KYBERNETIKA — VOLUME 29 (1993), NUMBER 5, PAGES 417–422

SOME REMARKS ON THE BRUNOVSKY CANONICAL FORM

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The Brunovsky canonical form is obtained via a module-theoretic approach which covers the time-varying case.

INTRODUCTION

Among the various canonical forms which were proposed for constant linear systems, the one due to Brunovsky [1] certainly is the most profound. It characterizes a dynamics modulo the group of static state feedbacks by a finite set of pure integrators. Its proof, which is quite computational, has been improved in various ways, and can be found in several textbooks (see, e. g., [12, 13, 20, 21] and the references therein). We here attempt to give a more algebraic and, hopefully, more intrinsic approach. It covers the time-varying case, which seems until now to have been left untouched.

We employ our module-theoretic framework [5], the corresponding filtrations [3, 4] and their connections with feedbacks. The uniqueness of the controllabity indices follows at once from some associated graduation.

A first draft of this result has already been presented [8].

1. THE BASIC FORMALISM

The ground field k is *differential* with respect to $d/dt =$ "$\cdot$" [14]. Denote by $k[d/dt]$ the set of linear differential operators of the type $\sum_{\text{finite}} a_\alpha \frac{d^\alpha}{dt^\alpha}$. This ring, which is in general noncommutative[1], nevertheless enjoys the property of being a *principal ideal* ring (see, e. g., [2]). The main properties of left $k[d/dt]$-modules mimic those of modules over commutative principal ideal rings [2].

Notation. The left $k[d/dt]$-module spanned by a set $w = \{w_i | i \in I\}$ is written $[w]$.

A *linear system* [5, 6] is a finitely generated left $k[d/dt]$-module. A *linear dynamics* D [5] is a linear system which contains a finite set $u = (u_1, \ldots, u_m)$, such that the

[1] It is commutative if, and only if, k is a field of *constants*.

quotient module $D/[u]$ is torsion. This dynamics can be given a Kalman state-variable representation [5]:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + B \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \tag{1}$$

where

- the dimension n of the (Kalman) state $x = (x_1, \ldots, x_n)$ is equal to the dimension of $D/[u]$ as a k-vector space;
- the matrices A and B have their entries in k and are of appropriate sizes.

A linear system is said to be *controllable* [5, 6] if, and only if, the associated module is *free*. A linear dynamics is *controllable* if, and only if, the corresponding linear system is controllable.

Assume for the sake of simplicity that the input u is *independent*, i. e., that the module $[u]$ is free. Formula (1) determines two *filtrations*[2] of the module D:

- The *(Kalman) input-state filtration* $\mathcal{F} = \{\mathcal{F}_\nu | \nu = 0, \pm 1, \pm 2, \ldots\}$ is an increasing sequence of k-vector spaces $\mathcal{F}_\nu$ such that

$$\mathcal{F}_\nu = \begin{cases} 0, \text{ if } \nu \leq -2 \\ \mathrm{span}_k(x), \text{ if } \nu = -1 \\ \mathrm{span}_k(x, u, \ldots, u^{(\nu)}), \text{ if } \nu \geq 0 \end{cases}$$

  where $\mathrm{span}_k(x, u, \ldots, u^{(\nu)})$ is the k-vector space spanned by the components of x, u and by the derivatives up to order ν of the components of u.

- The *(Kalman) state filtration* $\mathcal{X} = \{\mathcal{X}_\rho | \rho = 0, 1, 2, \ldots\}$ is a decreasing sequence of submodules

$$\mathcal{X}_\rho = [x^{(\rho)}].$$

The two filtrations $\mathcal{F}$ and $\mathcal{X}$ are obviously independent of the choice of the Kalman state x.

A *(regular) static state-feedback* [3] of the dynamics D is defined by a finite set $v = (v_1, \ldots, v_m)$ of elements in D, which plays the role of a *new input*, such that the filtration $\mathcal{G} = \{\mathcal{G}_\nu | \nu = 0, \pm 1, \pm 2, \ldots\}$, where

$$\mathcal{G}_\nu = \begin{cases} 0, \text{ if } \nu \leq -2 \\ \mathrm{span}_k(x), \text{ if } \nu = -1 \\ \mathrm{span}_k(x, v, \ldots, v^{(\nu)}), \text{ if } \nu \geq 0 \end{cases}$$

coincides with $\mathcal{F}$, i. e., for any ν, $\mathcal{F}_\nu = \mathcal{G}_\nu$.
One easily recovers the classic formulas:

$$\begin{pmatrix} \overline{x}_1 \\ \vdots \\ \overline{x}_n \end{pmatrix} = P \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \tag{2}$$

[2] *Filtrations* and the associated *graduations* are common algebraic tools [16, 18].

$$\begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} = F \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + G \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} \tag{3}$$

where

- $\overline{x} = (\overline{x}_1, \dots, \overline{x}_n)$ is another Kalman state,
- P, F and G are matrices over k of appropriate sizes,
- P and G are invertible.

It follows at once from the above definition that there exists a regular static state feedback between two dynamics D and $\tilde{D}$, with input-state filtrations $\mathcal{F}$ and $\tilde{\mathcal{F}}$, if, and only if, the two filtered modules D and $\tilde{D}$ are isomorphic.

Remark. Let us relate the above notion of feedback to the concept of automorphism. First notice that D may be viewed as a k-vector space with filtration $\mathcal{F}$. The quotient $D/\mathcal{F}_{-1}$ is a k-vector space which is canonically isomorphic to $[u]$, also considered as a k-vector space: We will not distinguish those two vector spaces. To $\mathcal{F}$ corresponds a filtration $\overline{\mathcal{F}}$ of $[u]$ defined by

$$\overline{\mathcal{F}}_\nu = \begin{cases} 0, \text{ if } \nu \not\leq 0 \\ \operatorname{span}_k(u, \dots, u^{(\nu)}), \text{ if } \nu \geq 0 \end{cases}$$

A *(regular) static state feedback* is a k-linear filtered automorphism Ψ of D, i.e., a k-linear automorphism which leaves the filtration $\overline{\mathcal{F}}$ invariant, such that the induced mapping on the graded k-vector space $\operatorname{gr}_{\overline{\mathcal{F}}}[u]$ is an automorphism of the graded module $\operatorname{gr}_{\overline{\mathcal{F}}}[u]$ over the graded ring $\operatorname{gr} k[d/dt]$. This abstract definition of the group of static state feedbacks (compare, e.g., with [21]) permits to recover (2) and (3). If k is a field of constants, the above definition may be slightly simplified: A static state feedback is a k-linear filtered automorphism of D, such that its restriction to $[u]$ is a $k[d/dt]$-linear automorphism which preserves $\overline{\mathcal{F}}$.

2. WELL FORMED DYNAMICS

The next result interprets in our formalism the classic condition stating that the rank of the matrix B in (1) is m.

Theorem 1. The following three conditions are equivalent:
(i) $\mathcal{X}_0 = D$;
(ii) $\operatorname{rk} \mathcal{X}_0 = m$;
(iii) $\operatorname{rk} B = m$.

Proof. (i) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii) are obvious.
(i) $\Rightarrow$ (iii): There exists a k-vector space $U \subseteq \operatorname{span}_k(u)$, $\dim U = \operatorname{rk} B = m' \leq m$, such that any element of U belongs to $\operatorname{span}_k(x, \dot{x})$. Straightforward calculations demonstrate the existence of a k-vector space U_1, such that

- $\dim U_1 = m - m'$,
- $\mathrm{span}_k(u) = U \oplus U_1$,
- $U_1 \cap [x] = \{0\}$.

$D/[u]$ and $[x]/[U]$ are isomorphic torsion modules. Thus, $\mathrm{rk}\, B = m$ implies $[x] = D$. □

A dynamics D, which satisfies one of those equivalent conditions, is said to be *well formed.*

Remark. Assume that D is *not* well formed, i.e., that $m' \nleq m$. The above proof demonstrates the existence of another basis $v = (v_1, \ldots, v_m)$ of $\mathrm{span}_k(u)$, such that $(v_1, \ldots, v_{m'})$ is a basis of U and $(v_{m'+1}, \ldots, v_m)$ a basis of U_1. The dynamics $[x]$ with input $(v_1, \ldots, v_{m'})$ is a well formed dynamics associated to D. Such an associated well formed dynamics is unique up to an obvious isomorphism. Notice that the correspondence between u and v is a trivial static state feedback.

3. THE BRUNOVSKY CANONICAL FORM

Take a controllable and well formed dynamics D with input $u = (u_1, \ldots, u_m)$. Associate to the state filtration $\mathcal{X}$ of D the graded module $\mathrm{gr}_{\mathcal{X}} D = \bigoplus \mathcal{X}_\rho / \mathcal{X}_{\rho+1}$ over the graded ring $\mathrm{gr}\; k[d/dt]$.

Lemma 1. The module $\mathrm{gr}_{\mathcal{X}} D$ is graded-free[3]. For any $\rho \geq 0$, $\mathcal{X}^\rho / \mathcal{X}^{\rho+1}$ is an m-dimensional k-vector space.

Proof. For any $\rho \geq \theta$, the derivation d/dt induces a k-linear mapping $d_\rho : \mathcal{X}_\rho / \mathcal{X}_{\rho+1} \to \mathcal{X}_{\rho+1} / \mathcal{X}_{\rho+2}$, which is obviously surjective. Assume that d_ρ is not injective. The existence of a non-zero element in $\ker d_\rho$ implies the existence of z in $\mathcal{X}_\rho$, $z \neq 0$, such that $\dot{z} = 0$, which contradicts the freeness of D. The d_ρ's thus are isomorphisms. The conclusions follow at once. □

Denote by $\mathrm{gr}_{\mathcal{X}} \xi$ the canonical image in $\mathrm{gr}_{\mathcal{X}} D$ of an element ξ in D. There exists a finite binary sequence $\mathcal{S} = (\nu_\alpha, \delta_\alpha)$ of strictly positive integers, such that

$$\dim(\mathrm{gr}_{\mathcal{X}} \mathrm{span}_k(u) \cap \mathcal{X}_{\nu_\alpha} / \mathcal{X}_{\nu_\alpha + 1}) = \delta_\alpha .$$

The above lemma indicates that the dynamics D can be brought by a static state feedback to a set of pure integrators

$$\tilde{x}_{\mu_\alpha}^{(\nu_\alpha)} = v_{\mu_\alpha} \tag{4}$$

where

- the $\mathrm{gr}_{\mathcal{X}} \tilde{x}_{\mu_\alpha}$'s are a basis of the k-vector space $\mathcal{X}_0 / \mathcal{X}_1$;
- the v_{μ_α}'s are the new control variables.

The preceding constructions yield the

[3] See [16, 18] for a definition of *graded-free*, or *free-graded*, modules

Lemma 2. The sequence $\mathcal{S}$ is unique and $\sum \delta_\alpha = m$, $\sum \delta_\alpha \nu_\alpha = n$.
$\mathcal{S}$ is the *Brunovsky sequence* of the dynamics D. The ν_α's are the *controllability*, or *Kronecker, indices*; they correspond to pure integrators (4) of orders ν_α which are repeated δ_α times.

Formula (4) defines the *Brunovsky canonical form* associated to D. Lemmas 1 and 2 yield the

Theorem 2. The Brunovsky sequence (resp. canonical form) constitutes a complete set of invariants with respect to the action of the group of static state feedbacks on a controllable and well formed dynamics.

Remark. Consider a dynamics D which is not necessarily controllable or well formed. Let T be the torsion submodule of D and $\theta : D \to D/T$ be the canonical epimorphism. The dynamics $\overline{D} = D/T$, with input $\overline{u} = (\overline{u}_1 = \theta u_1, \ldots, \overline{u}_m = \theta u_m)$, is controllable. The Brunovsky canonical form or the Brunovsky sequence of D, by definition, are those of the well formed dynamics associated to $\overline{D}$ (see the remark of Section 2).

Example. Take a controllable and well formed dynamics D with a single input u, i.e., $m = 1$. Choose a basis b of D. Notice that any other basis $\overline{b}$ is related to b by $\overline{b} = \varpi b$, where $\varpi \in k$, $\varpi \neq 0$. If $n = \dim D/[u]$, u is a k-linear combination of $b, \dot{b}, \ldots, b^{(n)}$. Set $x_1 = b, \ldots, x_n = b^{(n-1)}$. It yields the *controller form* (see, e.g., [10])

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ & \vdots & \\ \dot{x}_{n-1} & = & x_n \\ \dot{x}_n & = & \alpha_1 x_n + \ldots + \alpha_n x_1 + \beta u \end{array} \right.$$

where $\alpha_1, \ldots, \alpha_n, \beta \in k, \beta \neq 0$. The Brunovsky canonical form is obtained by a straightforward static state feedback

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ & \vdots & \\ \dot{x}_{n-1} & = & x_n \\ \dot{x}_n & = & v \end{array} \right.$$

4. CONCLUSION

The Brunovsky canonical form can easily been obtained for nonlinear dynamics which are linearizable by static state feedbacks [11, 17]. It has been further extended by Rudolph [19] to nonlinear dynamics which are *flat* [9] and *well formed* by means of *quasi-static* state feedbacks [3]. His result also is new for controllable and well formed linear dynamics as any basis of the corresponding free module can now serve for obtaining the Brunovsky form via a quasi-static feedback.

Our approach applies to constant [15] and time-varying discrete-time systems via the tools developed in [7].

(Received February 16, 1993.)

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