(s)}{\phi_1(s)}, \ldots, \frac{\alpha_r(s)}{\phi_r(s)}$, $r = \operatorname{rank} H(s)$. Then there exist rational matrices $X(s), Y(s)$ and $Z(s)$ such that the $(m+p) \times (n+q)$ matrix

$$G(s) = \begin{bmatrix} H(s) & X(s) \\ Y(s) & Z(s) \end{bmatrix}$$

has the Smith–McMillan invariants $\frac{\beta_1(s)}{\psi_1(s)}, \ldots, \frac{\beta_t(s)}{\psi_t(s)}$, $t = \operatorname{rank} G(s)$, if and only if

$$\beta_{j+p+q}(s) \geq \alpha_j(s) \geq \beta_j(s), \quad j = 1, 2, .., r$$

and

$$\psi_j(s) \geq \phi_j(s) \geq \psi_{j+p+q}(s), \quad j = 1, 2, .., r$$

where $\beta_j(s) := 0$ and $\psi_j(s) := 1$ for $j > t$.

Now we can derive a set of necessary conditions for decoupling. To that end, let $\psi_1(w) \geq \psi_2(w) \geq \ldots \geq \psi_m(w)$ be the invariant polynomials of $D_F(w)$. Then

$$\operatorname{diag}\left[\ \frac{1}{\psi_1(s)},\ \ \frac{1}{\psi_2(s)}, \ldots, \frac{1}{\psi_m(s)}\ \right]$$

is clearly the Smith–McMillan form of $\bar{D}(w)$. Let further

$$\operatorname{diag}\left[\ \frac{\epsilon_1(w)}{\phi_1(w)}, \ldots, \frac{\epsilon_p(w)}{\phi_p(w)}\ \right]$$

be the Smith–McMillan form of $Q^{-1}(w)\Lambda(w)$. It is clear that $\epsilon_i(w) = w^{k_i},\quad i = 1, 2, ..., p$ where k_i's are some non-negative integers. Now we apply Lemma 2 to the equation (8) to get

$$\psi_{j+2(m-p)}(w) \leq \phi_j(w) \leq \psi_j(w), \quad j = 1, 2, ..., p \tag{9}$$

and

$$\beta_j(w) \leq \epsilon_j(w) \leq \beta_{j+2(m-p)}(w), \quad j = 1, 2, ..., p, \tag{10}$$

where $\beta_j(w) = 1, \ \ j = 1, 2, ..., m$, $\beta_j(w) := 0$ for $j > m$ and $\psi_j(w) := 1$ for $j > m$.

Analysing (9) and (10), we can obtain the above divisibility conditions in a more convenient form. From (9), it follows that

$$\psi_j(w) \geq \phi_j(w) \geq \psi_{j+2(m-p)}(w), \quad j = 1, 2, ..., 2p - m, \tag{11}$$

and

$$\psi_j(w) \geq \phi_j(w), \quad j = 2p - m + 1, ..., p \tag{12}$$

since $\psi_j(w) := 1, \ \ j > m$. The divisibility conditions (10) imply

$$\beta_j(w) \leq \epsilon_{j+2(m-p)}(w), \quad j = 1, 2, ..., 2p - m,$$

and hence,

$$\epsilon_j(w) = 1, \quad j = 1, 2, ..., 2p - m$$

since $\beta_j(w) = 1, \ \ j = 1, 2, ..., m$ and $\beta_j(w) = 0$ for $j > m$. For $j = 2p - m + 1, ..., p$, it easily follows that

$$\epsilon_j(w) = w^{k_j} \tag{13}$$

with k_j's mentioned above.

Another conditions are implied by Lemma 1. The degrees of the polynomials $\psi_1(s), \psi_2(s), ..., \psi_m(s)$ satisfy the inequalities

$$\sum_{i=j}^{m} \deg \psi_i(s) \leq \sum_{i=j}^{m} c_i, \quad \text{for} \ \ j = 1, 2, ..., m, \tag{14}$$

where c_j's are the controllability indices of (1) and equality holds for $j = 1$.

It is to be noted that if $\psi_j(w)$'s are chosen such that $\psi_j(w)$ is not divisible by $w, \ \ j = 1, 2, ..., m$, then the conditions (14) are also necessary and sufficient for there to exist a matrix $D_F(w)$ satisfying (i) and (iii) of Proposition 1.

Unfortunately, the conditions (11), (12), and (13) are only necessary for (8) to hold, as follows from Lemma 2. To sum up, we state the following

Main Theorem. With the notation above, if there exists a solution to the Morgan problem, then (11), (12), (13) and (14) hold.

Remark 2. Even if the presented theory concerns the Morgan problem, i.e. the row-by-row and integrator decoupling problem, it can be readily seen that our setting is more general and can be used when trying to solve the problem of decoupling with stability, too. To this end, it is sufficient to consider the matrix $\Lambda(s)$ to be a diagonal matrix with strictly proper and stable rational functions on the main diagonal.

Remark 3. It is to be noted that the condition (13) corresponds to the necessary and sufficient condition given by Dion and Commault [6] for decoupling by dynamic state feedback.

4. CONCLUSIONS

We established new explicit and necessary conditions for the existence of a solution to the Morgan problem. Our approach is based on the matrix completion theory, a branch of linear algebra that is now intensively developed. It seems that the ideas coming from this theory could play an important role and help in solving many difficult questions of control theory.

The necessary conditions for decoupling established here hint that we shall need more information concerning the structure of the system (1) for that to derive necessary and sufficient conditions. Probably some deeper insight into this problem could be given when considering the Heymann theorem and the Loiseau theorem, see Heymann [8] and Loiseau [11]. But that is a challenge for some future work.

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